

Chapter 3 Interpolation and Polynomial Approximation

A census of the population of US is taken every 10 years

Year	1940	1950	1960	1970	1980	1990
Population (in thousands)	132,165	151,326	179,323	203,302	226,592	249,633

If we want to know the population in 1965 or 2010 we have to fit a function through the given data.

Def: The process of fitting a function through given data is called interpolation.

The most usual type of functions fitted through data are polynomials.

Def: The process of fitting a polynomial through given data is called polynomial interpolation

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

Polynomials are often used since they have a very good property:

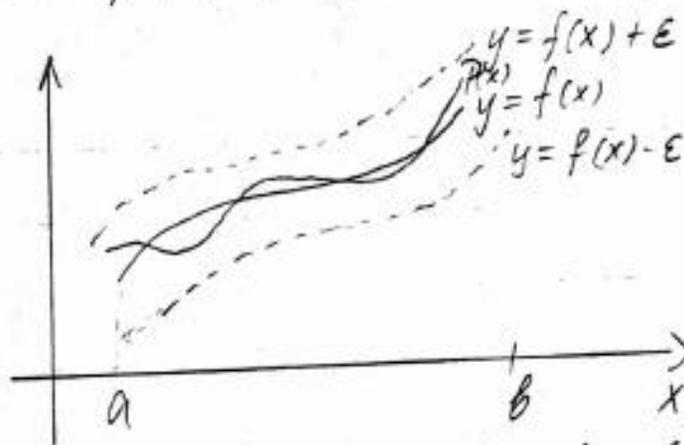
Given any function

$f(x)$ - continuous on $[a, b]$

$\epsilon > 0$ - tolerance

then there is a polynomial $P(x)$ which is closer to $f(x)$ than the tolerance

$$|f(x) - P(x)| < \epsilon \quad \forall x \in [a, b]$$



This fact is guaranteed by

Theorem 3.1 (Weierstrass Approximation Thm)

Suppose that f is defined and continuous on $[a, b]$. For each $\epsilon > 0$, there exists a polynomial $P(x)$, with the property

$$|f(x) - P(x)| < \epsilon \quad \forall x \text{ in } [a, b]$$

The most common way to approximate a function near a point is to use Taylor's polynomial.

3.1 Interpolation and the Lagrange Polynomial

1) Lagrange Interpolating Polynomial

Problem: Given the values of the function $f(x)$ at $n+1$ distinct points

$$\begin{array}{cccc} x_0 & x_1 & \dots & x_n \\ f_0 & f_1 & \dots & f_n \end{array}$$

where

$$f_i = f(x_i) \quad i=0, 1, \dots, n$$

Find a polynomial of degree n , $P(x)$ such that

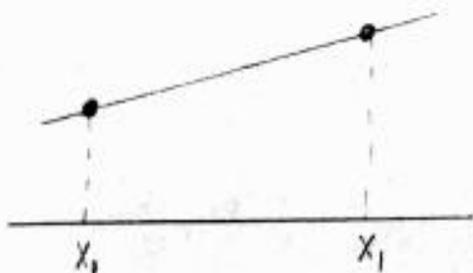
$$P(x_i) = f_i \quad i=0, 1, \dots, n.$$

Simplest case: Linear interpolation: Given

$$\begin{array}{cc} x_0 & x_1 \\ f_0 & f_1 \end{array}$$

Find a polynomial of degree one such that

$$P(x_0) = f_0 \quad P(x_1) = f_1$$



First we define linear functions

$L_0(x)$, $L_1(x)$
such that

$$L_0(x_0) = 1 \quad L_0(x_1) = 0$$

$$L_1(x_0) = 0 \quad L_1(x_1) = 1$$

namely,

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

then we define

$$P(x) = L_0(x)f_0 + L_1(x)f_1$$

$$\text{Thus, } P(x_0) = L_0(x_0)f_0 + L_1(x_0)f_1 = f_0$$

$$P(x_1) = L_0(x_1)f_0 + L_1(x_1)f_1 = f_1$$

Note: $P(x)$ is the unique linear function passing through the points (x_0, f_0) and (x_1, f_1) .

