3.3 Hermite Interpolation

1) Osculating Polynomials

So far we constructed polynomials which agree with the values of the function in $n+1$. Hermite's interpolation is about constructing a polynomial which agrees with the function and its derivative.

<u>Problem:</u> Given $n+1$ distinct points

$$x_0, \ldots, x_n$$

and given nonnegative integers

$$m_0, \ldots, m_n$$

with

$$m = \max\{m_0, \ldots, m_n\}$$

Assume $f \in C^m[a,b]$.

Find a polynomial of least degree which agrees with the function and all its derivatives of order $\leq m_i$ at x_i, that is

$$P^{(k)}(x_i) = f^{(k)}(x_i) \qquad \begin{array}{l} k = 0, \ldots, m_i \\ i = 0, \ldots, n \end{array}$$

The degree of the polynomial is at most equal to the number of conditions it has to satisfy minus one.

degree of $P(x) \leq$ # conditions $- 1$

$$\# \text{ conditions} = \sum_{i=0}^{n} (m_i + 1) = \sum_{i=0}^{n} m_i + n + 1$$

$$\Rightarrow \text{degree of polynomial} \leq \sum_{i=0}^{n-1} m_i + n$$

<u>Def.</u> A polynomial approximating f and its derivatives is called <u>osculating polynomial</u>.

<u>Def:</u> In the case when $m_i = 1$ for $i = 0, \ldots, n$ the approximating polynomial is called <u>Hermite polynomial</u>.

2) Using Newton's interpolatory divided-difference formula to form Hermit's polynomial.

Recall,

$$P(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, \ldots, x_k](x - x_0) \ldots (x - x_{k-1})$$

Theorem 3.6 $\Rightarrow$

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

where ξ in (a, b) defined by the points $x_0, \ldots, x_n$.

Suppose all points are the same
Then

$$f[\underbrace{x_0, \ldots, x_0}_{n+1}] = \frac{f^{(n)}(x_0)}{n!}$$

In particular

$$f[x_0, x_0] = f'(x_0)$$

Thus, originally we could compute the divided differences only with different points. Now, we know how to compute them with all points the same. Consequently, we can compute them with some points the same and some different

Ex: $$f[x_0, x_0, x_1] = \frac{f[x_0, x_0] - f[x_0, x_1]}{x_1 - x_0} = \frac{f'(x_0) - f[x_0}{x_1 - x_0}$$

How does this help us? From the distin points

$$x_0, \ldots, x_n$$

we construct the points

$$\underbrace{x_0, \ldots, x_0}_{\substack{m_0+1 \\ \text{times}}}, \underbrace{x_1, \ldots, x_1}_{\substack{m_1+1 \\ \text{times}}}, \ldots, \underbrace{x_n, \ldots, x_n}_{\substack{m_n+1 \\ \text{times}}}$$

Let's call these points

$$z_1, z_2, \ldots\ldots, z_M$$

where $M = \sum_{i=0}^{n} m_i$. Then, we can write the interpolating polynomial in the form

$$P(x) = f[z_0] + \sum_{k=1}^{M} f[z_0, \ldots, z_k](x - z_0)\ldots(x - z_{k-1})$$

Ex. Find a polynomial of degree 3 such tha

$$P(x_0) = f_0 \quad P'(x_0) = f_0' \quad P''(x_0) = f_0'' \quad P'''(x_0) = f_0'''$$

Thus, we have the points

x		x_0, x_0 I st DD	x_0, x_0, x_0 II nd DD	x_0, x_0, x_0, x_0 III rd DD
x_0	f_0			
x_0	f_0	f_0'		
x_0	f_0	f_0'	$\frac{f_0''}{2!}$	
x_0	f_0	f_0'	$\frac{f_0''}{2!}$	$\frac{f_0'''}{3!}$

$$P(x) = f_0 + f_0'(x - x_0) + \frac{f_0''}{2!}(x - x_0)^2 + \frac{f_0'''}{3!}(x - x_0)^3$$

Thus, Taylor's polynomial is a special case of Hermite Interpolation

Ex: Find $P(x)$ of degree 3 such that

$$P(x_0) = f_0 \quad P(x_1) = f_1 \qquad P(x_2) = f_2$$

$$P'(x_1) = f_1'$$

Assume $x_0 < x_1 < x_2$.

We use the points x_0, x_1, x_1, x_2

x	f	I^{st} DD	IInd DD	IIIrd DD.
x_0	f_0			
x_1	f_1	$f[x_0, x_1]$		
x_1	f_1	$f'(x_1)$	$f[x_0, x_1, x_1]$	
x_2	f_2	$f[x_1, x_2]$	$f[x_1, x_1, x_2]$	$f[x_0, x_1, x_1, x_2]$

where

$$f[x_0, x_1, x_1] = \frac{f'(x_1) - f[x_0, x_1]}{x_1 - x_0}$$

$$f[x_1, x_1, x_2] = \frac{f[x_1, x_2] - f'(x_1)}{x_2 - x_1}$$

$$P(x) = f_0 + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_1](x - x_0)(x - x_1)$$

$$+ f[x_0, x_1, x_1, x_2](x - x_0)(x - x_1)^2$$

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Given the function

$$f(x) = x \ln x$$

Use Hermite interpolation to construct an approximating polynomial for the following data

x	$f(x)$	$f'(x)$
8.3	17.56492	3.116256
8.6	18.50515	3.151762

Use the Hermite polynomial to approximate $f(8.4)$ and calculate the actual error.

x	$f(x)$	I^{st} DD	IInd DD	IIIrd DD
8.3	17.56492			
8.3	17.56492	3.116256		
8.6	18.50515	3.1341	0.05948	
8.6	18.50515	3.151762	0.058873	$-0.002022\bar{2}$

$$H(x) = 17.56492 + 3.116256(x-8.3) +$$

$$+ 0.05948(x-8.3)^2 - 0.0020222(x-8.3)^2(x-8.6)$$

$H(8.4) = 17.87714444$

$f(8.4) = 17.87714633$

Error $= 1.889134 * 10^{-6}$

Ex. For the function $f(x) = x^4$ compute the divided difference table for the points

$$0, 1, 1, 1, 2$$

Write down the osculating polynomial interpolating $f(x)$ at those points

x	$f(x)$	I DD	II DD	III DD	IV DD.
0	0				
1	1	1			
1	1	4	3		
1	1	4	6	3	
2	16	15	11	5	1

$f'(x) = 4x^3 \quad f'(1) = 4$

$f''(x) = 12x^2 \quad f''(1) = 12 \quad \frac{f''(1)}{2} = 6$

The osculating polynomial is

$$H_4(x) = 0 + 1 \cdot x + 3x(x-1) + 3x(x-1)^2 + 1 \cdot x(x-1)^3$$