

1.3 Algorithms and convergence

1) Convergence.

Def: Suppose $\{\beta_n\}_{n=1}^{\infty}$ is a sequence which converges to zero, and $\{\alpha_n\}_{n=1}^{\infty}$ converges to α . If there is a constant k

$$|\alpha_n - \alpha| \leq k |\beta_n| \text{ for large } n$$

we say that $\{\alpha_n\}_{n=1}^{\infty}$ converges to α with rate of convergence $O(\beta_n)$.

In most cases we use

$$\beta_n = \frac{1}{n^p} \quad \text{for some } p > 0.$$

② Ex. $\alpha_n = \frac{n+3}{n^2+1} \quad \alpha_n \rightarrow 0$

$$\alpha_n \leq \frac{n+n}{n^2} = \frac{2n}{n^2} = 2 \cdot \frac{1}{n} \quad \forall n \geq 3$$

$\Rightarrow \alpha_n \rightarrow 0$ with rate of convergence $O(\frac{1}{n})$

① Ex: $\alpha_n = \frac{5}{n^5+5} \leq \frac{5}{n^5} \Rightarrow \alpha_n \rightarrow 0$ with $O(\frac{1}{n^5})$

Ex. Find the rate of convergence of

#6^b

$$\lim_{n \rightarrow \infty} \sin \frac{1}{n^2} = 0$$

$$\left| \sin \frac{1}{n^2} \right| \leq \frac{1}{n^2}$$

Thus the rate of convergence is $O\left(\frac{1}{n^2}\right)$.

Def: Suppose $\lim_{h \rightarrow 0} G(h) = 0$ and $\lim_{h \rightarrow 0} F(h) = L$

If a positive constant K exists with

$|F(h) - L| \leq K|G(h)|$ for h -small
then we write

$$F(h) = L + O(G(h))$$

Usually: $G(h) = h^p$, where $p > 0$.

Ex. Find the rate of convergence of

#7c

$$\lim_{h \rightarrow 0} \frac{\sinh - h \cosh}{h} = 0$$

$$\sinh \approx h - \frac{h^3}{3!}$$

$$\cosh \approx 1 - \frac{h^2}{2!} + \frac{h^4}{4!}$$

$$\frac{\sinh - h \cosh}{h} \approx \frac{h - \frac{h^3}{3!} - h + \frac{h^3}{2!}}{h} = \frac{1}{3} h^2 = O(h^2)$$