Now we generalize to $n+1$ points.
Given

$$\begin{array}{ccccccc} x_0 & x_1 & \dots & \dots & x_n & & \\ f_0 & f_1 & & & f_n & & \end{array}$$

First, we construct polynomials

$L_{n,k}(x)$ for each $k=0, 1, \dots, n$
with properties

$$L_{n,k}(x_i) = 0 \text{ if } i \neq k$$

$$L_{n,k}(x_k) = 1$$

A polynomial of degree n which is zero
at the points

$$x_0, x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n$$

is

$$(x-x_0)(x-x_1)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)$$

To satisfy $L_{n,k}(x_k) = 1$ we must divide by
the value of this polynomial at $x = x_k$
Thus,

basis polynomial $\rightarrow L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)(x_k-x_1)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}$

Let $P(x)$ be a polynomial of degree n
satisfying

$$P(x_k) = f_k \quad k=0, 1, \dots, n$$

then

$$P(x) = f_0 L_{n,0}(x) + \dots + f_n L_{n,n}(x) = \sum_{k=0}^n f_k L_{n,k}(x)$$

Def. $P(x)$ is called the n^{th} Lagrange interpolating polynomial.

Note: $P(x)$ is unique!

2) Example.

Ex. Find the appropriate Lagrange interpolating polynomial using the table

x	x_0	x_1	x_2	x_3
	0	0.5	1	1.5
$f(x)$	1	2	3	4

Thus, we will build a polynomial of degree 3.

First we build the basis polynomials

$$L_{3,0}(x) = \frac{(x-0.5)(x-1)(x-1.5)}{(-0.5)(-1)(-1.5)}$$

$$L_{3,1}(x) = \frac{x(x-1)(x-1.5)}{(0.5)(0.5-1)(0.5-1.5)}$$

$$L_{3,2}(x) = \frac{x(x-0.5)(x-1.5)}{(1-0)(1-0.5)(1-1.5)}$$

$$L_{3,3}(x) = \frac{x(x-0.5)(x-1)}{(1.5-0)(1.5-0.5)(1.5-1)}$$

$$P(x) = L_{3,0}(x) + 2L_{3,1}(x) + 3L_{3,2}(x) + 4L_{3,3}(x)$$

3) The interpolation error

Theorem 3.3 Suppose x_0, x_1, \dots, x_n are $n+1$ distinct numbers in the interval $[a, b]$ and $f(x)$ has $n+1$ continuous derivatives. Then for each x in $[a, b]$, a number $\xi(x)$ in (a, b) exists with $E_n(x; f)$

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} (x-x_0)(x-x_1)\dots(x-x_n)$$

where $P(x)$ is the n^{th} Lagrange interpolating polynomial.

Def: $E_n(x; f) = f(x) - P(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} (x-x_0)\dots(x-x_n) - \text{error}$

Ex For the function

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$$f(x) = \cos x$$

let $x_0 = 0$, $x_1 = 0.6$, $x_2 = 0.9$. Construct the interpolation polynomial of degree at most two to approximate $f(0.45)$ and find the actual error. Use Theorem 3.3 to find the error bound for the error.

$$L_{2,0}(x) = \frac{(x-0.6)(x-0.9)}{(0-0.6)(0-0.9)} \quad L_{2,0}(0.45) = 0.125$$

$$L_{2,1}(x) = \frac{x(x-0.9)}{(0.6-0)(0.6-0.9)} \quad L_{2,1}(0.45) = 1.125$$

$$L_{2,2}(x) = \frac{x(x-0.6)}{(0.9-0)(0.9-0.6)} \quad L_{2,2}(0.45) = -0.25$$

Note $\cos 0 = 1$ $\cos 0.6 \approx 0.825$ $\cos 0.9 \approx 0.6216$

$$P(x) = L_{2,0}(x) + (\cos 0.6)L_{2,1}(x) + (\cos 0.9)L_{2,2}(x)$$

$$P(0.45) = 0.125 + (\cos 0.6) \cdot 1.125 + (\cos 0.9)(-0.25)$$

$$P(0.45) = 0.8981000747$$

Actual value = 0.9004471024

$$\text{Error} = \cos(0.45) - P(0.45) = 0.0023470276$$

Error bound:

