

3.2 Divided Differences

- a) A practical difficulty with Lagrange interpolation is that since the error term is difficult to apply, the degree of the polynomial needed for the desired accuracy is generally not known until after the computation.
- b) The work done in calculating n^{th} degree polynomial does not lessen the work for the computation of the $(n+1)^{\text{st}}$ degree polynomial.

1) Divided differences.

Let $P_n(x)$ be the n^{th} Lagrange polynomial such that

$$P_n(x_k) = f_k \quad k=0, \dots, n$$

We want to rewrite $P_n(x)$ in the form

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

for appropriate a_0, a_1, \dots, a_n - constants

Determining a_0 is easy

$$a_0 = P_n(x_0) = f_0 := f(x_0)$$

To determine a_1 we have

$$P_n(x_1) = a_0 + a_1(x_1 - x_0)$$

$$f_1 = f_0 + a_1(x_1 - x_0)$$

$$\Rightarrow a_1 = \frac{f_1 - f_0}{x_1 - x_0}$$

Def: The zeroth divided difference of the function f with respect to x_i is denoted by $f[x_i]$ and is defined as

$$f[x_i] = f(x_i)$$

The remaining divided differences are defined inductively.

Def: The first divided difference of f with respect to x_i, x_{i+1} is denoted by $f[x_i, x_{i+1}]$ and is defined as follows

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

Def: The second divided difference, $f[x_i, x_{i+1}, x_{i+2}]$ is defined as

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

Thus, if the $(k-1)^{st}$ divided differences $f[x_i, x_{i+1}, \dots, x_{i+k-1}], f[x_{i+1}, \dots, x_{i+k}]$ are given, then the k^{th} divided difference relative to $x_i, x_{i+1}, \dots, x_{i+k}$ is given by

$$f[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{f[x_{i+1}, \dots, x_{i+k}] - f[x_i, \dots, x_{i+k-1}]}{x_{i+k} - x_i}$$

The divided differences are usually computed in a table

x	$f(x)$	I st DD	II nd DD	III rd DD	IV th D.D.
x_0	f_0				
x_1	f_1	$f[x_0, x_1]$			
x_2	f_2	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$		
x_3	f_3	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$	
x_4	f_4	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_1, x_2, x_3, x_4]$	$f[x_0, x_1, x_2, x_3, x_4]$
\vdots					

Example: Compute the divided differences with the following data

x	$f(x)$	I st DD	II nd DD	III rd DD
0	3			
1	4	1		
2	7	3	1	
4	19	6	1	0

2) Interpolating with divided differences

If we want to write the interpolating polynomial in the form

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

we saw that

$$a_0 = f(x_0) = f_0 = f[x_0]$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1]$$

If we continue to compute we will get

$$a_k = f[x_0, x_1, \dots, x_k] \quad \forall k=0, 1, \dots, n$$

k^{th} divided difference of f relative to x_0, \dots, x_k

So $P_n(x)$ can be written as

$$(\star) \quad P_n(x) = f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) + \dots + f[x_0, x_1, \dots, x_n](x-x_0)\dots(x-x_{n-1})$$

Def: This formula is called Newton's interpolatory (forward) divided-difference formula.

Ex 1: #4/p.131 (a) Construct the interpolating polynomial of degree 4 for the points in the table

x	$f(x)$	I st DD	II nd DD	III rd DD	IV th DD
0.0	-6.000000				
0.1	-5.89483	1.0517			
0.3	-5.65014	1.22345	0.5725		
0.6	-5.17788	1.5742	0.7015	0.215	
1.0	-4.28172	2.2404	≈0.9517	0.278	0.06
1.1	-3.99583	2.8589	1.237	0.356625	0.0786

$$P_4(x) = -6 + 1.0517x + 0.5725x(x-0.1) + 0.215x(x-0.1)(x-0.3) + 0.063x(x-0.1)(x-0.3)(x-0.6)$$

(b) Add $f(1.1) = -3.99583$ to the table, and construct the interpolating polynomial of degree 5

The Vth DD is 0.063
 $0.078625 \rightarrow \approx 0.0142$

$$P_5(x) = P_4(x) + 0.0142x(x-0.1)(x-0.3)(x-0.6)(x-1)$$

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Notice

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The Mean Value Theorem implies that if f' exists

$$f[x_0, x_1] = f'(\xi)$$

for some ξ between x_0 and x_1 . The following theorem generalizes that

Theorem 3.6 Suppose f has n continuous derivatives and x_0, \dots, x_n are distinct numbers in $[a, b]$. Then $\xi \in (a, b)$ exists with

$$f[x_0, \dots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

If the interpolating nodes are reordered as x_n, x_{n-1}, \dots, x_0 a formula similar to (A) can be established

$$\begin{aligned} P_n(x) &= f[x_n] + f[x_n, x_{n-1}](x-x_n) + \dots \\ &\quad + f[x_n, \dots, x_0](x-x_n)\dots(x-x_1) \end{aligned}$$

Def: This form is called the Newton's backward divided-difference formula.

