Ex. A natural cubic spline is given by

\[ S(x) = \begin{cases} S_0(x) = 1 + \frac{1}{2} (x-1) - (x-1)^3 & 1 \leq x \leq 2 \\ S_1(x) = a + b(x-2) + (x-2)^2 + d(x-2)^3 & 2 < x \leq 3 \end{cases} \]

Determine \( a, b, c, d \).

Solution: \( S_0(2) = S_1(2) \) \( a = 1 + \frac{1}{2} - 1 = \frac{1}{2} \) \[ a = \frac{1}{2} \]

\[ S'(x) = \begin{cases} S_0'(x) = \frac{1}{2} - 3(x-1)^2 & 1 \leq x \leq 2 \\ S_1'(x) = b + 2c(x-2) + 3d(x-2)^2 & 2 < x \leq 3 \end{cases} \]

\[ S_0'(2) = S_1'(2) \] \[ \frac{1}{2} - 3 = b \] \[ b = -\frac{5}{2} \]

\[ S''(x) = \begin{cases} S_0''(x) = -6(x-1) & 1 \leq x \leq 2 \\ S_1''(x) = 2c + 6d(x-2) & 2 < x \leq 3 \end{cases} \]

\[ S_0''(2) = S_1''(2) \] \[ -6 = 2c \] \[ c = -3 \]

\[ S_0''(1) = 0 \text{ - satisfied} \] \[ S_1''(3) = -6 + 6d = 0 \] \[ d = 1 \]

Ex2) Use the most accurate three-point formula \( f^3 \) to determine each missing entry in the table.
<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$f'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>3.68</td>
<td>0.25</td>
</tr>
<tr>
<td>2.1</td>
<td>3.69</td>
<td>-0.05</td>
</tr>
<tr>
<td>2.2</td>
<td>3.67</td>
<td>-0.35</td>
</tr>
<tr>
<td>2.3</td>
<td>3.62</td>
<td>-0.65</td>
</tr>
</tbody>
</table>

\[
f'(x) = \frac{1}{2h} \left[ -3f(x) + 4f(x+h) - f(x+2h) \right] + \frac{h^2}{3} f^{(5)}(\xi) \]

\[
f''(x) = \frac{1}{2h} \left[ -f(x-h) + f(x+h) \right] - \frac{h^2}{6} f^{(4)}(\xi_1) \]

\[
f'''(x) = \frac{1}{2h} \left[ f(x-2h) - 4f(x-h) + 3f(x) \right] + \frac{h^2}{3} f^{(3)}(\xi_2) \]

\[
f''(x) = \frac{1}{0.2} \left[ -3 \cdot 3.68 + 4 \cdot 3.69 - 3.67 \right] = \frac{1}{0.2} \left[ -11.04 + 14.76 - 3.67 \right] = \frac{1}{0.2} \left( 0.05 \right) = \frac{5}{20} = \frac{1}{4} \]

\[
f'(2.1) = \frac{1}{0.2} \left[ 3.67 - 3.68 \right] = \frac{1}{0.2} \left( -0.01 \right) = -\frac{1}{20} \]

\[
f'(2.2) = \frac{1}{0.2} \left[ 3.62 - 3.69 \right] = \frac{1}{0.2} \left( -0.07 \right) = -\frac{7}{20} \]

\[
f'(2.3) = \frac{1}{0.2} \left[ 3.69 - 4 \cdot 3.67 + 3 \cdot 3.62 \right] = \frac{1}{0.2} \left[ -0.13 \right] = -0.65 \]
Ex. A centered difference is used for computing $f''(-2)$ with $h = 0.2$. Find an upper bound of the error if

$$f(x) = e^{\frac{x}{3}} + x^2$$

Solution:

$$\text{Error} = \frac{h^2}{6} f^{(3)}(\xi) \quad \text{where } \xi \in (2, 1.8)$$

$$f' = \frac{1}{3} e^{\frac{x}{3}} + 2x$$

$$f'' = \frac{1}{9} e^{\frac{x}{3}} + 2$$

$$f''' = \frac{1}{27} e^{\frac{x}{3}} \quad |f'''(\xi)| \leq \frac{1}{27} e^{-\frac{1.8}{3}} = \frac{1}{27} e^{-0.6}$$

$$\text{Error bound} = \frac{1}{5^2} \cdot \frac{1}{6} \cdot \frac{1}{27} e^{-0.6} = \frac{1}{27 \cdot 150} e^{-0.6}$$

