Competitive exclusion in a vector-host epidemic model with distributed delay*

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Abstract

A multi-strain model of a vector-borne disease with distributed delay in the vector and the host is investigated. It is shown that if the reproduction number of the model $\mathcal{R}_0 < 1$, the unique disease-free equilibrium is globally asymptotically stable. Without loss of generality strain one is assumed to have the largest reproduction number. In this case, the dominance equilibrium of strain one is shown to be locally stable. The basic reproduction number for a strain i ($\mathcal{R}_0^i$) is written as a product of the reproduction number of the vector ($\mathcal{R}_v^i$) and the reproduction number of the host ($\mathcal{R}_h^i$), that is $\mathcal{R}_0^i = \mathcal{R}_h^i \mathcal{R}_v^i$. Competitive exclusion principle is derived under the somewhat stronger condition that if strain one maximizes both the reproduction number of the host $\mathcal{R}_h^i < \mathcal{R}_h^1,\ i \neq 1$ and the reproduction number of the vector $\mathcal{R}_v^i < \mathcal{R}_v^1,\ i \neq 1$, strain one dominance equilibrium is globally asymptotically stable.

Key words: Vector-host, competitive exclusion, distributed delay, global stability, Lyapunov function.

AMS subject classifications: 92D30

1 Introduction

Vector-borne diseases are among the most complex infectious diseases to understand, model and control. Their complexity stems from the multi-staged life cycle of the vector, often influenced by the climate, the intricate evolution of the parasite, developing both in the vector and the

*This work was carried out with the support from NSF under grant DMS-0817789 and NSF of China grants 11271314 and 10911120387.

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host, and the fact that most vector-borne diseases have humans as well as non-human species as hosts. Many factors make controlling vector-borne diseases particularly challenging. This article focuses on one particular aspect, namely the evolution of the parasite. Parasites of vector-borne diseases exist in several species or serotypes. For instance, four species of the *Plasmodium* parasite cause malaria: *P. falciparum, P. ovale, P. vivax,* and *P.malariae*. Four serotypes of the dengue fever virus cause dengue. However, within these large classification groups, each pathogen can be antigenically diverse and may exist in multiple variants, called strains. Multi-strain models for vector-borne diseases have been considered before (see [5,7,8] and the references therein) but in most cases strains are assumed to interact through cross-immunity or other mechanism which generates coexistence.

For directly transmitted infections it is well known that in the simplest case when no trade off mechanism is present [22], the only possible outcome of the competition of n strains is competitive exclusion. In this case, the strain with the largest reproduction number outcompetes and eliminates the remaining strains, a result first established by Bremermann and Thieme [2]. In this article, we address the question whether this principle extends to vector-borne diseases. Because the dynamics of the vector is subjected to a significant influence from the incubation period of the pathogen within the vector, called extrinsic incubation period, and the incubation period within the host, called intrinsic incubation period, we incorporate distributed delay both in the vector and the host to account for these delays. Natural delays occurring in the dynamics of vector-borne diseases are typically incorporated as discrete delays or age-structure. As such they lead to backward bifurcation, oscillations [1,6,15,16,28,34] and even chaos [23].

In this article, we consider the outcome of the competition in the resulting multi-strain vector-borne model with distributed delay. Originally our hypothesis was that, analogously to directly transmitted diseases, the strain with the maximal reproduction number eliminates all the rest. However, the analysis revealed that we need a stronger condition for competitive dominance in the case of vector-borne diseases. In particular, we notice that the basic reproduction number for a strain i ($\mathcal{R}_0^i$) is written as a product of the reproduction number of the vector ($\mathcal{R}_v^i$) and the reproduction number of the host ($\mathcal{R}_h^i$), that is $\mathcal{R}_0^i = \mathcal{R}_h^i \mathcal{R}_v^i$. We find that for vector-borne diseases the competitive exclusion principle can be formulated as follows: *The strain that maximizes both the vector and the human reproduction numbers dominates in the population.* Such a strain will also have an overall maximal reproduction number. We should mention, however, that our competitive exclusion principle for vector-borne diseases is not complete. It would not give the outcome of the competition of the strains when different strains maximize the vector $\mathcal{R}_v$ and the human $\mathcal{R}_h$ reproduction numbers. For instance, if strain one maximizes the human reproduction number, that is $\mathcal{R}_h^1 > \mathcal{R}_h^i$ for all $i \neq 1$, but strain two maximizes the vector reproduction number, $\mathcal{R}_v^2 > \mathcal{R}_v^i$ for all $i \neq 2$, then our competitive exclusion principle does not imply which strain will dominate in the population. We hypothesize that the strain with the maximal basic reproduction number will dominate but different approach than the one we have applied here may be necessary to establish this result.

This paper is structured as follows. In section 2, we introduce a multi-strain version of the vector-borne disease model with two distributed delays. In section 3, we discuss the equilibria and establish their local stabilities. In section 4, we establish global stability of the disease-free equilibrium. In section 5, we verify competitive exclusion principle under the assumption that strain one maximizes both the human and the vector reproduction numbers. Section 6 summarizes our results.

2 Model formulation

As in the introduction, we assume the pathogen causes the vector-host transmission disease existence through multiple strains. The multi-strain model of the vector-host interaction divides the vector population under consideration into the following groups: susceptible vectors, denoted by S_v, and infected vectors, which are divided into n subgroups: those infected with a strain i, denoted by $I_{v_i}, i = 1, 2, \cdots, n$. If we let $N_v(t)$ be the total number of vectors at time t, we have $N_v(t) = S_v(t) + \sum_{i=1}^{n} I_{v_i}(t)$. Let $N_h(t)$ be the density of the total host population at time t, which is composed of the number of susceptible individuals $S_h(t)$, the number of infective individuals $I_{h_i}(t)$ with a strain i, $i = 1, 2, \cdots, n$, and the number of recovered or immune individuals $R_h(t)$. For the disease transmission, it is assumed that a susceptible host can receive the infection through contact with a strain i infective mosquitoes. Similarly, a susceptible mosquito can receive the infection through contact with strain i infectious human. Furthermore, it is assumed that a susceptible host and susceptible mosquito become exposed when they receive the infection from an infective individual and remain exposed for some period before becoming infectious.

Let Λ_h and Λ_v be the birth rate of the host population and the vectors, respectively. b is the average rate of biting on hosts by a single mosquito (number of bites per unit time), and the hosts are always sufficient in abundance; so that it is reasonable to assume that the biting rate b is constant. Thus, the number of bites on hosts per unit time per host is $\frac{b}{N_h}$. If β'_{h_i} is probability of infected bites with strain i on host individuals that produce an infection, $\frac{b}{N_h}\beta'_{h_i} S_h I_{v_i}$ is the infection force of the interaction between the infected mosquitos with strain i infected vectors I_{v_i} and the susceptible host S_h. As pointed out in the introduction, there exists extrinsic and intrinsic incubation period in mosquitos and hosts, respectively. Let τ be the extrinsic incubation period of the parasite in the vector population. Since only a proportion of vectors survive this incubation period, the force of infection on host individuals at the present time t is $\frac{b}{N_h}\beta'_{h_i} S_h(t) I_{v_i}(t-\tau) e^{-\mu_{v_i}\tau}$, where μ_{v_i} is the death rate of the vectors infected with strain i. In reality, however, the incubation period is not a number but an interval during which the maturation of the parasite occurs in different hosts and vectors. Hence, we assume that τ is a distributed parameter(see [3,19]). As a result, the force of infection becomes $\dfrac{b}{N_h}\beta'_{h_i} S_h(t) \displaystyle\int_0^{\tau_v} g'_{h_i}(\tau) I_{v_i}(t-\tau) e^{-\mu_{v_i}\tau} d\tau$ where the kernel function $g'_{h_i}(\tau)$ represents the distribution of the infectivity of the parasites in vector population where the time taken to become infectious is τ.

Similarly, let μ_h be death rate of host individuals, and α_{h_i} be recovery rate of infected host individuals with strain i. β'_{v_i} is probability of a susceptible vector bites on the infective host individuals with strain i that produce an infection. Based on the above facts and that the average total rate of contacts between host and vectors must be conserved [27, P_{667}], $\dfrac{b}{N_h}\beta'_{v_i} S_v(t) \displaystyle\int_0^{\tau_h} g'_{v_i}(\tau) I_{h_i}(t -$

$\tau)e^{-(\mu_h+\alpha_{h_i})\tau}d\tau$ gives the incidence of new cases of infection for the vector at the present time t. The kernel function $g'_{v_i}(\tau)$ expresses the infectivity on susceptible vectors during the intrinsic incubation period. Here, τ_v, τ_h are, respectively, the upper limits of the parasites incubation periods in hosts and vectors. The terms $e^{-\mu_{v_i}\tau}, e^{-(\mu_h+\alpha_{h_i})\tau}$ account for the probability of survival as infectious of host individuals during the intrinsic incubation period.

In more general sense, there would be possibility that the removed population is alive and bitten again by vectors, or multiple infectious bites on infected people may affect the disease progression. For simplicity, however, we assume that the recovered population will no longer be involved in the transmission process, and multiple infectious bites on infected hosts do not play a role in the progression of the disease. Under the above assumption, our multi-strain model with extrinsic and intrinsic distributed incubation periods can be formulated as follows:

$$\begin{aligned}
\frac{dS_v(t)}{dt} &= \Lambda_v - \sum_{i=1}^{n}\beta'_{v_i}\frac{bS_v(t)}{N_h(t)}\int_0^{\tau_h} g'_{v_i}(\tau)I_{h_i}(t-\tau)e^{-(\mu_h+\alpha_{h_i})\tau}d\tau - \mu_v S_v(t),\\
\frac{dI_{v_i}(t)}{dt} &= \beta'_{v_i}\frac{bS_v(t)}{N_h}\int_0^{\tau_h} g'_{v_i}(\tau)I_{h_i}(t-\tau)e^{-(\mu_h+\alpha_{h_i})\tau}d\tau - \mu_v I_{v_i}(t),\ i=1,\cdots,n,\\
\frac{dS_h(t)}{dt} &= \Lambda_h - \sum_{i=1}^{n}\beta'_{h_i}\frac{bS_h(t)}{N_h(t)}\int_0^{\tau_v} g'_{h_i}(\tau)I_{v_i}(t-\tau)e^{-\mu_{v_i}\tau}d\tau - \mu_h S_h(t),\\
\frac{dI_{h_i}(t)}{dt} &= \beta'_{h_i}\frac{bS_v(t)}{N_h(t)}\int_0^{\tau_v} g'_{h_i}(\tau)I_{v_i}(t-\tau)e^{-\mu_{v_i}\tau}d\tau - (\mu_h+\alpha_{h_i})I_{h_i}(t),\ i=1,\cdots,n,\\
\frac{dR_h(t)}{dt} &= \sum_{i=1}^{n}\alpha_{h_i}I_{h_i}(t) - \mu_h R_h(t).
\end{aligned} \tag{2.1}$$

Model (2.1) is equipped with the following initial conditions:

$$\begin{aligned}
&S_v(0)=S_{v_0},\ I_{v_i}(\theta)=\psi_{v_i}(\theta),\ S_h(0)=S_{h_0},\ I_{h_i}(\theta)=\psi_{h_i}(\theta),\ R_h(0)=R_{h_0},\ i=1,\cdots,n.\\
&S_{v_0}>0,\ S_{h_0}>0,\ R_{h_0}>0,\ \psi_{v_i}(\theta)\geq 0,\ \psi_{h_i}(\theta)\geq 0,\quad \theta\in[-h,0],\ h=\max\{\tau_v,\tau_h\}.
\end{aligned} \tag{2.2}$$

All parameters in model (2.1) are nonnegative. We define the following space of functions

$$X = \mathbb{R}^+ \times \prod_{i=1}^{n}(C([-h,0],\mathbb{R}^+)\times\mathbb{R}^+ \times \prod_{i=1}^{n}(C([-h,0],\mathbb{R}^+)\times\mathbb{R}^+,$$

where, the Banach space $C([-h,0],\mathbf{R})$ of continuous functions mapping the interval $[-h,0]$ into $\mathbf{R}$ is equipped with the sup-norm $||\psi|| = \sup_{-h\leq\theta\leq 0}|\psi(\theta)|$. By the standard theory of functional differential equations [11], it can be verified that solutions of (2.1) with nonnegative initial conditions (2.2) has a unique solution $(S_v(t), I_{v_i}(t), S_h(t), I_{h_i}(t), R_h(t))$ which remains nonnegative for all $t\geq 0$. Moreover, we can show the solutions of system (2.1) are ultimately uniformly bounded in X. In fact, it follows from the total vector population size $N_v(t)$ satisfying $N'_v(t) = \Lambda_v - \mu_v N_v(t)$ that

$$N_v(t) \to \frac{\Lambda_v}{\mu_v},\ \text{as}\ \ t\to\infty.$$

Similarly, we have

$$N_h(t) \to \frac{\Lambda_h}{\mu_h},\ \text{as}\ \ t\to\infty.$$

So, it is assumed that the total vector and host population sizes $N_v(t), N_h(t)$ are asymptotically constant.