Ex: Construct the interpolating polynomial of degree 4 using Newton's backward divided-difference formula using the data in Ex 1.

$$P(x) = -4.28172 + 2.2404(x-1) + 0.9517(x-1)(x-0.5) \\ + 0.278(x-1)(x-0.6)(x-0.3) \\ + 0.063(x-1)(x-0.6)(x-0.3)(x-0.1).$$

4) Error of interpolation with Divided Differences

The n^{th} degree polynomial generated by divided differences is the same as the one generated by Lagrange interpolation. Thus the error is the same.

$$E_n(x; f) = \frac{f^{(n+1)}(5)}{(n+1)!} (x-x_0) \dots (x-x_n)$$

$$\text{Recall } E_n(x; f) = f(x) - P_n(x)$$

Ex. For the function $f(x) = x^2 e^{-\frac{x}{2}}$ construct the divided difference table at the points

$$x_0 = 1.1 \quad x_1 = 2 \quad x_2 = 3.5 \quad x_3 = 5 \quad x_4 = 7.1$$

Find the Newton's forward divided

difference polynomials of degree 1, 2, 3.
 Find the error of the interpolates for
 $f(1.75)$. Found the error bounds for
 $E_1(x, f)$.

Solution: Here is the DD table for $f(x) = x^2 e^{-\frac{x}{2}}$

x_i	$f[x_i]$	I DD	II DD		
1.1	0.6981				
2	1.4715	0.8593			
3.5	2.1287	0.4381	-0.1755		
5	2.0521	-0.0511	-0.1631	0.0032	
7.1	1.4480	-0.2877	-0.0657	0.0191	0.0027

$$P_1(x) = 0.6981 + 0.8593(x - 1.1)$$

$$P_2(x) = P_1(x) - 0.1755(x - 1.1)(x - 2)$$

$$P_3(x) = P_2(x) + 0.0032(x - 1.1)(x - 2)(x - 3.5)$$

$$f(1.75) = 1.2766$$

$$P_1(1.75) = 0.6981 + 0.8593(1.75 - 1.1) = 1.25665$$

$$P_2(1.75) = P_1(1.75) - 0.1755(1.75 - 1.1)(1.75 - 2) = 1.2852$$

$$P_3(1.75) = P_2(1.75) + 0.0032(1.75 - 1.1)(1.75 - 2)(1.75 - 3.5) = 1.2861$$

The errors of the interpolation are

Degree	$P_n(1.75)$	Actual Error	$\max f^{(n+1)} $	Error Bound	Next-te rule
1	1.25665	0.01995	0.3679	0.03725	0.0285
2	1.2852	-0.0086			
3	1.2861	-0.0095			

Typically we can expect that a higher degree polynomial will approximate better but in this case the approximation of the degree 3 polynomial is worse than that of degree 2 polynomial

$$E_1(x, f) = \frac{f''(5)}{2!} (x-1.1)(x-2)$$

$$f'(x) = 2xe^{-\frac{x}{2}} - x^2 \cdot \frac{1}{2} e^{-\frac{x}{2}} = \left(2x - \frac{x^2}{2}\right) e^{-\frac{x}{2}}$$

$$f''(x) = \left(2-x\right) e^{-\frac{x}{2}} - \frac{1}{2} \left(2x - \frac{x^2}{2}\right) e^{-\frac{x}{2}} =$$

$$= \left(2-x-x+\frac{x^2}{4}\right) e^{-\frac{x}{2}} = \left(2-2x+\frac{x^2}{4}\right) e^{-\frac{x}{2}}$$

$$f'''(x) = \left(-2+\frac{x}{2}\right) e^{-\frac{x}{2}} - \frac{1}{2} \left(2-2x+\frac{x^2}{4}\right) e^{-\frac{x}{2}}$$

$$= \left(-3+\frac{3}{2}x-\frac{x^2}{2}\right) e^{-\frac{x}{2}} = 0$$

$$-\frac{x^2}{8} + \frac{3}{2}x - 3 = 0$$

$$x^2 - 12x + 24 = 0$$

$$x_{1,2} = 6 \pm \sqrt{12} \begin{cases} 2.5359 \\ 9.4641 \end{cases}$$

$$f''(1.1) = 0.05914$$

$$f''(2.5359) = -0.412$$

$$f''(7.1) = 0.01156$$

$$f''(2) = -0.3679$$

The interpolation for $P_1(x)$ is actually in the interval $[1.1, 2]$.