Ex. The quadrature formula

$$\int f(x) \, dx = c_0 f(-1) + c_1 f(0) + c_2 f(1)$$

is exact for all polynomials of degree $n$ or equal to $2$. Determine $c_0, c_1, c_2$

Solution:

$$\int_1^1 dx = 2 \quad c_0 + c_1 + c_2 = 2$$
\[ \int_0^1 x \, dx = 0 \quad -c_0 + c_2 = 0 \quad \Rightarrow \quad c_0 = c_2 \]

\[ \int_0^1 x^2 \, dx = \frac{2}{3} \quad c_0 + c_2 = \frac{2}{3} \quad 2c_2 = \frac{2}{3} \quad \Rightarrow \quad c_2 = \frac{1}{3} \]

From the first equation:

\[ c_1 = 2 - \frac{2}{3} = \frac{4}{3} \]

Ex. #8195

The Trapezoidal rule applied to

\[ \int_0^2 f(x) \, dx \]

gives value 5, and the Midpoint rule gives value 4. What value does Simpson's rule give?

Solution:

\[ \int_0^2 f(x) \, dx = \frac{2}{2} \left[ f(0) + f(2) \right] = 5 \]

\[ \Rightarrow \quad f(0) + f(2) = 5 \]

\[ h = 1 \]

\[ \int_0^1 f(x) \, dx = 2(1) \cdot f(1) = 4 \]

\[ \Rightarrow \quad f(1) = 2 \]

\[ h = \frac{1}{2} \]

\[ \int_0^2 f(x) \, dx = \frac{1}{3} \left[ f(0) + 4f(1) + f(2) \right] = \frac{1}{3} \left[ 5 + 4 \cdot 2 \right] = \frac{1}{3} \left[ 5 + 8 \right] = \frac{13}{3} \]
Ex. Set up a composite Simpson's method to evaluate the integral
\[ \int_{0}^{2} e^{2x} \sin 3x \, dx \quad \text{with } n = 8 \]
Do not evaluate to final answer.

Solution:

\[ \begin{array}{cccccc}
0 & 0.25 & 0.5 & 0.75 & 1.00 & 1.25 & 1.50 & 1.75 & 2 \\
\end{array} \]
Thus: \( h = 0.25 \)

\[ \int_{0}^{2} e^{2x} \sin 3x \, dx = \frac{0.25}{3} \left( e^{0} \sin 0 + 4 e^{2 \times 0.25} \sin 3(0.25) + 2 e^{2 \times 0.5} \sin 3(0.5) + 4 e^{2 \times 0.75} \sin 3(0.75) + 2 e^{2 \times 1.00} \sin 3(1.00) + 4 e^{2 \times 1.25} \sin 3(1.25) + 2 e^{2 \times 1.50} \sin 3(1.50) + e^{2 \times 1.75} \sin 3(1.75) \right) \]

Ex: Set up Gaussian quadrature with \( n = 3 \) to evaluate the integral:
\[ \int_{0}^{1} x^2 e^{-x} \, dx \]
Do not simplify your answer to a final value.
\[ x = \frac{1}{2} \left[ (b-a)t + a + b \right] \]
\[ x = \frac{1}{2} (t+1) \quad dx = \frac{1}{2} \, dt \]

\[ \int x^2 e^{-x} \, dx = \frac{1}{8} \int (t+1)^2 e^{-\frac{1}{2}(t+1)} \, dt \]

\[ = \frac{1}{8} \left[ e^{-\frac{1}{2} (t+1)} + 2e^{-\frac{1}{2} (t+1)} + 2e^{-\frac{1}{2} (t+1)} \right] \]
\[ = \frac{1}{8} \left[ 0.5555 \cdot (0.7746+1)^2 e^{-\frac{1}{2} (0.7746+1)} + 0.8889 \, e^{-\frac{1}{2} (0.7746+1)} \right. \]
\[ + \left. 0.5555 \cdot (-0.7746+1)^2 e^{-\frac{1}{2} (-0.7746+1)} \right] \]