Let

$$\Omega = \left\{ (S_v, I_{v_1}, \cdots, I_{v_n}, S_h, I_{h_1}, \cdots, I_{h_n}) \in |S_v(t) + \sum_{i=1}^{n} I_{v_i}(t) \leq \frac{\Lambda_v}{\mu_v}, S_h(t) + \sum_{i=1}^{n} I_{h_i}(t) \leq \frac{\Lambda_h}{\mu_h} \right\}.$$

It is easy to see that the set Ω is positively invariant for system (2.1). We note that $R_h(t)$ can be removed from the equations of system (2.1), it is sufficient to analyze the dynamical behavior of solutions to system (2.1) without the equation of $R_h(t)$. For simplicity, we introduce the following notation.

$$\beta_{v_i} = \frac{b}{N_h}\beta'_{v_i}, \quad \beta_{h_i} = \frac{b}{N_h}\beta'_{h_i}, \quad g_{h_i}(\tau) = e^{-\mu_{v_i}\tau} g'_{v_i}(\tau), \quad g_{v_i}(\tau) = g'_{v_i}(\tau) e^{-(\mu_h + \alpha_{h_i})\tau}, i = 1, 2, \cdots, n.$$

Furthermore, we impose the following assumptions:

Assumptions 1:

1. It is assumed that $g_{v_i}(\tau)$, and $g_{h_i}(\tau)$ are continuous on $[0, h]$, $i = 1, 2, \cdots, n$;
2. $g_{v_i}(\tau)$, and $g_{h_i}(\tau)$ satisfy
$$\int_0^{\tau_h} g_{v_i}(\tau) d\tau = a_{v_i}, \qquad \int_0^{\tau_v} g_{h_i}(\tau) d\tau = a_{h_i}, \quad i = 1, 2, \cdots, n;$$
3. $g_{v_i}(\tau) \geq 0, \quad g_{h_i}(\tau) \geq 0$ for $0 \leq \tau \leq h, \quad i = 1, 2, \cdots, n$. Here $h = \tau_h, \tau_v$ respectively.

In the next section, we investigate the dynamics of following system

$$\begin{aligned}
\frac{dS_v(t)}{dt} &= \Lambda_v - \sum_{i=1}^{n} \beta_{v_i} S_v(t) \int_0^{\tau_h} g_{v_i}(\tau) I_{h_i}(t-\tau) d\tau - \mu_v S_v(t), \\
\frac{dI_{v_i}(t)}{dt} &= \beta_{v_i} S_v(t) \int_0^{\tau_h} g_{v_i}(\tau) I_{h_i}(t-\tau) d\tau - \mu_v I_{v_i}(t), \ i = 1, \cdots, n. \\
\frac{dS_h(t)}{dt} &= \Lambda_h - \sum_{i=1}^{n} \beta_{h_i} S_h(t) \int_0^{\tau_v} g_{h_i}(\tau) I_{v_i}(t-\tau) d\tau - \mu_h S_h(t), \\
\frac{dI_{h_i}(t)}{dt} &= \beta_{h_i} S_h(t) \int_0^{\tau_v} g_{h_i}(\tau) I_{v_i}(t-\tau) d\tau - (\mu_h + \alpha_{h_i}) I_{h_i}(t), \ i = 1, \cdots, n.
\end{aligned} \tag{2.3}$$

From the above discussion, we have always the following proposition for system (2.3)

Proposition 2.1 *There exists a unique solution $(S_v(t), I_{v_i}(t), S_h(t), I_{h_i}(t))$ of system (2.3) with initial conditions (2.2). This solution is nonnegative for all $t \geq 0$. Moreover, all the solutions of system (2.3) are ultimately uniformly bounded in Ω.*

3 Equilibria and local stability

System (2.1) always has a unique disease-free equilibrium $\mathcal{E}_0$, which is given by

$$\mathcal{E}_0 = \left(\frac{\Lambda_v}{\mu_v}, \mathbf{0}, \frac{\Lambda_\mathrm{h}}{\mu_\mathrm{h}}, \mathbf{0} \right)$$

where $\mathbf{0} = (0, \cdots, 0)$ is a $n-$ dimensional vector of zeroes.

According to the definition of the basic reproduction number [30], which is the expected number of secondary cases produced in an entirely susceptible population by a typical infected individual during its entire infectious period, we define the reproduction number of strain i:

$$\mathcal{R}_0^i = \frac{a_{v_i} a_{h_i} \beta_{h_i} \beta_{v_i} \Lambda_h \Lambda_v}{(\mu_h + \alpha_{h_i}) \mu_h (\mu_v)^2}, \ i = 1, \cdots, n.$$

We notice that the disease cycle in vector-borne disease is consists of two transmission pathways: human-to-vector transmission and vector-to-human transmission. Each of these pathways is characterized with its own reproductive number. The reproductive number of human-to-vector transmission of strain i is given by

$$\mathcal{R}_h^i = \frac{a_{v_i} \beta_{v_i} \Lambda_v}{\mu_v (\mu_h + \alpha_{h_i})}$$

Similarly, the reproduction number of the vector-to-human pathway of strain i is given by

$$\mathcal{R}_v^i = \frac{a_{h_i} \beta_{h_i} \Lambda_h}{\mu_h \mu_v}, \ i = 1, 2, \cdots, n.$$

Obviously, the reproduction number for a strain i in system (2.3) is a product of the human and vector reproduction numbers:

$$\mathcal{R}_0^i = \mathcal{R}_h^i \mathcal{R}_v^i.$$

We define a reproduction number of the whole system (2.3)

$$\mathcal{R}_0 = \max\{\mathcal{R}_0^1, \mathcal{R}_0^2, \cdots, \mathcal{R}_0^n\}.$$

By direct calculation, we show that for each i, in system (2.3), there is a corresponding strain-i equilibrium $\mathcal{E}_i$ given by

$$\mathcal{E}_i = (S_{v_i}^*, 0, \cdots, 0, I_{v_i}^*, 0, \cdots, 0, S_{h_i}^*, 0, \cdots, 0, I_{h_i}^*, 0, \cdots, 0),$$

where the non-zero components $I_{v_i}^*$ and $I_{h_i}^*$ are in position $i+1$ and $n+j+1$, respectively. Moreover, equilibrium $\mathcal{E}_i$ exists if and only if $\mathcal{R}_0^i > 1$. The nonzero components of the equilibrium $\mathcal{E}_i$ are given by

$$S_{v_i}^* = \frac{\Lambda_v}{a_{v_i} \beta_{v_i} I_{h_i}^* + \mu_v}, \ S_h^* = \frac{\Lambda_h \mu_v (a_{v_i} \beta_{v_i} I_{h_i}^* + \mu_v)}{I_{h_i}^* (a_{h_i} a_{v_i} \beta_{h_i} \beta_{v_i} \Lambda_v + \beta_{v_i} \mu_v \mu_h) + \mu_h \mu_v^2},$$

$$I_{v_i}^* = \frac{a_{v_i} \beta_{v_i} I_{h_i}^* \Lambda_v}{\mu_v (a_{v_i} \beta_{v_i} I_{h_i}^* + \mu_v)}, \ I_{h_i}^* = \frac{\mu_h \mu_v^2 (\mathcal{R}_0^i - 1)}{\beta_{h_i} \beta_{v_i} a_{h_i} a_{v_i} \Lambda_v + a_{v_i} \beta_{v_i} \mu_h \mu_v}.$$

Now we investigate the stability of the equilibria in system (2.3), let us first linearizing system (2.3) at disease-free equilibrium $\mathcal{E}_0$. In particular, let $S_v(t) = \frac{\Lambda_v}{\mu_v} + x_v(t)$, $I_{v_i}(t) = y_{v_i}(t)$, $S_h(t) = \frac{\Lambda_h}{\mu_h} + x_h(t)$, $I_{h_i}(t) = y_{h_i}(t)$. We look for exponential solutions ($x_v(t) = \bar{x}_v e^{\lambda t}, y_{v_i}(t) = \bar{y}_{v_i} e^{\lambda t}, x_h(t) = \bar{x}_h e^{\lambda t}, y_{h_i}(t) = \bar{y}_{h_i} e^{\lambda t}$) of the time-dependent system for the perturbations. Thus, we obtain the following eigenvalue problem

$$\begin{cases} \lambda \bar{x}_v = -\dfrac{\Lambda_v}{\mu_v} \displaystyle\sum_{i=1}^{n} \beta_{v_i} \bar{y}_{h_i} \int_0^{\tau_h} g_{v_i}(\tau) e^{-\lambda \tau} d\tau - \mu_v \bar{x}_v, \\ \lambda \bar{y}_{v_i} = \bar{y}_{h_i} \dfrac{\Lambda_v}{\mu_v} \beta_{v_i} \displaystyle\int_0^{\tau_h} g_{v_i}(\tau) e^{-\lambda \tau} d\tau - \mu_v \bar{y}_{v_i}, \\ \lambda \bar{x}_h = -\dfrac{\Lambda_h}{\mu_h} \displaystyle\sum_{i=1}^{n} \beta_{h_i} \bar{y}_{v_i} \int_0^{\tau_v} g_{h_i}(\tau) e^{-\lambda \tau} d\tau - \mu_h \bar{x}_h, \\ \lambda \bar{y}_{h_i} = \bar{y}_{v_i} \dfrac{\Lambda_h}{\mu_h} \beta_{h_i} \displaystyle\int_0^{\tau_v} g_{h_i}(\tau) e^{-\lambda \tau} d\tau - (\mu_h + \alpha_{h_i}) \bar{y}_{h_i}. \end{cases} \tag{3.1}$$