$$|f''(\xi)| \leq 0.3679 \quad \xi \in [1.1, 2]$$

The maximum of

$$(x-1.1)(x-2)$$

is attained at the midpoint $\frac{1.1+2}{2} = 1.55$

$$|(x-1.1)(x-2)| \leq 0.2025$$

$$|E_1(x; f)| \leq \frac{|f''(\xi)|}{2!} |(x-1.1)(x-2)| \leq \frac{0.3679}{2} \cdot 0.2025 \\ = 0.03725$$

- Error Estimation when $f(x)$ -unknown:
The Next-term Rule

Often $f(x)$ is not known, and consequently we do not know bound for $f^{(n+1)}(\xi)$. As we saw

$$\frac{f^{(n+1)}(\xi)}{(n+1)!} = f[x_0, \dots, x_{n+1}]$$

Thus the $(n+1)^{\text{st}}$ DD is an estimate for the $(n+1)^{\text{st}}$ derivative of f . This means that the error of the interpolation is given approximately by the value of the next term that would be added

$E_n(x, f) \approx$ the value of the next term
that would be added to $P_n(x)$

Thus,

$$E_1(1.75; x^2 e^{-\frac{x}{2}}) \approx -0.1755(1.75-1.1)(1.75-2) = 0.02852$$

5) Interpolation formulas with equally spaced points. Ordinary differences

Def: The points x_0, x_1, \dots, x_n are equally spaced if
 $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \dots = x_n - x_{n-1} = h$
 ↑ step

Ex: a) $x_0 = 1 \quad x_1 = 1.5 \quad x_2 = 2 \quad x_3 = 2.5 \quad x_4 = 3$

$$x_1 - x_0 = 0.5 \quad x_2 - x_1 = 0.5 \dots$$

If the data are equally spaced, getting the interpolating polynomial is simpler. Also, when we compute the divided differences we would always divide by the same number. In this case it is more convenient to define „ordinary differences“

Def: The first forward difference $\Delta f(x_i)$ is defined as

$$\Delta f(x_i) = f(x_{i+1}) - f(x_i)$$

Then, $f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{\Delta f(x_i)}{h}$

Ex. Let $f(x) = x^3$. The first forward difference at the points $x_0 = 1, x_1 = 2$ is

$$\Delta f(x_0) = f(2) - f(1) = 2^3 - 1^3 = 7$$

Def. The second forward difference $\Delta^2 f(x_i)$ $\Delta^2 f(x_i)$ is defined as follows

$$\Delta^2 f(x_i) = \Delta f(x_{i+1}) - \Delta f(x_i)$$

Consequently, the divided difference is

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_{i+1}, x_i]}{x_{i+2} - x_i}$$

$$= \frac{1}{2h} \left[\frac{\Delta f(x_{i+1})}{h} - \frac{\Delta f(x_i)}{h} \right]$$

$$= \frac{1}{2h^2} \Delta^2 f(x_i)$$

Def. The $(k+1)^{st}$ forward difference $\Delta^{k+1} f(x_i)$ is defined as follows

$$\Delta^{k+1} f(x_i) = \Delta^k f(x_{i+1}) - \Delta^k f(x_i)$$

In general,

$$f[x_i, \dots, x_{i+k}] = \frac{1}{k! h^k} \Delta^k f(x_i)$$

Computing the differences is the same as computing the divided differences but we don't have to divide by anything

Ex. Compute the difference table for $f(x) = 2x^3$ for $x_0 = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.5, x_4 = 2, x_5 = 2.5, x_6 = 3$. Next, compute the divided difference table and compare them

Table of differences:

x_i	$f(x_i)$	$\Delta f(x_i)$	$\Delta^2 f(x_i)$	$\Delta^3 f(x_i)$	$\Delta^4 f(x_i)$	$\Delta^5 f(x_i)$
0	0					
0.5	0.25	0.25				
1	2.00	1.75	1.50			
1.5	6.75	4.75	3.00	1.50		
2	16.00	9.25	4.50	1.50	0.00	
2.5	31.25	15.25	6.00	1.50	0.00	
3	54.00	22.75	7.50	1.50	0.00	

Table of divided differences

x_i	$f(x_i)$	I DD	II DD	III DD	IV DD
0	0				
0.5	0.25	0.5			
1	2.00	3.5	3		
1.5	6.75	9.5	6	2	
2	16.00	18.5	9	2	0
2.5	31.25	30.5	12	2	0
3	54.00	45.5	15	2	0

Notice: 1) The IV DD of f are zero. That is because the IV DD approximates the fourth derivative of f so it is zero.