$$E(x; f) = \frac{f'''(\xi(x))}{3!} (x-0)(x-0.6)(x-0.9)$$

$$|E(x; f)| \leq \frac{|\sin \xi(x)|}{6} |x(x-0.6)(x-0.9)| \leq \frac{\sin 0.9}{6} 0.05704 = 0.0001493$$

$$f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x$$

Note: $\sin x$ is increasing in $[0, 0.9]$

Let $g(x) = x(x-0.6)(x-0.9)$ $g'(x) = 3(x^2 - x + 0.18)$
 $x_1 = 0.7645757311$
 $x_2 = 0.2354242689$
 $g(x_1) = 0.0570405184$
 $g(x_2) = -0.0170405184$

#15/120 construct the Lagrange interpolating polynomial of degree 2 for

$$f(x) = \sin(\ln x)$$

on the interval $[2, 2.6]$ with points

$$x_0 = 2.0 \quad x_1 = 2.4 \quad x_2 = 2.6$$

Find a bound for the absolute error on $[2, 2.6]$.

$$L_{2,0}(x) = \frac{(x-2.4)(x-2.6)}{(2-2.4)(2-2.6)}$$

$$L_{2,1}(x) = \frac{(x-2)(x-2.6)}{(2.4-2)(2.4-2.6)}$$

$$L_{2,2}(x) = \frac{(x-2.0)(x-2.4)}{(2.6-2.0)(2.6-2.4)}$$

$$P(x) = \sin(\ln 2) L_{2,0}(x) + \sin(\ln 2.4) L_{2,1}(x) \\ + \sin(\ln 2.6) L_{2,2}(x)$$

$$E(x; f) = \frac{f'''(\xi(x))}{3!} (x-2)(x-2.4)(x-2.6)$$

$$f'(x) = \cos(\ln x) \cdot \frac{1}{x}$$

$$f''(x) = -\sin(\ln x) \cdot \left(\frac{1}{x}\right)^2 - \cos(\ln x) \left(\frac{1}{x}\right)^2$$

$$= -[\sin(\ln x) + \cos(\ln x)] \cdot \left(\frac{1}{x}\right)^2$$

$$\begin{aligned}
 f'''(x) &= -[\cos(\ln x) - \sin(\ln x)] \cdot \frac{1}{x^3} \\
 &\quad + 2[\sin(\ln x) + \cos(\ln x)] \cdot \frac{1}{x^3} \\
 &= [3\sin(\ln x) + \cos(\ln x)] \cdot \frac{1}{x^3}
 \end{aligned}$$

To bound the 3rd derivative we find the fourth

$$\begin{aligned}
 f^{IV}(x) &= [3\cos(\ln x) - \sin(\ln x)] \cdot \frac{1}{x^4} \\
 &\quad - 3[3\sin(\ln x) + \cos(\ln x)] \cdot \frac{1}{x^4} \\
 &= -10\sin(\ln x) \cdot \frac{1}{x^4} < 0
 \end{aligned}$$

$\Rightarrow f'''(x)$ is decreasing on $[2, 2.6]$.

$$|f'''(x)| \leq [3\sin(\ln 2) + \cos(\ln 2)] \cdot \frac{1}{2^3} \leq 0.335765$$

Next, we need the maximum of

$$g(x) = (x-2)(x-2.4)(x-2.6)$$

$$\begin{aligned}
 g'(x) &= (x-2.4)(x-2.6) + (x-2)(x-2.6) + (x-2)(x-2.4) \\
 &= x^2 - 5x + 6.24 + x^2 - 4.6x + 5.2 + x^2 - 4.4x + 4.8 \\
 &= 3x^2 - 14x + 16.24 = 0
 \end{aligned}$$

$$x_1 = 2.5 \quad x_2 = 2.157$$

$$|g(x)| \leq |g(2.157)| \leq 0.0169$$

\Rightarrow

$$E(x; f) \leq \frac{0.335765}{3!} 0.0169 = 9.457 \cdot 10^{-4}$$