Solving the linear system (3.1), it is easy to obtain that $-\mu_v$, $-\mu_h$ are two negative characteristic roots of system (2.3) and other characteristic roots $(n-2)$ are determined by the following equation

$$[(\lambda+\mu_v)(\lambda+\mu_h+\alpha_{h_i})-\frac{\Lambda_v\Lambda_h\beta_{v_i}\beta_{h_i}a_{v_i}a_{h_i}}{\mu_v\mu_h}\int_0^{\tau_v}g_{h_i}(\tau)e^{-\lambda\tau}d\tau\int_0^{\tau_h}g_{v_i}(\tau)e^{-\lambda\tau}d\tau]^{n-2}=0. \quad (3.2)$$

We now show that all characteristic roots in (2.6) have negative real part. Note that Eq (3.2) is continuous functions of τ, and there is a continuity in the eigenvalues for $\tau\geq 0$. Using Rouche's Theorem [4, Theorem 9.17.4], we know that given the eigenvalues were all negative in the non-delay case, introducing a delay can cause a change in stability if and only if for some value of τ, there exists a purely imaginary root of equation (3.2), i.e., a pair of eigenvalues must cross the imaginary axis if they are to become positive. For the equation (3.2), we have the following lemma:

Lemma 3.1 *If $\mathcal{R}_0=\max\{\mathcal{R}_0^1,\cdots,\mathcal{R}_0^n\}<1$, the characteristic equation (3.2) has no purely imaginary roots.*

Proof. Assume the contrary. Set

$$Q(\lambda,\tau)=\lambda^2+(\mu_h+\alpha_{h_i}+\mu_v)\lambda+\mu_v(\mu_h+\alpha_{h_i})-\frac{\beta_{h_i}\beta_{v_i}\Lambda_v\Lambda_h}{\mu_h\mu_v}\int_0^{\tau_v}g_{h_i}(\tau)e^{-\lambda\tau}d\tau\int_0^{\tau_h}g_{v_i}(\tau)e^{-\lambda\tau}d\tau=0. \quad (3.3)$$

To proceed, we first consider equation (3.3) with $\tau=0$. It is easy to verify that all roots in Eq. (3.3) have negative real parts for $\mathcal{R}_0^i<1$ and $\tau=0$. For $\tau>0$, we show that $Q(\lambda,\tau)=0$ has no purely imaginary roots. Set

$$\begin{aligned}
&F_1(\lambda,\tau)=\int_0^{\tau_v}g_{h_i}(\tau)e^{-\lambda\tau}d\tau\int_0^{\tau_h}g_{v_i}(\tau)e^{-\lambda\tau}d\tau,\\
&F_2(\lambda)=\lambda^2+\lambda(\mu_h+\alpha_{h_i}+\mu_h)+\mu_v(\mu_h+\alpha_{h_i}),\\
&F_3=\frac{\beta_{h_i}\beta_{v_i}\Lambda_v\Lambda_h}{\mu_h\mu_v}.
\end{aligned}$$

Let $\lambda=i\omega$ $(\omega>0)$ be a root of equation (3.3). Thus,we have

$$F_1(i\omega,\tau)=\int_0^{\tau_v}g_{h_i}(\tau)e^{-i\omega\tau}d\tau\int_0^{\tau_h}g_{v_i}(\tau)e^{-i\omega\tau}d\tau.$$

Thus, we have $|F_1(i\omega,\tau)|\leq a_{h_i}a_{v_i}$. From equation (3.3), we have $F_2(i\omega)-F_3F_1(i\omega,\tau)=0$. Therefore, we have

$$\left|\frac{F_2(i\omega)}{F_3}\right|=|F_1(i\omega,\tau)|\leq a_{h_i}a_{v_i}.$$

That is, $|F_2(i\omega)|\leq a_{h_i}a_{v_i}|F_3|$. Thus, we have

$$\begin{aligned}
[\mu_v(\mu_h+\alpha_{h_i})-\omega^2]^2+(\mu_h+\alpha_{h_i}+\mu_v)^2\omega^2 &= \omega^4+[(\mu_h+\alpha_{h_i})^2+\mu_v^2]\omega^2+\mu_v^2(\mu_h+\alpha_{h_i})^2\\
&\leq a_{h_i}^2a_{v_i}^2\beta_{h_i}^2\beta_{v_i}^2\left(\frac{\Lambda_h\Lambda_v}{\mu_h\mu_v}\right)^2.
\end{aligned} \quad (3.4)$$

It follows from (3.4) that we obtain

$$\omega^4+[(\mu_h+\alpha)^2+\mu_v^2]\omega^2+\mu_v^2(\mu_h+\alpha_{h_i})^2(1-(\mathcal{R}_0^i)^2)\leq 0. \quad (3.5)$$

Obviously, if $\mathcal{R}_0^i<1, i=1,2,\cdots,n$, it is impossible for (3.5) to hold. Hence, the linearized system (3.2) can not have any roots with positive real part.

Therefore, from the above discussion, we have established the following result:

Theorem 3.1 *If $\mathcal{R}_0 = \max\{\mathcal{R}_0^1, \cdots, \mathcal{R}_0^n\} < 1$, then the disease-free equilibrium $\mathcal{E}_0$ of system (2.3) is locally asymptotically stable. If $\mathcal{R}_0 > 1$, it is unstable.*

Now we shall establish the local stability of the single-strain equilibrium $\mathcal{E}_i$ for a fixed i. We have the following result

Theorem 3.2 *Assume for a fixed i, $\mathcal{R}_0^i > 1$. If $\mathcal{R}_0^j < \mathcal{R}_0^i$ for all $j \neq i$, then the strain-i equilibrium $\mathcal{E}_i$ is locally asymptotically stable. Otherwise, if there exists k_0 such $\mathcal{R}_{k_0} > \mathcal{R}_i$, then the single-strain equilibrium $\mathcal{E}_i$ is unstable.*

Proof. To simplify the presentation, without loss of generality, we shall assume that $i = 1$, *i.e.*, $\mathcal{R}_0^1 > 1$, and $\mathcal{R}_0^i < \mathcal{R}_0^1$, for $i = 2, \cdots, n$. We linearize system (2.3) around the strain-one equilibrium $\mathcal{E}_1$. Let $S_v(t) = S_{v_1}^* + x_{v_1}(t)$, $I_{v_1}(t) = I_{v_1}^* + y_{v_1}(t)$, $S_h(t) = S_{h_1}^* + x_{h_1}(t)$, $I_{h_1}(t) = I_{h_1}^* + y_{h_1}(t)$, $I_{v_i}(t) = y_{v_i}(t)$, $I_{h_i}(t) = y_{h_i}(t)$, for $i = 2, \cdots, n$. We obtain the following linearized system

$$
\begin{aligned}
\frac{dx_{v_1}(t)}{dt} &= -\beta_{v_1} S_{v_1}^* \int_0^{\tau_h} g_{v_1}(\tau) y_{h_1}(t-\tau) d\tau - \beta_{v_1} I_{h_1}^* a_{v_1} x_{v_1}(t) - S_{v_1}^* \sum_{i=2}^{n} \int_0^{\tau_h} \beta_{v_i} g_{v_i}(\tau) y_{h_i}(t-\tau) d\tau \\
&\quad -\mu_v x_{v_1}(t), \\
\frac{dy_{v_1}(t)}{dt} &= \beta_{v_1} S_{v_1}^* \int_0^{\tau_h} g_{v_1}(\tau) y_{h_1}(t-\tau) d\tau + \beta_{v_1} I_{h_1}^* a_{v_1} x_{v_1}(t) - \mu_v y_{v_1}(t), \\
\frac{dy_{v_i}(t)}{dt} &= \beta_{v_i} S_{v_1}^* \int_0^{\tau_h} g_{v_i}(\tau) y_{h_i}(t-\tau) d\tau - \mu_v y_{v_i}(t), \ i = 2, \cdots, n, \\
\frac{dx_{h_1}(t)}{dt} &= -\beta_{h_1} S_{h_1}^* \int_0^{\tau_v} g_{h_1}(\tau) y_{v_1}(t-\tau) d\tau - \beta_{h_1} I_{v_1}^* a_{h_1} x_{h_1}(t) - S_{h_1}^* \sum_{i=2}^{n} \int_0^{\tau_v} \beta_{h_i} g_{h_i}(\tau) y_{v_i}(t-\tau) d\tau \\
&\quad -\mu_h x_{h_1}(t), \\
\frac{dy_{h_1}(t)}{dt} &= \beta_{h_1} S_{h_1}^* \int_0^{\tau_v} g_{h_1}(\tau) y_{v_1}(t-\tau) d\tau + \beta_{h_1} I_{h_1}^* a_{h_1} x_{h_1}(t) - (\mu_h + \alpha_{h_1}) y_{h_1}(t), \\
\frac{dy_{h_i}(t)}{dt} &= \beta_{h_i} S_{h_1}^* \int_0^{\tau_v} g_{h_i}(\tau) y_{v_i}(t-\tau) d\tau - (\mu_h + \alpha_{h_i}) y_{h_i}(t), \ i = 2, \cdots, n.
\end{aligned}
\tag{3.6}
$$

We notice that the linearized equations for strains $k = 2, 3, \cdots, n$ can separate from the whole system. Thus, by direct calculation, we obtain the following characteristic equations

$$
(\lambda + \mu_h + \alpha_{h_i})(\lambda + \mu_v) - \beta_{v_i} \beta_{h_i} S_{h_1}^* S_{v_1}^* \int_0^{\tau_h} g_{v_i}(\tau) e^{-\lambda\tau} d\tau \int_0^{\tau_v} g_{h_i}(\tau) e^{-\lambda\tau} d\tau = 0. \tag{3.7}
$$

Similar to proof in Lemma 3.1, it is easy to show that if $\mathcal{R}_0^i < R_0^1, i = 2, 3, \cdots, n$, Eq (3.7) has no purely imaginary roots and all eigenvalues of these equations have negative real parts.

