5.2 Euler's Method

1) Euler's Method

Consider the initial-value problem (IVP)

$$\begin{cases} y' = f(t,y) & a \le t \le b \\ y(a) = \alpha \end{cases}$$

Assume: this IVP has a unique solution

<u>Problem</u>: Obtain an approximation of the solution $y(t)$.

Actually, a continuous $y(t)$ will not be obtained. Instead, approximate values of y will be generated at various values of t in $[a,b]$.

Def. The values of t in $[a,b]$ – t_i for which we will obtain an approximation of $y(t_i)$ are called <u>mesh points</u>.

We will take the mesh points equally spaced. We divide the interval $[a,b]$ into N subintervals of equal length

$$h = \frac{b-a}{N}$$

$$t_i = a + ih = t_{i-1} + h \qquad i = 0, 1, \dots, N$$

Suppose $y(t)$ has 2 continuous derivatives. Then

$$y(t_{i+1}) = y(t_i + h) = y(t_i) + y'(t_i)h + \frac{h^2}{2} y''(\xi_i)$$

ξ_i in (t_i, t_{i+1})

From the IVP we have

$$y'(t_i) = f(t_i, y(t_i))$$

Thus,

$$y(t_{i+1}) = y(t_i) + h f(t_i, y(t_i)) + \underbrace{\frac{h^2}{2} y''(\xi_i)}_{\text{local error}}$$

Euler's method: Construct w_i such that

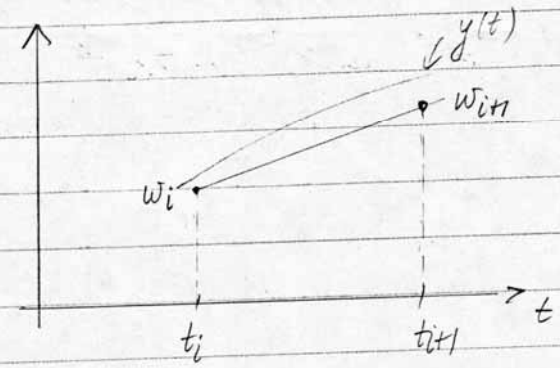
$$w_i \approx y(t_i) \qquad i = 1, 2, \dots, N$$

from the formula

$$(1) \qquad \begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h f(t_i, w_i) \quad i = 0, 1, \dots, N-1 \end{cases}$$

Def. Equation (1) is called <u>the difference equation</u> associated with Euler's Method.

Geometric interpretation of Euler's Method



Since $w_i = y(t_i)$ $f(t_i, w_i) \approx y'(t_i)$ and

$$w_{i+1} = w_i + hf(t_i, w_i)$$

then from the point $w_i \approx y(t_i)$ we draw a line with slope $f(t_i, w_i)$ which approximates the actual slope $y'(t_i)$ to get w_{i+1}.

2) Error in Euler's Method.

We saw that at each point we are neglecting the term

$$\left| \frac{h^2}{2} y''(\xi_i) \right| \leq \frac{M}{2} h^2$$

Thus, the error at each point is $O(h^2)$. However, as we take the approximation as a starting point for

the next step (in the next subinterval) thus, from interval to interval the error at each point piles up and the global error is much larger. Thus, we can only be sure that

$$|y(t_i) - w_i| \le Ch \qquad i = 0, \dots, N$$

where C is a constant. Thus the global error of Euler's method is only $O(h)$.

Theorem 5.9. Suppose

1) f is continuous
2) f satisfies Lipschitz condition with constant L on
$D = \{(t,y) : a \le t \le b, \ -\infty < y < \infty\}$
3) Constant M exists with

$$|y''(t)| \le M \text{ for all } t \text{ in } [a,b]$$

Let $y(t)$ be the unique solution of the IVP

$$\begin{cases} y' = f(t,y) & a \le t \le b \\ y(a) = \alpha \end{cases}$$

and $w_0, \dots, w_N$ be the approximation generated by Euler's Method. Then

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{(b-a)L} - 1\right] \qquad i = 0, \dots, N.$$

Note: 1) The error depends on the length of the time interval $[a,b]$. Thus, the error grows if we want to compute for larger and larger intervals of time. This is typical for solving ODEs.

2) Errors of order $O(h)$ are relatively large and Euler's method is not used in practice to find the solution of ODE.

3) In the error bound the upper bound of y'' participates. Thus, we have to be able to get M. It can be done as follows

$$y'(t) = f(t, y(t))$$

$$y''(t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} \cdot y' = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} \cdot f(t, y(t))$$

From here it is sometimes possible to obtain bound of y''.

Ex: $y' = \cos(t+y)$

$$y'' = -\sin(t+y) - \sin(t+y) \cdot y' =$$
$$= -\sin(t+y) - \sin(t+y) \cdot \cos(t+y)$$
$$\Rightarrow |y''| \le |\sin(t+y)| + |\sin(t+y)\cos(t+y)| \le 1+1 = 2.$$

Ex. #3,4/264 Use Euler's method to approximate the solution for the IVP

$$y' = -(y+1)(y+3) \qquad 0 \le t \le 2$$
$$y(0) = -2$$

with $h = 0.2$.

(Exact solution is $y(t) = -3 + \frac{2}{1+e^{-2t}}$)

Solution: Euler's method is

$$\begin{cases} w_0 = -2 \\ w_{i+1} = w_i - 0.2(w_i+1)(w_i+3) \end{cases}$$
$$i = 0, \ldots, 9$$

i	t_i	w_i	$y(t_i)$	Error
0	0	−2	−2	0
1	0.2	−1.8	−1.80262468	−0.002625
2	0.4	−1.608	−1.620051038	−0.012051
3	0.6	−1.4387328	−1.462950433	−0.024218
4	0.8	−1.301736974	−1.33596323	−0.034226
5	1.0	−1.199251225	−1.238405844	−0.0391546
6	1.2	−1.127490945	−1.166345393	−0.0388544
7	1.4	−1.079745355	−1.114648352	−0.034903
8	1.6	−1.049119077	−1.078331446	−0.029124
9	1.8	−1.029953983	−1.053193987	−0.02324
10	2	−1.01815183 8	−1.03597242	−0.01782058

Ex #5c/264

Given the IVP

$$y' = \frac{2}{t}y + t^2e^t \qquad 1 \le t \le 2$$

$$y(1) = 0$$

with exact solution

$$y(t) = t^2(e^t - e)$$

Compute the value of h necessary for

$$|y(t_i) - w_i| \le 0.1 \quad \text{for all } i.$$

<u>Solution</u>: From Theorem 5.9 we know

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{(b-a)L} - 1\right] \underset{\text{want}}{\le} 0.1$$

$a = 1$, $b = 2$. We need M, L.

To find M we have to use the exact sol.

$$y' = 2t(e^t - e) + t^2e^t$$

$$\begin{aligned} y'' &= 2(e^t - e) + 2te^t + 2te^t + t^2e^t \\ &= 2(e^t - e) + 4te^t + t^2e^t \\ &\le 2(e^2 - e) + 4 \cdot 2e^2 + 4e^2 = \\ &= 14e^2 - 2e \le 98.02 \implies M = 98.02 \end{aligned}$$

To find L:

$$|f(t,y_1)-f(t,y_2)| = \left|\frac{2}{t}\right| |y_1-y_2| \leq 2|y_1-y_2|$$

$$\Rightarrow L = 2$$

$$\frac{98.02\,h}{2\cdot 2}\left(e^2-1\right) < 0.1$$

$$24.505\,h\left(e^2-1\right) < 0.1$$

$$h < 0.6387\cdot 10^{-3}$$