**Ex.** Determine constants \( a, b, c, d \) and \( e \) that will produce a quadrature formula

\[ \int f(x) \, dx = af(-1) + bf(0) + cf(1) + df(-1) + ef(1) \]

that has degree of precision 4.

\[ f(x) = 1 \quad f'(x) = 0 \]
\[ \int 1 \, dx = 2 \quad a+b+c = 2 \]
\[ f(x) = x \quad f'(x) = 1 \]
\[ \int x\,dx = \frac{x^2}{2} \quad a(-1) + c + d + e = 0 \]
\[ f(x) = x^2 \quad f'(x) = 2x \]
\[ \int x^2\,dx = \frac{x^3}{3} \quad a + c - 2d + 2e = \frac{2}{3} \]
\[ f(x) = x^3 \quad f'(x) = 3x^2 \]
\[ \int x^3\,dx = 0 \quad a(-1) + c + 3d + 3e = 0 \]
\[ f(x) = x^4 \quad f'(x) = 4x^3 \]
\[ \int x^4\,dx = \frac{x^5}{5} \quad a + c - 4d + 4e = \frac{2}{5} \]

Thus, we get the system

(1) \[ a + b + c = 2 \]
(2) \[ -a + c + d + e = 0 \]
(3) \[ a + c - 2d + 2e = \frac{2}{3} \]
(4) \[ -a + c + 3d + 3e = 0 \]
(5) \[ a + c - 4d + 4e = \frac{2}{5} \]
Subtracting (4) - (2) we get
\[ 2(d + e) = 0 \Rightarrow d + e = 0 \Rightarrow e = -d \]
Hence, from (2) we get \(-a + c = 0 \Rightarrow c = a\)

Eliminating \(c\) and \(e\) from (3) and (5)

\[
\begin{align*}
2a - 2d - 2d &= \frac{2}{3} \\
2a - 4d - 4d &= \frac{2}{5}
\end{align*}
\]

\[
\begin{align*}
2a - 4d &= \frac{2}{3} \\
2a - 8d &= \frac{2}{5}
\end{align*}
\]

\[
\begin{align*}
a - 2d &= \frac{1}{3} \\
a - 4d &= \frac{1}{5}
\end{align*}
\]

\[-2d - (-4d) = \frac{1}{3} - \frac{1}{5}
\]

\[2d = \frac{2}{15} \quad \boxed{d = \frac{1}{15}}\]

\[a = \frac{1}{3} + \frac{2}{15} = \frac{7}{15} \quad c = \frac{7}{15} \quad e = -\frac{1}{15}\]

From (1) \[\frac{14}{15} + b = 2 \quad b = 2 - \frac{14}{15} = \frac{16}{15}\]
Ex. Determine the values of \( n \) and \( h \) required to approximate

\[
\int_0^x dx
\]

within \( 10^{-6} \) using Composite Simpson's rule.

Solution: The error of composite Simpson's rule is

\[
E^s(f) = -\frac{b-a}{180} h^4 f''''(\xi)
\]

where \( \xi \) is in \((a, b)\).

Here \( a = 0, b = 1 \).

\[
f(x) = e^x
\]

\[
|f''''(x)| = |e^x| = e
\]

\[
|E^s(f)| = \frac{1}{180} h^4 e \leq 10^{-6}
\]

We solve for \( h \)

\[
h^4 \leq \frac{180 \cdot 10^{-6}}{e} = 6.62183 \times 10^{-5}
\]

\[
h \leq 0.09020278861
\]