Therefore, the stability of $\mathcal{E}_1$ depends on the eigenvalue of the following system

$$
\begin{cases}
\lambda x_{v_1} = -\beta_{v_1} S_{v_1}^* y_{h_1} \int_0^{\tau_h} g_{v_1}(\tau) e^{-\lambda\tau} d\tau - \beta_{v_1} a_{v_1} I_{h_1}^* x_{v_1} - \mu_v x_{v_1} \\
\lambda y_{v_1} = \beta_{v_1} S_{v_1}^* y_{h_1} \int_0^{\tau_h} g_{v_1}(\tau) e^{-\lambda\tau} d\tau + \beta_{v_1} a_{v_1} I_{h_1}^* x_{v_1} - \mu_v y_{v_1}, \\
\lambda x_{h_1} = -\beta_{h_1} S_{h_1}^* y_{v_1} \int_0^{\tau_v} g_{h_1}(\tau) e^{-\lambda\tau} d\tau - \beta_{h_1} a_{h_1} I_{v_1}^* x_{h_1} - \mu_v x_{h_1}, \\
\lambda \bar{y}_{h_1} = \beta_{h_1} S_{h_1}^* y_{v_1} \int_0^{\tau_v} g_{h_1}(\tau) e^{-\lambda\tau} d\tau + \beta_{h_1} a_{h_1} I_{v_1}^* x_{h_1} - (\mu_h + \alpha_{h_1}) y_{h_1}.
\end{cases}
\tag{3.8}
$$

By looking for exponential solutions of (3.8), we obtain the following the characteristic equation

$$\begin{aligned}&(\lambda+\mu_h+\alpha_{h_1})(\lambda+\mu_v+a_{v_1}\beta_{v_1}I^*_{h_1})(\lambda+\mu_h+a_{h_1}\beta_{h_1}I^*_{v_1})\\&-(\lambda+\mu_h)\beta_{h_1}S^*_{h_1}\int_0^{\tau_v}g_{h_1}(\tau)e^{-\lambda\tau}d\tau\beta_{v_1}S^*_{v_1}\int_0^{\tau_h}g_{v_1}(\tau)e^{-\lambda\tau}d\tau=0.\end{aligned}\tag{3.9}$$

Now we show that Eq (3.9) has no purely imaginary roots. In fact, we divide both sides by $(\lambda+\mu_h)$, then we set

$$\begin{aligned}LHS&\overset{def}{=}\frac{(\lambda+a_{v_1}\beta_{v_1}I^*_h+\mu_v)(\lambda+\mu_h+a_{h_1}\beta_{h_1}I^*_v)(\lambda+\mu_h+\alpha_{h_1})}{(\lambda+\mu_h)}\\RHS&\overset{def}{=}\beta_{h_1}\beta_{v_1}S^*_{h_1}\left(\frac{\Lambda_v}{\mu_v}-I^*_{v_1}\right)\int_0^{\tau_v}e^{-\lambda\tau}g_{h_1}(\tau)d\tau\int_0^{\tau_h}e^{-\lambda\tau}g_{v_1}(\tau)d\tau.\end{aligned}\tag{3.10}$$

If λ is a root with $\Re\lambda\geq 0$, it follows from (3.10) that

$$|LHS|\geq(a_{v_1}\beta_{v_1}I^*_{h_1}+\mu_v)(\alpha_{h_1}+\mu_h),\quad |RHS|\leq a_{h_1}a_{v_1}\beta_{h_1}\beta_{v_1}S^*_{h_1}(\frac{\Lambda_v}{\mu_v}-I^*_v)=a_{h_1}a_{v_1}\beta_{h_1}\beta_{v_1}S^*_{h_1}S^*_{v_1}.$$

Using the equilibrium point$(S^*_{v_1},I^*_{v_1},S^*_{h_1},I^*_{v_1})$ satisfies the corresponding equation, we obtain

$$\begin{aligned}|LHS|\geq(a_{v_1}\beta_{v_1}I^*_{h_1}+\mu_v)(\alpha_{h_1}+\mu_h)&=(a_{v_1}\beta_{v_1}I^*_{h_1}+\mu_v)\frac{a_{h_1}\beta_{h_1}S^*_{h_1}I^*_{v_1}}{I^*_{h_1}}\\&=(a_{v_1}\beta_{v_1}I^*_{h_1}+\mu_v)\frac{a_{h_1}\beta_{h_1}S^*_{h_1}}{I^*_{h_1}}\frac{a_{v_1}\beta_{v_1}S^*_{v_1}I^*_{h_1}}{\mu_v}\\&>a_{h_1}a_{v_1}\beta_{h_1}\beta_{v_1}S^*_{h_1}S^*_{v_1}\geq|RHS|.\end{aligned}$$

It is a contradiction. Hence, using Rouche's Theorem [4, Theorem 9.17.4], we only show that Eq. (3.9) can not have any roots with positive real part. Therefore, the strain-one equilibrium $\mathcal{E}_1$ is locally asymptotically stable for $\mathcal{R}^1_0>1$ and $\mathcal{R}^i_0<\mathcal{R}^1_0, i=2,3,\cdots,n$. Otherwise $\mathcal{E}_1$ is unstable.

□

4 Global stability of the disease-free equilibrium

In the previous section, we have shown that equilibria are locally stable, *i.e.* given the conditions on the parameters, if the initial conditions are close enough to the equilibrium, the solution will converge to that equilibrium. In this section, our objective is to extend these results to global results. Now we first establish the global stability of the disease-free equilibrium. We have the following result:

Theorem 4.1 *If $\mathcal{R}_0=\max\{\mathcal{R}^1_0,\cdots,\mathcal{R}^n_0\}<1$, the disease-free equilibrium $\mathcal{E}_0$ is globally asymptotical stable.*

Proof From the first and the third equations of system (2.3), we obtain

$$S'_v(t)\leq\Lambda_v-\mu_vS_v(t),\quad S'_h(t)\leq\Lambda_h-\mu_hS_h(t).\tag{4.1}$$

It follows from the above inequalities that

$$\limsup_t S_v(t)\leq\frac{\Lambda_v}{\mu_v},\quad \limsup_t S_h(t)\leq\frac{\Lambda_h}{\mu_h}.\tag{4.2}$$

Thus, from system (2.3) and inequalities (4.2), we have

$$\begin{aligned}
\frac{dI_{v_i}(t)}{dt} &\leq \beta_{v_i}\frac{\Lambda_v}{\mu_v}\int_0^{\tau_h} g_{v_i}I_{h_i}(t-\tau)d\tau - \mu_v I_{v_i}(t),\\
\frac{dI_{h_i}(t)}{dt} &\leq \beta_{h_i}\frac{\Lambda_h}{\mu_h}\int_0^{\tau_v} g_{h_i}I_{v_i}(t-\tau)d\tau - (\mu_h+\alpha_{h_i})I_{h_i}(t), \quad i=1,\cdots,n.
\end{aligned} \tag{4.3}$$

From (4.3), we obtain that

$$I_{v_i}(t) \leq e^{-\mu_v t}I_{v_i}(0) + \beta_{v_i}\frac{\Lambda_v}{\mu_v}\int_0^t e^{-\mu s}\int_0^{\tau_h} g_{v_i}(\tau)I_{h_i}(s-\tau)d\tau ds.$$

Thus, from the above expression, for $i=1,\cdots,n$, we have

$$\limsup_t I_{v_i}(t) \leq \beta_{v_i}a_{v_i}\frac{\Lambda_v}{\mu_v}\int_0^\infty e^{-\mu_v s}ds \ \limsup_t I_{h_i}(t) \leq \beta_{v_i}a_{v_i}\frac{\Lambda_v}{\mu_v^2}\limsup_t I_{h_i}(t). \tag{4.4}$$

Similarly, for $i=1,\cdots,n$ we also have

$$\limsup_t I_{h_i}(t) \leq a_{h_i}\beta_{h_i}\frac{\Lambda_h}{\mu_h}\int_0^\infty e^{-(\mu_h+\alpha_{h_i})s}ds \ \limsup_t I_{v_i}(t) \leq a_{h_i}\beta_{h_i}\frac{\Lambda_h}{\mu_h}\frac{1}{\mu_h+\alpha_{h_i}}\limsup_t I_{v_i}(t). \tag{4.5}$$

From (4.4) and (4.5), we have

$$\limsup_t I_{v_i}(t) \leq a_{v_i}a_{h_i}\beta_{v_i}\frac{\Lambda_v}{\mu_v^2}\beta_{h_i}\frac{\Lambda_h}{\mu_h}\frac{1}{\mu_h+\alpha_{h_i}}\limsup_t I_{v_i}(t) \leq \mathcal{R}_0\limsup_t I_{v_i}(t). \tag{4.6}$$

Since $\mathcal{R}_0 < 1$ and $I_{v_i}(t), i=1,\cdots,n$ is bounded, it follows from the above expression that

$$\limsup_t I_{v_i}(t) = 0, \ i=1,\cdots,n.$$

Similarly, from (4.5), we have

$$\limsup_t I_{h_i}(t) = 0, \ i=1,\cdots,n.$$

Therefore, $(I_{v_i}(t), I_{h_i}(t)) \to (0,0)$ as $t\to\infty$. From the above discussion and Theorem 2.1, we can conclude that the disease-free equilibrium E_0 is globally asymptotically stable for $\mathcal{R}_0 < 1$. This completes the proof of Theorem 4.1. □

5 Principle of competitive exclusion

For directly transmitted diseases, Bremermann and Thieme [2] establish that in the simplest scenario when competitive exclusion is the only outcome, the strain with the largest reproduction number persists, while the remaining strains die out. While extending the Bremermann and Thieme [2] result to vector-borne diseases remains an open problem (*in the case of vector-borne diseases, we seem to need a stronger condition for competitive exclusion*), we were able to show competitive exclusion under the somewhat stronger condition that the winning strain maximizes both the human and the vector reproduction numbers, not just the overall reproduction number.

Mathematically speaking, establishing the competitive exclusion principle means establishing the global stability of the strain one equilibrium $\mathcal{E}_1$. From Theorem 2.2, we obtain that under some conditions the equilibrium $\mathcal{E}_1$ is locally asymptotically stable. In this section, we are ready to

establish the global stability of the strain one equilibrium $\mathcal{E}_1$ by constructing Lyapunov functional, similar to the one used in [9,10,13,14,18,20,21,24,25,31,32]. We note that for constructing Lyapunov function, such as $f(x) = x - 1 - \ln x$, here $f(x)$ is not defined if $x = 0$. Thus, in order to construct our Lyapunov function, we must first show that when $\mathcal{R}_0 > 1$, the disease persist in the form of strain one. Now we apply the persistence theory by Hale and Waltmann [12] for infinite-dimensional systems to show the uniform persistence of strain one in system (2.3). The methods and techniques have been recently employed by other authors (see [26,29,33]).

To proceed, we introduce the following notation and terminology. Let X be a complete metric space with metric d. Suppose that T is a continuous semiflow on X, that is, a continuous mapping $T : [0, \infty) \times X \to X$ with the following properties:

$$T_t \circ T_s = T_{t+s}, t, s > 0, T_0(x) = x, x \in X,$$

where T_t denotes the mapping from X to X given by $T_t(x) = T(t, x)$. The distance $d(x, Y)$ of a point $x \in X$ to a subset Y of X is defined by

$$d(x, Y) = \inf_{y \in Y} d(x, y).$$

Recall that the positive orbit $\gamma^+(x)$ through x is defined as $\gamma^+(x) = \cup_{t>0}\{T(t)x\}$, and its $\omega-$limit set is $\omega(x) = \cap_{s\geq 0}\overline{\cup_{t\geq s}\{T(t)x\}}$. Define by $W^s(E)$ the strong stable set of a compact invariant set E as $W^s(E) = \{x : x \in X, \omega(x) \neq \phi, \omega(x) \subset E\}$. Denote by $T(t), t \geq 0$, the family of solution operators corresponding to (2.3). The following assumption must be satisfied

(H1): Let $X_0 \subset X, X^0 \subset X, X_0 \cap X^0 = \emptyset$. X_0 is open and dense in X. Moreover, the C_0-semigroup $T(t)$ on X satisfies

$$T(t) : X_0 \to X_0,$$

$$T(t) : X^0 \to X^0.$$

Let $T_b(t) = T(t)|X_0$ and A_b be the global attractor for $T_b(t)$. The following result is discussed in [12, Theorem 4.2]:

Lemma 5.1 *Suppose that $T(t)$ satisfies $(H1)$ and the following conditions:*

(i) There is a $t_0 > 0$ such that $T(t)$ is compact for $t > t_0$;

(ii) $T(t)$ is point dissipative in X;

(iii) $\widetilde{A_b} = \bigcap_{x\in A_b} \Omega(x)$ is isolated and has an acyclic covering $\widetilde{M}$, where

$$\widetilde{M} = \{M_1, M_2, \cdots, M_n\};$$

(iv) $W^s(M_i) \cap X_0 = \emptyset$ for $i = 1, 2, \cdots, n$.

