

Chapter 2 Solutions of Equations in One Variable

In this chapter we shall study methods for solving (approximately) the equation

$$f(x) = 0 \quad p \begin{cases} \nearrow \text{root} \\ \rightarrow \text{zero} \\ \searrow \text{solution} \end{cases}$$

If p is a solution of this equation, then we shall use the so-called iterative methods which generate

x_0, x_1, x_2, \dots
such that

$$\lim_{n \rightarrow \infty} x_n = p$$

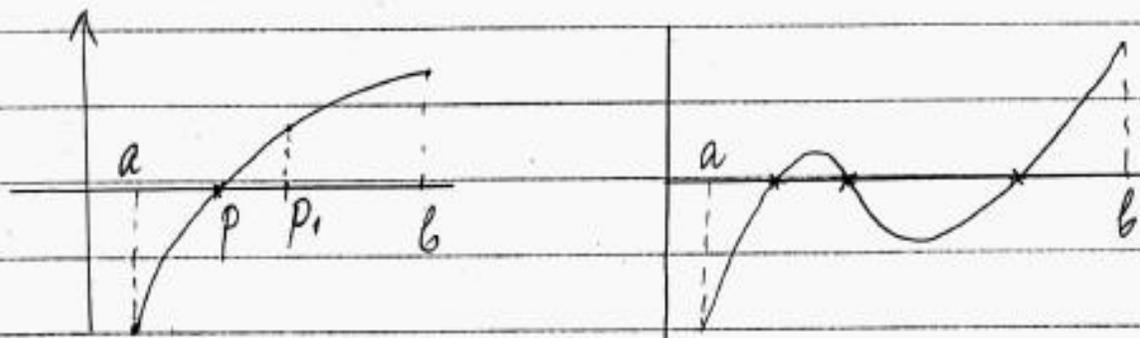
2.1 Bisection Method

1) The Bisection Method

The Bisection Method is based on the Intermediate Value Theorem.

Suppose: $f(x)$ - continuous on $[a, b]$
with $f(a) < 0$ $f(b) > 0$
or $f(a) > 0$ $f(b) < 0$

Under these conditions there is at least root in the interval (a, b)



The method consists in dividing the interval in half and taking the half that contains the root.

Set $a_1 = a$, $b_1 = b$. Let p_1 - midpt of $[a_1, b_1]$

$$p_1 = \frac{a_1 + b_1}{2}$$

If $f(p_1) = 0$ then $p = p_1$ and we are done.

If $\text{sign } f(a_1) = \text{sign } f(p_1)$ then $p \in (p_1, b_1)$ and we set $a_2 = p_1$, $b_2 = b_1$ and $\text{sign } f(a_2) \neq \text{sign } f(b_2)$.

If $\text{sign } f(b_1) = \text{sign } f(p_1)$ then $p \in (a_1, p_1)$ and we set $a_2 = a_1$, $b_2 = p_1$

Then we repeat the process for $[a_2, b_2]$

2) Stopping criteria.

The bisection method generates a sequence

$$p_n = \frac{a_n + b_n}{2}$$

The main question is: When do we stop?
Given some tolerance ϵ we may compute

$$p_1, p_2, \dots, p_N$$

until one of the following criteria is met

a) $\left| \frac{b_n - a_n}{2} \right| < \epsilon$ Only works for the Bisection method

b) $|p_n - p_{n-1}| < \epsilon$

This however, does not guarantee that the sequence converges.

Ex: $p_n = \sum_{k=1}^n \frac{1}{k}$ $p_n - p_{n-1} = \frac{1}{n} \rightarrow 0 \quad n \rightarrow \infty$

however, $p_n \rightarrow \infty$

c) $|f(p_N)| < \epsilon$

This may result in a poor approximation of p_N

Ex. $f(x) = (x-1)^{10}$ $p_n = 1 + \frac{1}{n}$

$$|f(p_2)| = \left(\frac{1}{2}\right)^{10} = \frac{1}{1024} < \frac{1}{1000} = 10^{-3}$$

Thus

$$|f(p_2)| < 10^{-3}$$

$p_2 = 1.5$ is a poor approximation of the root 1.

d) $\frac{|p_N - p_{N-1}|}{|p_N|} < \epsilon$

Best stopping criterion

This is closest to computing the relative error. Note, we can't compute exactly the relative error because we don't know the solution

Note: One should try to find the smallest possible interval with a root.

3) Example

#3^a

Example: Use the Bisection method to find solution accurate within 10^{-2} for

$$x^3 - 7x^2 + 14x - 6 = 0$$

on the interval $[0, 1]$

$$f(0) = -6 \quad f(1) = 2$$

n	a_n	b_n	p_n	$f(p_n)$
1	0	1	0.5	-0.625
2	0.5	1	0.75	0.984375
3	0.5	0.75	0.625	0.259766
4	0.5	0.625	0.5625	-0.161865
5	0.5625	0.625	0.59375	0.054
6	0.5625	0.59375	0.578125	-0.0526
7	0.578125	0.59375	0.5859375	0.001031
8	0.578125	0.5859375	0.58203125	-0.025716
9	0.58203125	0.5859375		

$$\frac{|p - p_7|}{|p|} \leq \frac{|a_8 - b_8|}{|a_8|} = \frac{0.0078125}{0.578125} = 0.01351$$

$p_7 = 0.5859375$ approximates the root.

4) Convergence of the Bisection method. Number of iterations

Theorem 2.1 Suppose $f \in C[a, b]$ and $f(a)f(b) < 0$.
The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$\begin{matrix} (*) \\ (*) \end{matrix} \quad |p_n - p| \leq \frac{b-a}{2^n} \quad \text{with } n \geq 1$$

Proof:

$$|p_n - p| \leq \frac{b_n - a_n}{2} = \frac{b-a}{2^{n-1}} \cdot \frac{1}{2} = \frac{b-a}{2^n}$$

- Consequently the sequence $\{p_n\}_{n=1}^{\infty}$

converges to p with rate of convergence $O\left(\frac{1}{2^n}\right)$. Thus

$$p_n = p + O\left(\frac{1}{2^n}\right)$$

Q.E.D

Note: $\begin{matrix} (*) \\ (*) \end{matrix}$ gives only an upper bound for the error. The actual error could be much smaller.

Theorem 2.1 gives a method for determining the number of necessary iterations.

Given a tolerance ϵ we find n such that

$$\frac{b-a}{2^n} \leq \epsilon \rightarrow 2^n \geq \frac{b-a}{\epsilon}$$

$$n \ln 2 \geq \ln \frac{b-a}{\epsilon}$$

$$n \geq \frac{\ln \frac{b-a}{\epsilon}}{\ln 2}$$

#.13 Example: Use Theorem 2.1 to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-4} to the solution of $x^3 - x - 1 = 0$ lying in the interval $[1, 2]$

$$\frac{2-1}{2^n} \leq 10^{-4} \rightarrow 2^n \geq 10^4$$

$$\ln 2^n \geq \ln 10^4$$

$$n \geq \frac{\ln 10^4}{\ln 2} \approx 13.2877$$

$$n \geq 14$$