5.2 Euler's Method

1) Euler's Method

Consider the initial-value problem (IVP)

\[
\begin{align*}
  y' &= f(t, y) \\
  y(a) &= \alpha
\end{align*}
\]

Assume: this IVP has a unique solution

Problem: Obtain an approximation of the solution \( y(t) \).

Actually, a continuous \( y(t) \) will not be obtained. Instead, approximate values of \( y \) will be generated at various values of \( t \) in \([a, b]\).

Def. The values of \( t \) in \([a, b]\) - \( t_i \) for which we will obtain an approximation of \( y(t_i) \) are called mesh points.

We will take the mesh points equally spaced. We divide the interval \([a, b]\) into \( N \) subintervals of equal length

\[ h = \frac{b-a}{N} \]
Suppose $y(t)$ has 2 continuous derivatives. Then

$$y(t_{i+1}) = y(t_i) + y'(t_i)h + \frac{h^2}{2} y''(\xi_i)$$

where $\xi_i$ is in $(t_i, t_{i+1})$.

From the IVP we have

$$y'(t_i) = f(t_i, y(t_i))$$

Thus,

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i)$$

(Euler's method: Construct $w_i$ such that $w_i = y(t_i)$ $i = 1, 2, \ldots, N$)

From the formula

(1) \[
\begin{align*}
W_0 &= x \\
W_{i+1} &= W_i + hf(t_i, W_i) \quad i = 0, 1, \ldots, N-1
\end{align*}
\]

Definition: Equation (1) is called the difference equation associated with Euler's Method.
Geometric interpretation of Euler's Method

Since \( w_i = y(t_i) \) \( f(t_i, w_i) \approx y'(t_i) \) and

\[ w_{i+1} = w_i + hf(t_i, w_i) \]

then from the point \( w_i \approx y(t_i) \) we draw a line with slope \( f(t_i, w_i) \) which approximates the actual slope \( y'(t_i) \) to get \( w_{i+1} \).

2) Error in Euler's Method.

We saw that at each point we are

neglecting the term

\[ \frac{h^2}{2} y''(t_i) \leq \frac{M}{2} h^2 \]

Thus, the error at each point is \( O(h^2) \). However, as we take the approximation as a starting point for
the next step (in the next subinterval)
thus from interval to interval the
error at each point piles up and
the global error is much larger.
Thus, we can only be sure that

\[ |y(t_i) - w_i| \leq Ch \quad i = 0, \ldots, N \]

where \( C \) is a constant. Thus the
global error of Euler's method
is only \( O(h) \).

Theorem 5.9. Suppose
1) \( f \) is continuous
2) \( f \) satisfies Lipschitz condition with constant \( L \) on
\[ D = \{ (t, y) : a \leq t \leq b, -\infty < y < \infty \} \]
3) Constant \( M \) exists with
\[ |y''(t)| \leq M \quad \text{for all } t \text{ in } [a, b] \]

Let \( y(t) \) be the unique solution of
the IVP

\[
\begin{cases}
y' = f(t, y) \quad a \leq t \leq b \\
y(a) = 2
\end{cases}
\]

and \( w_0, \ldots, w_N \) be the approximation
generated by Euler's Method. Then

\[ |y(t_i) - w_i| \leq \frac{hM}{2L} \left[ e^{L(t_i-a)} - 1 \right] \quad i = 0, \ldots, N. \]
Note: 1) The error depends on the length of the time interval \([a, b]\).
Thus, the error grows if we want to compute for larger and larger intervals of time.
This is typical for solving ODEs.

2) Errors of order \(O(h)\) are relatively large and Euler's method is not useful in practice to find the solution of ODE.

3) In the error bound, the upper bound of \(y''\) participates. Thus, we have to be able to get \(M\). It can be done as follows:

\[
y'(t) = f(t, y(t))
\]

\[
y''(t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} \cdot y' = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} \cdot f(t, y(t))
\]

From here it is sometimes possible to obtain bound of \(y''\).

Ex. \(y' = \cos(t + y)\)

\[
y'' = -\sin(t + y) - \sin(t + y) \cdot y' = -\sin(t + y) - \sin(t + y) \cdot \cos(t + y)
\]

\[
\Rightarrow |y''| \leq |\sin(t + y)| + |\sin(t + y) \cdot \cos(t + y)| \leq 1 + 1 = 2.
\]
Ex. Use Euler's method to approximate the solution for the IVP

\[ y' = -(y+1)(y+3) \quad 0 \leq t \leq 2 \]
\[ y(0) = -2 \]

with \( h = 0.2 \).

(Exact solution is \( y(t) = -3 + \frac{2}{1 + e^{-2t}} \)).

Solution: Euler's method is

\[
\begin{align*}
\omega_0 &= -2 \\
\omega_{i+1} &= \omega_i - 0.2 (\omega_i + 1)(\omega_i + 3) \\
& \quad \text{for } i = 0, \ldots, 9
\end{align*}
\]

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Given the IVP

\[ y' = \frac{2}{t} y + t^2 e^t \quad 1 < t < 2 \]

\[ y(1) = 0 \]

with exact solution

\[ y(t) = t^2 (e^t - e) \]

Compute the value of \( h \) necessary for

\[ |y(t_i) - y_i| \leq 0.1 \quad \text{for all } i \]

Solution. From Theorem 5.9 we know

\[ |y(t_i) - y_i| \leq \frac{h M}{2L} \left[ e^{(b-a)L} - 1 \right] \leq 0.1 \]

\( a = 1, \ b = 2 \). We need \( M, L \).

To find \( M \) we have to use the exact sol.

\[ y' = 2t (e^t - e) + t^2 e^t \]

\[ y'' = 2(e^t - e) + 2te^t + t^2 e^t \]

\[ = 2(e^t - e) + 4te^t + \frac{t^2}{e^t} \]

\[ \leq 2(e^2 - e) + 4e^2 + 4e^2 \]

\[ = 14e^2 - 2e \leq 98.02 \Rightarrow M = 98.02 \]

To find \( L \):
\[ |f(t, y_1) - f(t, y_2)| = \frac{1}{2} |y_1 - y_2| \leq \frac{1}{2} L |y_1 - y_2| \]

\[ \Rightarrow L = 2 \]

\[ \frac{98.02 \ h}{2.2} (e^2 - 1) < 0.1 \]

\[ 24.505 \ h (e^2 - 1) < 0.1 \]

\[ h < 0.6387 \cdot 10^{-3} \]