Then $T(t)$ is a uniform repeller with respect to X_0, i.e., there is an $\eta > 0$ such that for any $x \in X^0$, $\liminf_{t\to+\infty} d(T(t), X_0) \geq \eta$.

We use the above lemma to establish the following result:

Theorem 5.1 *Assume that $\mathcal{R}_0 > 1$. Then the disease perisist in the population. If $\mathcal{R}_0^1 > 1$ and strains two to n approach zero, that is*

$$\limsup_{t\to\infty} I_{h_i}(t) = 0, \quad \limsup_{t\to\infty} I_{v_i}(t) = 0, \quad i = 2, 3, \cdots, n.$$

then, for all initial conditions that belong to Ω_0, strain one can persists. In particular, there exists a constant $\gamma > 0$ such that

$$\lim_{t\to+\infty}\inf I_{v_1}(t) > \gamma, \quad \lim_{t\to+\infty}\inf I_{h_1}(t) > \gamma.$$

Proof. Under the assumptions of Theorem 5.1, system has two nonnegative equilibrium $\mathcal{E}_0$ and $\mathcal{E}_1$. Let ϕ be a vector of initial conditions for system (2.1). We assume the function in the vector ϕ are continuous. Let

$$\hat{X} = \mathbb{R}^+ \times \prod_{i=1}^{n}(C([-h,0],\mathbb{R}^+) \times \mathbb{R}^+ \times \prod_{i=1}^{n}(C([-h,0],\mathbb{R}^+),$$

where, the Banach space $C([-h,0],\mathbb{R}^+)$ of continuous functions mapping the interval $[-h,0]$ into $\mathbb{R}^+$ equipped with the sup-norm $||\psi|| = \sup_{-h\le\theta\le 0}|\psi(\theta)|$. Set

$$\begin{aligned} X^0 &= \{\phi \in \hat{X} : \psi_{v_1}(\theta) > 0, \psi_{h_1}(\theta) > 0, \psi_{v_i}(\theta) = 0, \psi_{h_i}(\theta) = 0, \theta < 0, i = 2, \cdots n\}, \\ X_0 &= \{\phi \in \hat{X} : \psi_{v_i}(\theta) = 0, \psi_{h_i}(\theta) = 0, \theta \le 0, i = 1, \cdots n.\} \end{aligned}$$

Obviously, we have $X = X^0 \bigcup X_0$. It suffices to show that there exists an $\eta > 0$ such that for any solution u_t of system (2.3) initiating from X^0, $\liminf_{t\to\infty} d(U_t, X_0) \ge \eta$. To this end, we have to verity below that the conditions of Lemma 5.1 are satisfied.

First, it follows that both X and X^0 are positively invariant. Clearly, ∂X_0 is relatively closed in X. It is easy to verify that system (2.3) is point dissipative. Set

$$M_\partial = \{\phi \in X : T(t)\phi \text{ satisfies system (2.3) and } \ T(t)\phi \in \partial X, \forall t \ge 0.\}$$

We now claim that $M_\partial = \{(\frac{\Lambda_v}{\mu_v}, \ \mathbf{0}, \frac{\Lambda_h}{\mu_h}, \mathbf{0})\}$, where $\mathbf{0} = (0, \cdots, 0)$ is a $n-$ dimensional vector of zeroes. Assuming $\phi \in M_\partial$ it suffices to show that $I_{v_i}(t) = I_{h_i}(t) = 0, \ i = 1, \cdots, n, \ \forall t \ge 0$. Assume that on the contrary, there exists $t_0 > 0$ such that *(i)* for some strain-i_0, we have $I_{v_{i_0}}(t_0) > 0$ while $I_{h_i}(t_0) = 0, i = 1, \cdots, n$ or *(ii)* for some strain-j_0, we have $I_{h_{j_0}}(t_0) > 0$ while $I_{v_i}(t_0) = 0, i = 1, \cdots, n$

In case (i), from the third equation of (2.3), we have

$$\dot{I}_{h_{i_0}}(t)|_{t=t_0} = \beta_{h_{i_0}} S_h(t_0) \int_0^{\tau_v} g_{h_{i_0}}(\tau) I_{v_{i_0}}(t_0 - \tau) d\tau > 0.$$

Hence, there is a sufficiently small constant ε_0 such that $I_{h_{i_0}}(t) > 0, \forall t \in (t_0, t_0 + \varepsilon_0)$. On the other hand, from $I_{v_{i_0}}(t_0) > 0$, we obtain a positive ε_1 $(0 < \varepsilon_1 < \varepsilon_0)$ such that $I_{h_{i_0}}(t) > 0, \forall t \in (t_0, t_0 + \varepsilon_1)$. Thus, we obtain $I_{v_{i_0}}(t) > 0, I_{h_{i_0}}(t) > 0, \forall \in (t_0, t_0 + \varepsilon)$. This is in contradiction with the assumption that $(S_v(t), I_{v_i}(t), S_h(t), I_{h_i}(t)) \in M_\partial, \forall t \ge 0$. Similarly, we can show the case (ii) does not hold.

Let $\Omega_2 = \bigcup_{x\in Y_2} \omega(x)$, where Y_2 is the global attractor of $T(t)$ restricted to ∂X. We now show that $\Omega_2 = \{\mathcal{E}_0\}$. In fact, it follows from $\Omega_2 \subseteq M_\partial$ and the first and third equations of (2.3). we have $\lim_{t\to+\infty} S_v(t) = \frac{\Lambda_v}{\mu_v}$, $\lim_{t\to+\infty} S_v(t) = \frac{\Lambda_v}{\mu_v}$. Thus, $\{\mathcal{E}_0\}$ is the isolated invariant set in X.

Finally we need to show that $W^s(\mathcal{E}_0) \bigcap X^0 = \emptyset$.

Assume that on contrary. Thus, there exists a solution $U_t \in X^0$ such that

$$\begin{aligned} &\lim_{t\to+\infty} S_v(t) = \frac{\Lambda_v}{\mu_v}, \ \lim_{t\to+\infty} S_h(t) = \frac{\Lambda_h}{\mu_h}, \\ &\lim_{t\to+\infty} I_{v_i}(t) = 0, \quad \lim_{t\to+\infty} I_{h_i}(t) = 0. \ i = 1, \cdots, n. \end{aligned}$$

Hence, for each $\varepsilon > 0$ there exists a time T_0 such that for each $t > T_0$ we have

$$I_{v_i}(t) < \varepsilon, \quad I_{h_i}(t) < \varepsilon. \quad i = 1, \cdots, n.$$

By shifting the dynamical system, we may assume that the above inequalities are satisfied for $t \geq 0$. From the first and third equations of (2.3) and taking into account the above inequalities, we have

$$\frac{dS_v(t)}{dt} \geq \Lambda_v - \sum_{i=1}^{n} \beta_{v_i} a_{v_i} \varepsilon S_v(t) - \mu_v S_v(t), \quad \frac{dS_h(t)}{dt} \geq \Lambda_h - \sum_{i=1}^{n} \beta_{h_i} a_{h_i} S_h(t) - \mu_v S_h(t). \tag{5.1}$$

Therefore, from (5.1), we have

$$\limsup_{t\to\infty} S_v(t) \geq \liminf_{t\to\infty} S_v(t) \geq \frac{\Lambda_v}{\mu_v + \sum_{i=1}^{n} \beta_{v_i} a_{v_i} \varepsilon}$$

$$\limsup_{t\to\infty} S_h(t) \geq \liminf_{t\to\infty} S_h(t) \geq \frac{\Lambda_h}{\mu_h + \sum_{i=1}^{n} \beta_{h_i} a_{h_i} \varepsilon}.$$

Thus, there exists a $T_1 > T_0$ such that, for any $t > T_1$, and using the second equation of system (2.3) and the inequality above, we have

$$\frac{dI_{v_i}(t)}{dt} \geq \frac{\beta_{v_i} \Lambda_v}{\mu_v + \sum_{i=1}^{n} \beta_{v_i} a_{v_i} \varepsilon} \int_0^{\tau_h} g_{v_i}(\tau) I_{h_i}(t-\tau) d\tau - \mu_v I_{v_i}(t). \tag{5.2}$$

Similarly, there exists a $T_2 > T_0$, such that for any $t \geq T_2$, we have

$$\frac{dI_{h_i}(t)}{dt} \geq \frac{\beta_{h_i} \Lambda_h}{\mu_h + \sum_{i=1}^{n} \beta_{h_i} a_{h_i} \varepsilon} \int_0^{\tau_v} g_{h_i}(\tau) I_{v_i}(t-\tau) d\tau - (\mu_h + \alpha_{h_i}) I_{h_i}(t). \tag{5.3}$$

By the mean value theorem for integrals, we know that for any time t, there is a ξ_t such that

$$\int_0^{\tau_h} g_{v_i}(\tau) I_{h_i}(t-\tau) d\tau = a_{v_i} I_{h_i}(\xi_t), \quad \text{for} \quad t - h < \xi_t < t. \tag{5.4}$$

Therefore, from (5.2)-(5.4), we obtain

$$\begin{aligned} \frac{dI_{v_i}(t)}{dt} &\geq \frac{a_{v_i} \beta_{v_i} \Lambda_v}{\mu_v + \sum_{i=1}^{n} \beta_{v_i} a_{v_i} \varepsilon} I_{h_i}(\xi_t) - \mu_v I_{v_i}(t), \\ \frac{dI_{h_i}(t)}{dt} &\geq \frac{\beta_{h_i} \Lambda_h}{\mu_h + \sum_{i=1}^{n} \beta_{h_i} a_{h_i} \varepsilon} \int_0^{\tau_v} g_{h_i}(\tau) I_{v_i}(t-\tau) d\tau - (\mu_h + \alpha_{h_i}) I_{h_i}(t). \end{aligned} \tag{5.5}$$

Notice that $\mathcal{R}_1 = \dfrac{a_{h_1} a_{v_1} \beta_{h_1} \beta_{v_1} \Lambda_h \Lambda_v}{(\mu_h + \alpha_{h_1}) \mu_h \mu_v^2} > 1$. Thus, we may assume we have chosen sufficiently small ε in such a way that we have

$$\frac{a_{h_1} a_{v_1} \beta_{h_1} \beta_{v_1} \Lambda_h \Lambda_v}{(\mu_h + \alpha_{h_1})(\mu_h + \sum_{i=1}^{n} \beta_{h_i} a_{h_i} \varepsilon)(\mu_v + \sum_{i=1}^{n} \beta_{v_i} a_{v_i} \varepsilon) \mu_v} > 1. \tag{5.6}$$

Since $(I_{v_i}(t), I_{h_i}(t)) \to (0,0), i = 1, \cdots, n$ as $t \to \infty$, then by a standard comparison argument, the solution $(x_{v_i}(t), y_{h_i}(t))$ of the following equation

$$\begin{aligned} \frac{dx_{v_i}(t)}{dt} &= a_{v_i} \beta_{v_i} \frac{\Lambda_v}{\mu_v + \sum_{i=1}^{n} \beta_{v_i} a_{v_i} \varepsilon} y_{h_i}(\xi_t) - \mu_v x_{v_i}(t), \\ \frac{dy_{h_i}(t)}{dt} &= \beta_{h_i} \frac{\Lambda_h}{\mu_h + \sum_{i=1}^{n} \beta_{h_i} a_{h_i} \varepsilon} \int_0^{\tau_v} g_{h_i}(\tau) x_{v_i}(t-\tau) d\tau - (\mu_h + \alpha_{h_i}) y_{h_i}(t), \ i = 1, \cdots, n. \end{aligned} \tag{5.7}$$

with initial conditions $x_{v0}(T_1) = I_{v_i}(T_1),\ y_{n0}(T_2) = I_{h_i}(T_2)$ has to converge to $(0,0)$ as well for inequality (5.6). On the other hand, for $i = 1, \cdots, n$, if we set,

$$W_i(t) = \frac{\beta_{h_i}\Lambda_h}{\mu_v(\mu_h + \sum_{i=1}^n \beta_{h_i} a_{h_i}\varepsilon)} \int_0^{\tau_v} g_{h_i}(\tau) x_{v_i}(t-\tau)d\tau + y_{h_i}(t) + (\mu_h + \alpha_{h_i}) \int_{\xi_t}^t y_{h_i}(s)ds, \quad (5.8)$$