$$2) I^{st} DD = \frac{I^{st} \text{ difference}}{h} \quad (h=0.5)$$

$$II^{nd} DD = \frac{II^{nd} \text{ difference}}{h(2h)} \quad (0.5 \text{ here})$$

$$III^{rd} DD = \frac{III^{rd} \text{ difference}}{h(2h)(3h)} \quad (0.75 \text{ here})$$

An interpolating polynomial of degree n can be written in terms of the ordinary differences

$$P_n(x_0+sh) = P_n(s) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!} \Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!} \Delta^3 f(x_0) \\ + \dots + \frac{s(s-1)\dots(s-n+1)}{n!} \Delta^n f(x_0)$$

where $s = \frac{x-x_0}{h}$. This formula is called

Newton forward-difference formula.

Ex. Given the table of x_i and $f(x_i)$, compute the forward differences to order 4. Find $f(0.73)$ from a cubic interpolating polynomial

x_i	$f(x_i)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0	0				
0.2	0.203	0.203			
0.4	0.423	0.220	0.017		
0.6	0.684	0.261	0.041	0.024	
0.8	1.030	0.346	0.085	0.044	0.020
1.0	1.557	0.527	0.181	0.096	0.052
1.2	2.572	1.015	0.488	0.307	0.211

Since 0.73 falls between 0.6 and 0.8 and the closest 4 entries are

$$\begin{matrix} 0.4 & 0.6 & 0.8 & 1 \\ x_0 & x_1 & x_2 & x_3 \end{matrix}$$

$$P_3(s) = 0.423 + 0.261s + 0.085 \frac{s(s-1)}{2} + 0.096 \frac{s(s-1)(s-2)}{6}$$

$$\text{Since } x=0.73, s = \frac{0.73-0.4}{0.2} = \frac{0.33}{0.2} = 1.65$$

$$\begin{aligned} P_3(1.65) &= 0.423 + (0.261)(1.65) + 0.085 \frac{(1.65)(0.65)}{2} \\ &\quad + 0.096 \frac{(1.65)(0.65)(-0.35)}{6} = \end{aligned}$$

$$= 0.423 + 0.4306 + 0.0456 - 0.006 = 0.893$$

Note: The function actually is $f(x) = \tan x$
So $f(0.73) = 0.895$. Thus, the error is 0.002.

Similarly, we can define backward differences. Consider the points

$$x_n, x_{n-1}, \dots, x_0$$

Def: The first backward difference at x_i is defined as

$$\nabla f(x_i) = f(x_i) - f(x_{i-1})$$

Note: $\nabla f(x_i) = \Delta f(x_{i-1})$

Def: The k^{th} backward difference at x_i is defined as

$$\nabla^k f(x_i) = \nabla^{k-1} f(x_i) - \nabla^{k-1} f(x_{i-1})$$

Newton's backward-difference formula

$$P_n(x_0+sh) = P_n(s) = f(x_n) + s \nabla f(x_n) + \frac{s(s+1)}{2!} \nabla^2 f(x_n) + \frac{s(s+1)(s+2)}{3!} \nabla^3 f(x_n)$$

$$+ \dots + \frac{s(s+1)\dots(s+n-1)}{n!} \nabla^n f(x_n)$$

Ex

#2/3/13
 Use Newton's forward-difference formula to construct the interpolating polynomial of degree 3. Use Newton's backward-difference formula to construct the interpolating polynomial of degree 3. Use either polynomial to approximate $f(-\frac{1}{3})$.

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	
-0.75	-0.0718125				
-0.5	-0.02475	0.0470625			
-0.25	0.3349375	0.3596875	0.312625		
0	1.101	0.7660625	0.406375	0.09375	

$$P_3(x_0 + sh) = \hat{P}_3(s) = -0.0718125 + 0.0470625s$$

$$+ 0.312625 \frac{s(s-1)}{2!} + 0.09375 \frac{s(s-1)(s-2)}{3!}$$

$$P_3(x_3 + sh) = 1.101 + 0.7660625s + 0.406375 \frac{s(s+1)}{2!}$$

$$+ 0.09375 \frac{s(s+1)(s+2)}{3!}$$

$$\text{If } x = -\frac{1}{3} \quad 0 + s \cdot \frac{1}{4} = -\frac{1}{3} \quad s = -\frac{4}{3}$$

$$P_3\left(-\frac{4}{3}\right) = 1.101 + 0.7660625\left(-\frac{4}{3}\right) + 0.406375 \frac{(-\frac{4}{3})(-\frac{1}{3})}{2!}$$

$$+ 0.09375 \frac{(-\frac{4}{3})(-\frac{1}{3})(\frac{2}{3})}{3!} = 0.1745185$$