4.7 Gaussian Quadrature

The quadrature formulas in the previous section were obtained by approximating the integrand function with a polynomial and integrating the polynomial instead. Such formulas are called Newton-Cotes formulas.

All Newton-Cotes formulas use values of the function at equally spaced points.

In the trapezoidal rule, we approximate with the line joining the values at the endpoints. This is most cases, however, this is not the best line.

Clearly approximating the integral with this line will be much accurate and possibly exact.
Gaussian quadrature chooses the points for evaluation in an optimal (rather than equally spaced) way.

**Problem:** Choose the nodes $x_1, \ldots, x_n$ in $[a, b]$ and the coefficients $c_1, \ldots, c_n$ so that the error of the approximation

$$\int_a^b f(x) \, dx \approx \sum_{i=1}^n c_i f(x_i)$$

be as small as possible.

**Criterion for accuracy:** We want to choose $x_1, \ldots, x_n, c_1, \ldots, c_n$ so that the formula is exact for the largest class of polynomials.

How many parameters to choose? $2n$

A formula is exact for any polynomial of degree $n$ iff it is exact for $1, x, x^2, \ldots, x^n$.

Since we have $2n$ parameters to choose, we can pose $2n$ conditions.
Thus we want the formula to be exact for
\[1, x, x^2, \ldots, x^{2n-1}\]

Thus, a formula will be exact for all polynomials of degree \(2n-1\).

Ex. Suppose we are approximating
\[\int f(x)dx \approx cf(x) + g(x)dx\]

and we want to choose \(c_1, c_2, x_1, x_2\) so that the formula is exact (we have equality instead approximate equality) for the polynomials
\[1, x, x^2, x^3\]

Thus,
\[\int x^0 dx = c_1 + c_2 \Rightarrow c_1 + c_2 = 1\]
\[\int x^1 dx = c_1 x + c_2 x^2 \Rightarrow c_1 x + c_2 x^2 = 0\]
\[\int x^2 dx = c_1 x^2 + c_2 x^3 \Rightarrow c_1 x^2 + c_2 x^3 = \frac{2}{3}\]
\[\int x^3 dx = c_1 x^3 + c_2 x^4 \Rightarrow c_1 x^3 + c_2 x^4 = 0\]
Legendre polynomials \( P_0(x), P_1(x), \ldots, P_n(x) \)...

where

\[ \text{deg } P_n(x) = n \]

have the property that

\[ \int P_0(x) P_n(x) \, dx = 0 \]

for every polynomial \( R(x) \) with

\[ \text{deg } R(x) \leq n-1 \]

The first few Legendre polynomials are

\[ P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - \frac{1}{3} \]

\[ P_3(x) = x^3 - \frac{3}{5} x, \quad P_4(x) = x^4 - \frac{6}{7} x^2 + \frac{3}{35} \]

The roots of these polynomials are distinct and lie in \((-1, 1)\). They are taken as \( x_0, \ldots, x_n \).

It also turns out that \( c_0, \ldots, c_n \) can be determined

\[ c_i = \int L_i(x) \, dx \]

where \( L_i(x) \) is the basis Lagrange polynomial.
Thus, we get a system for $c, d, x, x_2$. We get:

$$c = \frac{1}{2}, \quad x = -\frac{\sqrt{3}}{3}, \quad x_2 = \frac{\sqrt{3}}{3}.$$

This gives the approximation formula:

$$\int f(x) \, dx = f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3})$$

This formula has a degree of precision 3, that is, the formula is exact for all polynomials of degree 3 or less.

Note: We don't need $f$ to find $c_0, c_1, x, x_2$. It turns out that we can find $x_1, \ldots, x_n$ and $c_0, \ldots, c_n$ so that

$$\int f(x) \, dx \approx c_0 f(x_1) + \ldots + c_n f(x_n)$$

and the formula is exact for polynomials of degree $2n-1$ and this can be done (as in the example above) for any function $f$.

It turns out that the nodes $x_1, \ldots, x_n$ are the roots of the $n$th Legendre polynomial.
Actually, the values of \( x_1, x_2, \ldots, x_n, c_1, \ldots, c_n \) are tabulated. See Table 4.11 (p. 225).

If we want to use Gauss quadrature to approximate
\[
\int_a^b f(x) \, dx
\]
over an arbitrary interval \([a, b]\), we first have to change variables so that the interval becomes \([-1, 1]\).

\[
t = \frac{2x-a-b}{b-a}, \quad x = \frac{1}{2} [(b-a)t + a+b]
\]

Thus, we have
\[
\int_a^b f(x) \, dx = \int_{-1}^1 f(\frac{(b-a)t + a+b}{2}) \cdot \frac{b-a}{2} \, dt
\]

Ex. Use Gaussian quadrature with \( n = 3 \) to approximate the integral. Compare your result to the exact value of the integral

\[
\int_0^\frac{\pi}{2} x^2 \sin x \, dx
\]
First we change variables

\[ x = \frac{1}{2} \left[ \frac{\pi}{4} t + \frac{\pi}{4} \right] = \frac{\pi}{8} (t+1), \quad -1 \leq t \leq 1 \]