Obviously, it follows from the solution $(x_{v_i}(t), y_{h_i}(t)) \to (0,0)$ that $W_i(t) \to 0$ as $t \to \infty$. In particular, we have $W_1(t) \to 0$ as $t \to \infty$. However, by directly calculating the derivative in (5.8), we obtain

$$\frac{dW_i(t)}{dt} = \left[\frac{a_{h_i}a_{v_i}\beta_{h_i}\beta_{v_i}\Lambda_h\Lambda_v}{(\mu_h + \sum_{i=1}^n \beta_{h_i}a_{h_i}\varepsilon)(\mu_v + \sum_{i=1}^n \beta_{v_i}a_{v_i}\varepsilon)} - (\mu_h + \alpha_{h_i})\right] y(\xi_t). \quad (5.9)$$

Thus, for the above expression (5.9), if we consider $i = 1$ and use (5.6), then we have $W_1(t) \to \infty$ as $t \to \infty$. This is a contradiction.

Therefore, we have $W^s(\mathcal{E}_0) \bigcap X^0 = \emptyset$. This completes the proof of Theorem 5.1. □

Now we are able to state our main result.

Theorem 5.2 *Assume that $\mathcal{R}_1 > 1$, $\mathcal{R}_h^j < \mathcal{R}_h^1$ and $\mathcal{R}_v^j < \mathcal{R}_v^1$ for $j = 2, \cdots, n$. Then, equilibrium $\mathcal{E}_1$ is globally asymptotically stable.*

Proof. Because of the complexity of the expressions, we define the Lyapunov functional in components and take the derivative of each component separately. Set $f(x) = x - 1 - \ln x,\ x > 0$. The components of the Lyapunov's functional are defined as follows:

$$\begin{aligned}
&W_{S_v}(t) = a_{h_1} f\left(\frac{S_v(t)}{S_{v_1}^*}\right), \quad W_{I_{v_1}}(t) = a_{h_1} f\left(\frac{I_{v1(t)}}{I_{v_1}^*}\right),\\
&W_{+I_{v_1}}(t) = a_{h_1}\int_0^{\tau_h} \phi_1(\tau) f\left(\frac{I_{h_1}(t-\tau)}{I_{h_1}^*}\right)d\tau,\\
&W_{+I_{v_i}}(t) = a_{h_1}\int_0^{\tau_h} \phi_j(\tau) I_{h_i}(t-\tau)d\tau, \quad i = 2, \cdots, n,
\end{aligned} \quad (5.10)$$

where, $\phi_j(\tau) = \int_\tau^{\tau_h} g_{v_j}(\sigma)d\sigma,\ j = 1, \cdots, n.$

We calculate the derivative of the above expressions in (5.10) along system (2.3). We obtain

$$\begin{aligned}
\frac{dW_{S_v}(t)}{dt} &= a_{h_1}\frac{S_v(t) - S_{v_1}^*}{S_v(t)}\frac{dS_v(t)}{S_{v_1}^* dt}\\
&= a_{h_1}\frac{S_v(t) - S_{v_1}^*}{S_v(t)S_{v_1}^*}\left[\Lambda_v - \sum_{i=1}^n \beta_{v_i}S_v(t)\int_0^{\tau_h} g_{v_i}(\tau)I_{h_i}(t-\tau)d\tau - \mu_v S_v(t)\right]\\
&= a_{h_1}\left(\frac{1}{S_{v_1}^*} - \frac{1}{S_v(t)}\right)\left[\beta_{v_1}S_{v_1}^* I_{h_1}^* a_{v_1} + \mu_v S_{v_1}^* - \sum_{i=1}^n \int_0^{\tau_h} \beta_{v_i}S_v(t)g_{v_i}(\tau)I_{h_i}(t-\tau)d\tau - \mu_v S_v\right]\\
&= -\frac{a_{h_1}\mu_v(S_v(t) - S_{v_1}^*)^2}{S_v(t)S_{v_1}^*} + \beta_{v_1}I_{h_1}^* a_{h_1}\int_0^{\tau_h} g_{v_1}(\tau)\left(1 - \frac{S_{v_1}^*}{S_v(t)}\right)\left(1 - \frac{S_v(t)I_{h_1}(t-\tau)}{S_{v_1}^* I_{h_1}^*}\right)d\tau\\
&\quad -a_{h_1}\sum_{i=2}^n \int_0^{\tau_h} \beta_{v_i}g_{v_i}(\tau)\left(1 - \frac{S_{v_1}^*}{S_v(t)}\right)\frac{S_v(t)I_{h_i}(t-\tau)}{S_{v_1}^*}d\tau.
\end{aligned}$$

The above equations can reduced to the following

$$\begin{aligned}\frac{dW_{S_v}(t)}{dt} &= -\frac{\mu_v a_{h_1}(S_v(t)-S_{v_1}^*)^2}{S_v(t)S_{v_1}^*} + \beta_{v_1} I_{h_1}^* a_{h_1}\int_0^{\tau_h} g_{v_1}(\tau)\Big(1-\frac{S_v(t)}{S_{v_1}^*}\frac{I_{h_1}(t-\tau)}{I_{h_1}^*} \\ &\quad -\frac{S_{v_1}^*}{S_v(t)}+\frac{I_{h_1}(t-\tau)}{I_{h_1}^*}\Big)d\tau - a_{h_1}\sum_{i=2}^{n}\int_0^{\tau_h}\beta_{v_i}g_{v_i}(\tau)\Big(\frac{S_v(t)I_{h_i}(t-\tau)}{S_{v_1}^*}-I_{h_i}(t-\tau)\Big)d\tau.\end{aligned} \tag{5.11}$$

$$\begin{aligned}\frac{dW_{I_{v_1}}(t)}{dt} &= \frac{a_{h_1}(I_{v_1}(t)-I_{v_1}^*)}{I_{v_1}(t)}\frac{dI_{v_1}(t)}{I_{v_1}^* dt} \\ &= \frac{a_{h_1}(I_{v_1}(t)-I_{v_1}^*)}{I_{v_1}(t)I_{v_1}^*}\Big[\beta_{v_1}S_v(t)\int_0^{\tau_h} g_{v_1}(\tau)I_{h_1}(t-\tau)d\tau - \mu_v I_{v_1}(t)\Big] \\ &= \frac{a_{h_1}\beta_{v_1}S_{v_1}^* I_{h_1}^*}{I_{v_1}^*}\int_0^{\tau_h} g_{v_1}(\tau)\Big(1+\frac{S_v(t)I_{h_1}(t-\tau)}{S_{v_1}^* I_{h_1}^*}-\frac{I_{v_1}(t)}{I_{v_1}^*}-\frac{S_v(t)I_{h_1}(t-\tau)}{S_{v_1}^* I_{h_1}^*}\frac{I_{v_1}^*}{I_{v_1}(t)}\Big)d\tau.\end{aligned} \tag{5.12}$$

Differentiating the time derivative of $W_{+I_{v_1}}(t)$ with respective to t, we obtain

$$\begin{aligned}\frac{dW_{+I_{v_1}}(t)}{dt} &= a_{h_1}\frac{d}{dt}\int_0^{\tau_h}\phi_1(\tau)f\Big(\frac{I_{h_1}(t-\tau)}{I_{h_1}^*}\Big)d\tau \\ &= -a_{h_1}\int_0^{\tau_h}\phi_1(\tau)df\Big(\frac{I_{h_1}(t-\tau)}{I_{h_1}^*}\Big) \\ &= -a_{h_1}\phi_1(\tau)f\Big(\frac{I_{h_1}(t-\tau)}{I_{h_1}^*}\Big)\Big|_0^{\tau_h} + a_{h_1}\int_0^{\tau_h} f\Big(\frac{I_{h_1}(t-\tau)}{I_{h_1}^*}\Big)d\phi_1(\tau).\end{aligned}$$

Notice that $\lim_{\tau\to h}\phi_1(\tau)=0$ and $f\Big(\frac{I_{h_1}(t-\tau)}{I_{h_1}^*}\Big)$ is bounded. It follows that $\lim_{\tau\to h}\phi_1(\tau)\Big(\frac{I_{h_1}(t-\tau)}{I_{h_1}^*}\Big)=0$. Moreover, $\frac{d\phi_1(\tau)}{d\tau}=-g_{v_1}(\tau)$. Hence, we have

$$\begin{aligned}\frac{dW_{+I_{v_1}}(t)}{dt} &= a_{h_1}\phi_1(0)f\Big(\frac{I_{h_1}(t)}{I_{h_1}^*}\Big) - a_{h_1}\int_0^{\tau_h} g_{v_1}(\tau)f\Big(\frac{I_{h_1}(t-\tau)}{I_{h_1}^*}\Big)d\tau \\ &= a_{h_1}\int_0^{\tau_h} g_{v_1}(\tau)\Big(f\Big(\frac{I_{h_1}(t)}{I_{h_1}^*}\Big)-f\Big(\frac{I_{h_1}(t-\tau)}{I_{h_1}^*}\Big)\Big)d\tau.\end{aligned} \tag{5.13}$$

Similarly, we have

$$\frac{dW_{+I_{v_i}}(t)}{dt} = a_{h_1}\int_0^{\tau_h} g_{v_i}(\tau)(I_{h_i}(t)-I_{h_i}(t-\tau))d\tau, \quad i=2,\cdots n. \tag{5.14}$$

Set

$$\begin{aligned}W_v(t) &= \frac{1}{\beta_{v_1}I_{h_1}^*}f\Big(\frac{S_v(t)}{S_{v_1}^*}\Big)+\frac{I_{v_1}^*}{\beta_{v_1}S_{v_1}^*I_{h_1}^*}f\Big(\frac{I_{v_1}(t)}{I_{v_1}^*}\Big)+\frac{1}{\beta_{v_1}S_{v_1}^*I_{h_1}^*}\sum_{i=2}^{n}I_{v_i}(t)+W_{+I_{v_1}}(t) \\ &\quad +\sum_{i=2}^{n}\frac{\beta_{v_i}}{\beta_{v_1}I_{h_1}^*}W_{+I_{v_i}}(t).\end{aligned} \tag{5.15}$$

