3.3 Hermite Interpolation

Osculating Polynomials
So far we constructed polynomials which agree with the values of the function in $n+1$. Hermite's interpolation is about constructing a polynomial which agrees with the function and its derivative.

Problem: Given $n+1$ distinct points $x_0, \ldots, x_n$ and given nonnegative integers $m_0, \ldots, m_n$ with $m = \max\{m_0, \ldots, m_n\}$

Assume $f \in C^m[a, b]$. Find a polynomial of least degree which agrees with the function and all its derivatives of order $\leq m_i$ at $x_i$, that is

$$P^{(k)}(x_i) = f^{(k)}(x_i) \quad k=0, \ldots, m_i \quad i=0, \ldots, n$$

The degree of the polynomial is at most equal to the number of conditions it has to satisfy minus one.
degree of $P(x) \leq \#\text{conditions} - 1$

$\#\text{conditions} = \sum_{i=0}^{n} (m_i + 1) = \sum_{i=0}^{n} m_i + n + 1$

$\leq \text{degree of polynomial} \leq \sum_{i=0}^{n} m_i + n$

**Def.** A polynomial approximating $f$ and its derivatives is called an *osculating polynomial*.

**Def.** In the case when $m_i=1$ for $i=0, \ldots, n$ the approximating polynomial is called a *Hermite* polynomial.

2) Using Newton's interpolatory divided-difference formula to form Hermat's polynomial.

Recall,

$$P(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, \ldots, x_k](x-x_0) \cdots (x-x_{k-1})$$

**Theorem 3.6** $\Rightarrow$

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

where $\xi$ in $(a,b)$ defined by the points $x_0, \ldots, x_n$. 
Suppose all points are the same
Then
\[
\frac{f[x_0, \ldots, x_0]}{n+1} = \frac{f^{(n)}(x_0)}{n!}
\]

In particular
\[
f[x_0, x_0] = f'(x_0)
\]

Thus, originally we could compute the divided differences only with different points. Now, we know how to compute them with all points the same. Consequently, we can compute them with some points the same and some different.

Ex:
\[
f[x_0, x_0, x_1] = \frac{f[x_0, x_0] - f[x_0, x_1]}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_1 - x_0}
\]

How does this help us? From the distinct points
\[x_0, \ldots, x_n\]
we construct the points
\[
\frac{x_0, \ldots, x_0}{M_0 + 1}, \frac{x_1, \ldots, x_1}{M_1 + 1}, \ldots, \frac{x_n, \ldots, x_n}{M_n + 1}
\]

Let's call these points
\[ z_1, z_2, \ldots, z_M \]

where \( M = \sum_{i=0}^{M} m_i \). Then, we can write the interpolating polynomial in the form

\[ P(x) = f(z_0) + \sum_{k=1}^{M} f[z_0, \ldots, z_k](x-z_0)\ldots(x-z_{k-1}) \]

Ex. Find a polynomial of degree 3 such that:

\[ P(x_0) = f_0, \quad P'(x_0) = f'_0, \quad P''(x_0) = f''_0, \quad P'''(x_0) = f'''_0 \]

Thus, we have the points

\[
\begin{array}{c|cccc}
X & x_0 & x_0 & x_0 & x_0 \\
\hline
x_0 & f_0 & f'_0 & f''_0 & f'''_0 \\
\end{array}
\]

\[ P(x) = f_0 + f'_0 (x-x_0) + f''_0 \frac{(x-x_0)^2}{2!} + f'''_0 \frac{(x-x_0)^3}{3!} \]

Thus, Taylor's polynomial is a special case of Hermite Interpolation.
Ex: Find $P(x)$ of degree 3 such that

\[ P(x_0) = f_0 \quad P(x_1) = f_1 \quad P(x_2) = f_2 \]

\[ P'(x_1) = f'_1 \]

Assume $x_0 < x_1 < x_2$.

We use the points $x_0, x_1, x_1, x_2$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f$</th>
<th>$1^{st}$ DD</th>
<th>$2^{nd}$ DD</th>
<th>$3^{rd}$ DD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0$</td>
<td>$f_0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>$f_1$</td>
<td>$f[x_0, x_1]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>$f_1$</td>
<td></td>
<td>$f(x_1)$</td>
<td></td>
</tr>
<tr>
<td>$x_2$</td>
<td>$f_2$</td>
<td>$f[x_1, x_2]$</td>
<td>$f[x_0, x_1, x_1]$</td>
<td>$f[x_0, x_1, x_1, x_2]$</td>
</tr>
</tbody>
</table>

where

\[
\begin{align*}
  f[x_0, x_1, x_1] &= \frac{f'(x_1) - f[x_0, x_1]}{x_1 - x_0} \\
  f[x_1, x_1, x_2] &= \frac{f[x_1, x_2] - f'(x_1)}{x_2 - x_1}
\end{align*}
\]

\[
P(x) = f_0 + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_1](x-x_0)(x-x_1) + f[x_0, x_1, x_1, x_2](x-x_0)(x-x_1)^2
\]
Given the function \( f(x) = x \ln x \), use Hermite interpolation to construct an approximating polynomial for the following data:

\[
\begin{array}{|c|c|c|}
\hline
x & f(x) & f'(x) \\
\hline
8.3 & 17.56492 & 3.116256 \\
8.6 & 18.50575 & 3.151762 \\
\hline
\end{array}
\]

Use the Hermite polynomial to approximate \( f(8.4) \) and calculate the actual error.

\[
\begin{array}{|c|c|c||c||c|}
\hline
x & f(x) & f^{1\text{st}} DD & f^{2\text{nd}} DD & f^{3\text{rd}} DD \\
\hline
8.3 & 17.56492 & 3.116256 & & \\
8.3 & 17.56492 & & 0.05948 & \\
8.6 & 18.50575 & 3.1341 & 0.058873 & -0.002022 \\
8.6 & 18.50575 & 3.151762 & & \\
\hline
\end{array}
\]

\[ H(x) = 17.56492 + 3.116256(x-8.3) + 0.05948(x-8.3)^2 - 0.002022(x-8.3)^2(x-8.6) \]

\[ H(8.4) = 17.87714444 \]

\[ f(8.4) = 17.87714633 \]

Error = 1.889134 \times 10^{-6}
For the function $f(x) = x^4$ compute the divided difference table for the points $0, 1, 1, 1, 2$.

Write down the osculating polynomial interpolating $f(x)$ at those points.

\[
\begin{array}{cc|cccc}
X & f(x) & I DD & I I DD & I I I DD & I I I I DD \\
0 & 0 & & & & \\
1 & 1 & & 1 & & \\
1 & 1 & & 4 & & 3 \\
1 & 1 & & 4 & & 6 \\
2 & 16 & & 15 & & 11 \\
\end{array}
\]

\[
f'(x) = 4x^3 \quad f'(1) = 4 \\
f''(x) = 12x^2 \quad f''(1) = 12 \quad \frac{f'''(1)}{2} = 6
\]

The osculating polynomial is

\[H_4(x) = 0 + 1x + 3x(x-1) + 3x(x-1)^2 + 1x(x-1)^3\]