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \sin x \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{\pi}{8} (t+1) \right)^2 \sin \frac{\pi}{8} (t+1) \cdot \frac{\pi}{8} \, dt
\]

\[
= \frac{\pi^3}{512} \int_{-1}^{1} (t+1)^2 \sin \frac{\pi}{8} (t+1) \, dt
\]

From table 411 we have

\[ t_1 = 0.774596692, \quad t_2 = 0, \quad t_3 = 0.774596692 \]

\[ \ell_1 = \frac{5}{9}, \quad \ell_2 = \frac{8}{9}, \quad \ell_3 = \frac{5}{9} \]

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \sin x \, dx \approx \frac{5}{9} \int_{-1}^{1} (t+1)^2 \sin \frac{\pi}{8} (t+1) \, dt
\]

\[ + \frac{\pi^3}{512} \sin \frac{\pi}{8} \]

\[ + \frac{\pi^2}{9} \left( 0.774596692 + 1 \right) \sin \frac{\pi}{8} \left( 0.774596692 + 1 \right)
\]

\[ = 1.511056695 \times 10^{-8} + 0.002059997 \]

\[ + 0.068002781 = 0.088753856
\]

The exact value is 0.0887538456

Error \approx -1.43 \times 10^{-6}
Ex. Determine the parameters \( a, b, c, d, e \) so that the formula

\[
\int x^2 \, dx = a f(0) + b f(1) + c f(2) + d f(3) + e f(4)
\]

has highest order of precision

\[
\int x \, dx = a + b + c \quad a + b + c = 2
\]

\[
\int x^2 \, dx = b + 2c + d + e \quad b + 2c + d + e = 2
\]

\[
\int x^3 \, dx = b + 4c + 4e \quad b + 4c + 4e = \frac{8}{3}
\]

\[
\int x^4 \, dx = b + 8c + 12e \quad b + 8c + 12e = 4
\]

\[
\int x^5 \, dx = b + 16c + 32e \quad b + 16c + 32e = \frac{32}{5}
\]

We get \( a = \frac{7}{15} \), \( b = \frac{16}{15} \), \( c = \frac{7}{15} \), \( d = \frac{1}{15} \), \( e = -\frac{1}{15} \).
Ex. Use Gauss quadrature to evaluate
\[ \int_{0}^{\frac{\pi}{2}} \sin x \, dx \] with \( n = 2 \).

Then use Trapezoidal and Simpson's
rule and compare the errors of 3
methods.

Solution: \( x = \frac{1}{2} \left[ \frac{\pi}{2} t + \frac{\pi}{2} \right] = \frac{\pi}{4} \, (t+1) \), \( dx = \frac{\pi}{4} \, dt \),

\[ \int_{0}^{\frac{\pi}{2}} \sin x \, dx = \frac{\pi}{4} \int_{-1}^{0} \sin \left( \frac{\pi}{4} (t+1) \right) \, dt \]

From table 4.11 the nodes are
\( t_1 = -0.57735 \), \( t_2 = 0.57735 \),
and the weights are
\[ c_1 = 1 \quad c_2 = 1 \]

Thus,
\[ \frac{\pi}{4} \int_{-1}^{0} \sin \left( \frac{\pi}{4} (t+1) \right) \, dt = \frac{\pi}{4} \left( 1 \sin \frac{\pi}{4} (-0.57735) + \right. \]
\[ \left. + 1 \sin \frac{\pi}{4} (0.57735) \right) = 0.99847 \]

Exact value is
\[ \int_{0}^{\frac{\pi}{2}} \sin x \, dx = \left. -\cos x \right|_{0}^{\frac{\pi}{2}} = -(-1) - (0) = 1 \]

Actual error = 1.53 \cdot 10^{-3}
Using Trapezoidal rule we have \((n=2)\)
\[
\int_{0}^{\pi} \sin x \, dx = \frac{\pi}{8} \left( \sin 0 + 2\sin \frac{\pi}{4} + \sin \frac{\pi}{2} \right) = 0.948
\]
\[\text{Error} = 0.07194 = 5.194 \times 10^{-2}\]

Using Simpson’s rule we have \((n=3)\)
\[
\int_{0}^{\pi} \sin x \, dx = \frac{\pi}{12} \left[ \sin 0 + 4\sin \frac{\pi}{4} + \sin \frac{\pi}{2} \right] = \frac{\pi}{12} \left[ 4\cdot \frac{\pi}{2} + 1 \right] = 1.00228
\]
Actual error = \(-0.00228 = -2.28 \times 10^{-3}\)

Remark: Trapezoidal rule has two evaluations of \(f\) (as many as Gaussian method has) but gives much worse approximation. Simpson’s rule gives approximately the same approximation (only slightly worse) but it has 3 evaluations of the function.
Conclusion: Gaussian quadrature are among the best.