Thus, by (5.11)-(5.14), we obtain

$$
\begin{aligned}
\frac{dW_v(t)}{dt} = & -\frac{a_{h_1}\mu_v(S_v(t)-S_{v_1}^*)^2}{S_v(t)S_{v_1}^*\beta_{v_1}I_{h_1}^*} + a_{h_1}\int_0^{\tau_h} g_{v_1}(\tau)\Big(1-\frac{S_v(t)}{S_{v_1}^*}\frac{I_{h_1}(t-\tau)}{I_{h_1}^*}-\frac{S_{v_1}^*}{S_v(t)}+\frac{I_{h_1}(t-\tau)}{I_{h_1}^*}\Big)d\tau \\
& +\frac{a_{h_1}}{\beta_{v_1}I_{h_1}^*}\sum_{i=2}^{n}\int_0^{\tau_h}\beta_{v_i}g_{v_i}(\tau)\Big(I_{h_i}(t-\tau)-\frac{S_v(t)I_{h_i}(t-\tau)}{S_{v_1}^*}\Big)d\tau \\
& +a_{h_1}\int_0^{\tau_h} g_{v_1}(\tau)\Big(1+\frac{S_v(t)I_{h_1}(t-\tau)}{S_{v_1}^*I_{h_1}^*}-\frac{I_{v_1}(t)}{I_{v_1}^*}-\frac{S_v(t)I_{h_1}(t-\tau)}{S_{v_1}^*I_{h_1}^*}\frac{I_{v_1}^*}{I_{v_1}(t)}\Big)d\tau \\
& +\sum_{i=2}^{n}\Big(\frac{a_{h_1}\beta_{v_i}S_v(t)}{\beta_{v_1}S_{v_1}^*I_{h_1}^*}\int_0^{\tau_h} g_{v_i}(\tau)I_{h_i}(t-\tau)d\tau-\frac{\mu_v a_{h_1}}{\beta_{v_1}S_{v_1}^*I_{h_1}^*}I_{v_i}(t)\Big) \\
& +\int_0^{\tau_h} a_{h_1}g_{v_1}(\tau)\Big(\frac{I_{h_1}(t)}{I_{h_1}^*}-\ln\frac{I_{h_1}(t)}{I_{h_1}^*}-\frac{I_{h_1}(t-\tau)}{I_{h_1}^*}+\ln\frac{I_{h_1}(t-\tau)}{I_{h_1}^*}\Big)d\tau \\
& +\sum_{i=2}^{n}\frac{a_{h_1}\beta_{v_i}}{\beta_{v_1}I_{h_1}^*}\int_0^{\tau_h} g_{v_i}(\tau)(I_{h_i}(t)-I_{h_i}(t-\tau))d\tau \\
= & -\frac{a_{h_1}\mu_v(S_v(t)-S_{v_1}^*)^2}{S_v(t)S_{v_1}^*\beta_{v_1}I_{h_1}^*}+\int_0^{\tau_h} a_{h_1}g_{v_1}(\tau)\Big(2-\frac{S_{v_1}^*}{S_v(t)}-\frac{I_{v_1}(t)}{I_{v_1}^*}-\frac{S_v(t)I_{v_1}^*}{S_{v_1}^*I_{v_1}}\frac{I_{h_1}(t-\tau)}{I_{h_1}^*}+\frac{I_{h_1}(t)}{I_{h_1}^*} \\
& -\ln\frac{I_{h_1}(t)}{I_{h_1}^*}+\ln\frac{I_{h_1}(t-\tau)}{I_{h_1}^*}\Big)d\tau+\sum_{i=2}^{n}\frac{a_{h_1}\beta_{v_i}}{\beta_{v_1}I_{h_1}^*}\int_0^{\tau_h} g_{v_i}(\tau)I_{h_i}(t)d\tau-\sum_{i=2}^{n}\frac{a_{h_1}\mu_v}{\beta_{v_1}I_{h_1}^*S_{v_1}^*}I_{v_i}(t).
\end{aligned}
\tag{5.16}
$$

Similarly, set

$$
\begin{aligned}
& W_{S_h}(t)=a_{v_1}f\Big(\frac{S_h(t)}{S_{h_1}^*}\Big),\ W_{+I_{h_1}}(t)=\int_0^{\tau_v} a_{v_1}\varphi_1(\tau)f\Big(\frac{I_{v_1}(t-\tau)}{I_{v_1}^*}\Big)d\tau, \\
& W_{I_{h_1}}(t)=a_{v_1}f\Big(\frac{I_{h_1}(t)}{I_{h_1}^*}\Big),\quad W_{+I_{h_i}}(t)=\int_0^{\tau_v} a_{v_1}\varphi_j(\tau)I_{v_i}(t-\tau)d\tau,\quad i=2,\cdots,n,
\end{aligned}
\tag{5.17}
$$

where, $\varphi_j(\tau)=\int_{\tau}^{\tau_v} g_{h_j}(\sigma)d\sigma,\ j=1,\cdots,n.$

Calculating the derivative of the expressions in (5.17), respectively, along system (2.3), we obtain

$$
\begin{aligned}
\frac{dW_{S_h}(t)}{dt} & =\frac{a_{v_1}(S_h(t)-S_{h_1}^*)}{S_h(t)}\frac{dS_h(t)}{S_{h_1}^*dt} \\
& =-\frac{\mu_h a_{v_1}(S_h(t)-S_{h_1}^*)^2}{S_h(t)S_{h_1}^*}+a_{v_1}\beta_{h_1}I_{v_1}^*\int_0^{\tau_v} g_{h_1}(\tau)\Big(1-\frac{S_h(t)}{S_{h_1}^*}\frac{I_{v_1}(t-\tau)}{I_{v_1}^*}-\frac{S_{h_1}^*}{S_h(t)}+\frac{I_{v_1}(t-\tau)}{I_{v_1}^*}\Big)d\tau \\
& \quad -\sum_{i=2}^{n}\int_0^{\tau_v} a_{v_1}\beta_{h_i}g_{h_i}(\tau)\Big(\frac{S_h(t)I_{v_i}(t-\tau)}{S_{h_1}^*}-I_{v_i}(t-\tau)\Big)d\tau.
\end{aligned}
\tag{5.18}
$$

$$
\begin{aligned}
\frac{dW_{I_{h_1}}(t)}{dt} &= \frac{a_{v_1}(I_{h_1}(t)-I^*_{h_1})}{I_{h_1}(t)}\frac{dI_{h_1}(t)}{I^*_{h_1}dt} \\
&= \frac{a_{v_1}(I_{h_1}(t)-I^*_{h_1})}{I^*_{h_1}I_{h_1}(t)}[\beta_{h_1}S_h(t)\int_0^{\tau_v} g_{h_1}(\tau)I_{v_1}(t-\tau)d\tau-(\mu_h+\alpha_{h_1})I_{h_1}(t)] \\
&= \frac{a_{v_1}(I_{h_1}(t)-I^*_{h_1})}{I^*_{h_1}I_{h_1}(t)}[\beta_{h_1}S_h(t)\int_0^{\tau_v} g_{h_1}(\tau)I_{v_1}(t-\tau)d\tau-\frac{\beta_{h_1}S^*_{h_1}I^*_{v_1}a_{h_1}}{I^*_{h_1}}I_{h_1}(t)] \\
&= \frac{\beta_{h_1}S^*_{h_1}I^*_{v_1}a_{v_1}}{I^*_{h_1}}\int_0^{\tau_v} g_{h_1}(\tau)\left(1+\frac{S_h(t)I_{v_1}(t-\tau)}{S^*_{h_1}I^*_{v_1}}-\frac{I_{h_1}(t)}{I^*_{h_1}}-\frac{S_h(t)I_{v_1}(t-\tau)}{S^*_{h_1}I^*_{v_1}}\frac{I^*_{h_1}}{I_{h_1}(t)}\right)d\tau. \\
\frac{dW_{+I_{h_1}}(t)}{dt} &= \int_0^{\tau_v} a_{v_1}g_{h_1}(\tau)\left(f\left(\frac{I_{v_1}(t)}{I^*_{v_1}}\right)-f\left(\frac{I_{v_1}(t-\tau)}{I^*_{v_1}}\right)\right)d\tau \\
&= \int_0^{\tau_v} a_{v_1}g_{h_1}(\tau)\left(\frac{I_{v_1}(t)}{I^*_{v_1}}-\ln\frac{I_{v_1}(t)}{I^*_{v_1}}-\frac{I_{v_1}(t-\tau)}{I^*_{v_1}}+\ln\frac{I_{v_1}(t-\tau)}{I^*_{v_1}}\right)d\tau. \\
\frac{dW_{+I_{h_i}}(t)}{dt} &= \int_0^{\tau_v} a_{v_1}g_{h_i}(\tau)(I_{v_i}(t)-I_{v_i}(t-\tau))d\tau, \quad i=2,\cdots,n.
\end{aligned}
\tag{5.19}
$$

Set

$$
\begin{aligned}
W_h(t) &= \frac{1}{\beta_{h_1}I^*_{v_1}}f\left(\frac{S_h(t)}{S^*_{h_1}}\right)+\frac{I^*_{h_1}}{\beta_{h_1}S^*_{h_1}I^*_{v_1}}f\left(\frac{I_{h_1}(t)}{I^*_{h_1}}\right)+\sum_{i=2}^{n}\frac{1}{\beta_{h_1}S^*_{h_1}I^*_{v_1}}I_{h_i}(t)+W_{+I_{h_1}}(t) \\
&\quad+\sum_{i=2}^{n}\frac{\beta_{h_i}}{\beta_{h_1}I^*_{v_1}}W_{+I_{h_i}}(t).
\end{aligned}
\tag{5.20}
$$

By (5.18)-(5.20) and the system (2.3), we obtain

$$
\begin{aligned}
\frac{dW_h(t)}{dt} &= -\frac{a_{v_1}\mu_h(S_h(t)-S^*_{h_1})^2}{\beta_{h_1}S^*_{h_1}I^*_{v_1}S_h(t)}+\int_0^{\tau_v} a_{v_1}g_{h_1}(\tau)\left(2-\frac{S^*_{h_1}}{S_h(t)}-\frac{I_{h_1}(t)}{I^*_{h_1}}-\frac{S_h(t)I^*_{h_1}}{S^*_hI_{h_1}}\frac{I_{v_1}(t-\tau)}{I^*_{v_1}}+\frac{I_{v_1}(t)}{I^*_{v_1}}\right. \\
&\quad\left.-\ln\frac{I_{v_1}(t)}{I^*_{v_1}}+\ln\frac{I_{v_1}(t-\tau)}{I^*_{v_1}}\right)d\tau+\sum_{i=2}^{n}\frac{a_{v_1}\beta_{h_i}}{\beta_{h_1}I^*_{v_1}}\int_0^{\tau_v} g_{h_i}(\tau)I_{v_i}(t)d\tau-\sum_{i=2}^{n}\frac{a_{v_1}(\mu_h+\alpha_{h_i})}{\beta_{h_1}S^*_{h_1}I^*_{v_1}}I_{h_i}(t).
\end{aligned}
\tag{5.21}
$$

Set

$$
W(t)=W_v(t)+W_h(t).
$$

From (5.16) and (5.21), we obtain

$$
\begin{aligned}
\frac{dW(t)}{dt} &= -\frac{\mu_va_{h_1}(S_v(t)-S^*_{v_1})^2}{\beta_{v_1}S^*_{v_1}I^*_{h_1}S_v(t)}-\frac{\mu_ha_{v_1}(S_h(t)-S^*_{h_1})^2}{\beta_{h_1}S^*_{h_1}I^*_{v_1}S_h(t)}+\int_0^{\tau_h} a_{h_1}g_{v_1}(\tau)\left(2-\frac{S^*_{v_1}}{S_v(t)}-\frac{I_{v_1}(t)}{I^*_{v_1}}\right. \\
&\quad\left.-\frac{S_v(t)I^*_{v_1}}{S^*_{v_1}I_{v_1}}\frac{I_{h_1}(t-\tau)}{I^*_{h_1}}+\frac{I_{h_1}(t)}{I^*_{h_1}}-\ln\frac{I_{h_1}(t)}{I^*_{h_1}}+\ln\frac{I_{h_1}(t-\tau)}{I^*_{h_1}}\right)d\tau+\int_0^{\tau_v} a_{v_1}g_{h_1}(\tau)\left(2-\frac{S^*_{h_1}}{S_h(t)}\right. \\
&\quad\left.-\frac{I_{h_1}(t)}{I^*_{h_1}}-\frac{S_{h_1}(t)I^*_{h_1}}{S^*_{h_1}I_{h_1}}\frac{I_{v_1}(t-\tau)}{I^*_{v_1}}+\frac{I_{v_1}(t)}{I^*_{v_1}}-\ln\frac{I_{v_1}(t)}{I^*_{v_1}}+\ln\frac{I_{v_1}(t-\tau)}{I^*_{v_1}}\right)d\tau \\
&\quad+\sum_{i=2}^{n}\frac{a_{h_1}\beta_{h_i}}{\beta_{h_1}I^*_{v_1}}\int_0^{\tau_v} g_{h_i}(\tau)I_{v_i}(t)d\tau-\sum_{i=2}^{n}\frac{a_{v_1}(\mu_h+\alpha_{h_i})}{\beta_{h_1}S^*_{h_1}I^*_{v_1}}I_{h_i}(t)+\sum_{i=2}^{n}\frac{a_{v_1}\beta_{v_i}}{\beta_{v_1}I^*_{h_1}}\int_0^{\tau_h} g_{v_i}(\tau)I_{h_i}(t)d\tau \\
&\quad-\sum_{i=2}^{n}\frac{a_{h_1}\mu_v}{\beta_{v_1}I^*_{h_1}S^*_{v_1}}I_{v_i}(t).
\end{aligned}
\tag{5.22}
$$

It follows from equation (5.22) that

$$\begin{aligned}\frac{dW(t)}{dt} &= -\frac{\mu_v a_{h_1}(S_v(t)-S^*_{v_1})^2}{\beta_{v_1}S^*_{v_1}I^*_{h_1}S_v(t)} - \frac{\mu_h a_{v_1}(S_h(t)-S^*_{h_1})^2}{\beta_{h_1}S^*_{h_1}I^*_{v_1}S_h(t)} - \int_0^{\tau_h} a_{h_1}g_{v_1}(\tau)\Big[\frac{S^*_{v_1}}{S_v(t)} - 1 \\ &\quad - \ln\frac{S^*_{v_1}}{S_v(t)} + \frac{S_v(t)I^*_{v_1}}{S^*_{v_1}I_{v_1}}\frac{I_{h_1}(t-\tau)}{I^*_{h_1}} - 1 - \ln\frac{S_v(t)I^*_{v_1}}{S^*_{v_1}I_{v_1}}\frac{I_{h_1}(t-\tau)}{I^*_{h_1}} + \frac{I_{h_1}(t)}{I^*_{h_1}} - \frac{I_{v_1}(t)}{I^*_{v_1}}\Big]d\tau \\ &\quad + \int_0^{\tau_h} a_{h_1}g_{v_1}(\tau)\Big(\ln\frac{I_{h_1}(t)}{I^*_{h_1}} - \ln\frac{I^*_{v_1}}{I_{v_1}(t)}\Big)d\tau - \int_0^{\tau_v} a_{v_1}g_{h_1}(\tau)\Big[\frac{S^*_{h_1}}{S_h(t)} - 1 \\ &\quad - \ln\frac{S^*_{h_1}}{S_h(t)} + \frac{S_h(t)I_{v_1}(t-\tau)I^*_{h_1}}{S^*_{h_1}I^*_{v_1}I_{h_1}} - 1 - \ln\frac{S_h(t)I_{v_1}(t-\tau)I^*_{h_1}}{S^*_{h_1}I^*_{v_1}I_{h_1}}\Big]d\tau \\ &\quad + \int_0^{\tau_v} a_{v_1}g_{h_1}(\tau)\Big(\frac{I_{v_1}(t)}{I^*_{v_1}} - \frac{I_{h_1}(t)}{I^*_{h_1}} + \ln\frac{I_{v_1}(t)}{I^*_{v_1}} - \ln\frac{I^*_{h_1}}{I_{h_1}(t)}\Big)d\tau \\ &\quad + \sum_{i=2}^{n}\Big(\frac{a_{v_1}a_{v_i}\beta_{v_i}}{\beta_{v_1}I^*_{h_1}} - \frac{a^2_{v_1}(\mu_h+\alpha_{h_i})}{(\mu_h+\alpha_{h_1})I^*_{h_1}}\Big)I_{h_i}(t) + \sum_{i=2}^{n}\frac{a^2_{h_1}}{I^*_{v_1}}\Big(\frac{a_{h_i}\beta_{h_i}}{a_{h_1}\beta_{h_1}} - 1\Big)I_{v_i}(t).\end{aligned} \tag{5.23}$$

Using the equality $\int_0^{\tau_v} g_{h_1}(\tau)d\tau = a_{h_1}, \int_0^{\tau_h} g_{v_1}(\tau)\ d\tau = a_{v_1}$, we obtain

$$\begin{aligned}&\Big(\int_0^{\tau_h} a_{h_1}g_{v_1}(\tau)d\tau - \int_0^{\tau_v} a_{v_1}g_{h_1}(\tau)\Big)\Big(\frac{I_{h_1}(t)}{I^*_{h_1}} - \frac{I_{v_1}(t)}{I^*_{v_1}}\Big) = 0, \\ &\Big(\int_0^{\tau_h} a_{h_1}g_{v_1}(\tau) - \int_0^{\tau_v} a_{v_1}g_{h_1}(\tau)d\tau\Big)\Big(\ln\frac{I_{h_1}(t)}{I^*_{h_1}} - \ln\frac{I^*_{v_1}}{I_{v_1}(t)}\Big) = 0.\end{aligned} \tag{5.24}$$

Using $f(x) = x - 1 - \ln x,\ x > 0$ and the expression of $\mathcal{R}^i_h$ and $\mathcal{R}^i_v$, from (5.23) we obtain

$$\begin{aligned}\frac{dW(t)}{dt} &= -\frac{\mu_v a_{h_1}(S_v(t)-S^*_{v_1})^2}{\beta_{v_1}S^*_{v_1}I^*_{h_1}S_v(t)} - \frac{\mu_h a_{v_1}(S_h(t)-S^*_{h_1})^2}{\beta_{h_1}S^*_{h_1}I^*_{v_1}S_h(t)} - \int_0^{h} a_{h_1}g_{v_1}(\tau)\Big[f\Big(\frac{S^*_{v_1}}{S_v(t)}\Big) \\ &\quad + f\Big(\frac{S_{v_1}(t)I^*_{v_1}}{S^*_{v_1}I_{v_1}}\frac{I_{h_1}(t-\tau)}{I^*_{h_1}}\Big)\Big]d\tau - \int_0^{h} a_{v_1}g_{h_1}(\tau)\Big[f\Big(\frac{S^*_{h_1}}{S_h(t)}\Big) + f\Big(\frac{S_h(t)I^*_{h_1}}{S^*_{h_1}I_{h_1}}\frac{I_{v_1}(t-\tau)}{I^*_{v_1}}\Big)\Big]d\tau \\ &\quad + \sum_{i=2}^{n}\frac{a^2_{h_1}}{I^*_{v_1}}\Big(\frac{\mathcal{R}^i_v}{\mathcal{R}^1_v} - 1\Big)I_{v_i}(t) + \sum_{i=2}^{n}\frac{a^2_{v_1}(\mu_h+\alpha_{h_i})}{(\mu_h+\alpha_{h_1})I^*_{h_1}}\Big(\frac{\mathcal{R}^i_h}{\mathcal{R}^1_h} - 1\Big)I_{h_i}(t).\end{aligned}$$

Since $f(x) \geq 0$ for $x > 0$, $\mathcal{R}^i_h < R^1_h$ and $\mathcal{R}^i_v < R^1_v,\ i \neq 1$ we have $\frac{dW(t)}{dt} \leq 0$. Let

$$\bar{\Omega} = \{(S_v, I_{v_1}, \cdots, I_{v_n}, S_h, I_{h_1}, \cdots, I_{h_n}) \in X | \frac{W(t)}{dt} = 0\}.$$

It is easy to verify that $\frac{dW(t)}{dt} = 0$ if and only if $S_v(t) = S^*_{v_1},\ I_{v_1}(t) = S^*_{v_1},\ S_h(t) = S^*_{h_1}, I_{h_1}(t) = I^*_{h_1},\ I_{v_i}(t) = 0,\ I_{h_i}(t) = 0,\ i = 2, \cdots n$. Hence, the largest invariant set in $\bar{\Omega}$ is the singleton $\{\mathcal{E}_1\}$. By LaSalle invariance principle and Theorem 3.2, we show that equilibrium $\mathcal{E}_1$ is globally asymptotically stable. □

6 Discussion

In this paper we introduce a multi-strain model of a vector-borne disease with distributed delay in the vector and the host. We define the basic reproduction number of the disease as the maximum

of the reproduction numbers of each strain. We show that if $\mathcal{R}_0 < 1$ the disease-free equilibrium is locally and globally stable, that is the number of infected with each strain goes to zero. Furthermore, we show that if $\mathcal{R}_0 > 1$, then the disease persists.

Our main objective with the article was to extend the competitive exclusion result established by Bremermann and Thieme [2]. This principle states that in the basic ODE model when multiple strains compete only the strain with the largest reproduction number persists, if its reproduction number is larger than one, and the other strains die out. In this case coexistence is not possible outside of the trivial scenario when the reproduction numbers of two different strains are equal.

To prove the competitive exclusion result for the multi-strain vector-borne model with distributed delay, we need to prove global stability of the a single-strain equilibrium. We approached the problem using a Lyapunov function. However, we were able to establish only the following weaker result: It is well known that the reproduction number in vector-borne diseases splits into a vector reproduction number and host reproduction number, namely $\mathcal{R}_0^i = \mathcal{R}_h^i \mathcal{R}_v^i$. Our result states the the strain that maximizes both the vector reproduction number and the human reproduction number will dominate in the population and eliminate the rest. In particular, if we assume that $\mathcal{R}_h^i < \mathcal{R}_h^1$ and $\mathcal{R}_v^i < \mathcal{R}_v^1$ for $i = 2, \ldots, n$ then strain one will dominate and eliminate the rest. In this case coexistence does not occur. In fact, it can be shown that model (2.3) does not have a coexistence equilibrium if all strain reproduction numbers are different.

Our result does not resolve the competition outcome in the case when different strains maximize the human and the vector reproduction numbers. In this sense the question whether complete competitive exclusion holds in the vector-host model is still an open question. We surmise that because coexistence equilibrium is not present, competitive exclusion in this case also occurs with the strain with maximal reproduction number eliminating the rest. Establishing this result, however, may need a different approach.

From the prospective of public health, public health efforts will work best if directed to monitoring the vector and host reproduction numbers of the multiple strains in vector-borne diseases. Control measures may need to be applied to the strain(s) that maximize the vector and the host reproduction numbers.

Acknowledgements The authors wish to thank J.M.Cushing (Editor-in-Chief), the handling editor and the anonymous referees for their useful feedback and helpful comments and suggestions which have improved the paper.

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