Multi-strain persistence induced by host age structure

Zhipeng Qiu[1*] Xuezhi Li[2†] Maia Martcheva[3‡]

[1]*Department of Applied Mathematics, Nanjing University of Science and Technology, Nanjing, 210094, P R China*

[2]*Department of Mathematics, Xinyang Normal University, Xinyang, 464000, P R China*

[3]*Department of Mathematics, University of Florida, Gainesville, FL 32611-8105,USA*

Abstract. In this paper, we formulate an age-structured epidemic model with two competing strains. The model incorporates disease-induced mortality so that the population can not be assumed to be in a stationary demographic state. We derive explicit expressions of the basic and invasion reproduction numbers for strain one and two, respectively. Analytical results of the model show that the existence and local stability of boundary equilibria can be determined by the reproduction numbers to some extent. Subsequently, under the condition that both invasion reproduction numbers are larger than one, the coexistence of two competing strains is rigorously proved by the theory of uniform persistence of infinite dimensional dynamical systems. However, the results for the corresponding age-independent model show that the two competing strains can not coexist. This implies that age structure can lead to the coexistence of the strains. Numerical simulations are further conducted to confirm and extend the analytic results.

Key Words. Epidemic model with two competing strain; age structure; coexistence; reproduction number; competition.

Mathematics subject classifications(2000): 92D30, 92D25, 35L60

1 Introduction

Epidemiological studies have shown that the susceptibility to many diseases often varies with age. For example, children of school age usually exhibit higher susceptibility to influenza than adults [1]. Host age structure is often an important factor which affects the dynamics of disease transmission. In this article we incorporate age structure into an epidemic model and study its impact on the competition of two strains.

During the past decades, the age-structured epidemic models have been extensively studied by many authors (see, for example, [2, 3, 4, 5] and references therein). These studies have enriched

*nustqzp@mail.njust.edu.cn

†xzli66@sina.com

‡maia@ufl.edu

our knowledge of epidemic models with age-structure. However, most of the models neglect the disease-induced death rate so that the population can be assumed to be in a stationary demographic state. It is this assumption that simplifies significantly the analysis of the models. Actually, the disease-induced mortality is inevitable component of many diseases, especially deadly infectious diseases, such as pandemic influenza [6], West Nile virus[7], dengue virus[8], HIV and others. Investigating the impact of the disease-induced mortality on the dynamics of diseases, particularly multi-strain diseases is a neglected activity of significant importance.

Few papers have investigated the impact of host age and disease-induced mortality on the dynamics of multi-strain interactions. In reality, however, many of the diseases which are still significant public health problem, are caused by more than one antigenically different strains of the causative agent [9]. These diseases include killer-diseases such as malaria, tuberculosis and HIV/AIDS. Therefore, it is necessary to study age-structured epidemic model with multiple strains. In this paper we formulate an age-structured model with two competing strains, and we incorporate the disease-induced mortality into the model. Then we study the dynamics of the model, and further investigate how the age-structured affects the interactions between two competing strains.

The dynamics of the pathogen-host interactions involving multiple strains has fascinated researchers for a long time [9]. The competitive exclusion principle is a classic result in this field, which states that no two species can indefinitely occupy the same ecological niche [10]. Using a multiple-strain ODE epidemic model, Bremermann and Thieme [11] proved that the principle of competitive exclusion is valid under the assumption that infection with one strain precludes additional infections with the other strains. However, it is a common phenomenon that multiple strains coexist in nature. For instance, dengue fever has four different serotypes, often coexisting in the same geographical region [9]. It follows from the competitive exclusion principle that there must be some heterogeneity in the ecological niche. Identifying the factors that allow multiple strains to coexist is an important topic in theoretical biology. Studies have pointed out to a number of mechanisms, such as superinfection [8, 12], co-infection [13, 16], partial cross-immunity [17], density dependent host mortality [18], that can lead to coexistence of strains. In a recent article Martcheva *et. al.* [14] suggested that host age-structure coupled with disease-induced mortality may lead to coexistence of competing pathogens. This result was obtained numerically. In this article we investigate a similar scenario but we prove rigorously that if both invasion reproduction numbers are larger than one the two strains will persist in the age-structured model. To our knowledge this is the first result on persistence of multiple strains in a partial differential equation model. Persistence of a single pathogen has been addressed in multiple articles and it follows from the fact that the reproduction number is greater than one [15].

This paper is organized as follows. In the next section, we introduce our age-structured epidemic model with two competing strains. In Section 3 we consider the age independent case. In Section 4, we discuss the existence of equilibria for the age-structured model and investigate the local stability of boundary equilibria. Section 5 is devoted to deriving sufficient conditions for the persistence of the two competing strains. In section 6, we present several numerical simulations which support and extend our theoretical results. In the conclusions (section 7) we discuss our results.

2 Model formulation

In this section, we formulate an age structured epidemic model with two competing strains. We start with a typical Gurtin-MacCamy model which describes the dynamics of an age-structured population without disease.

Let $n(a,t)$ denote the age density of the total population at age a and time t. Without the disease, we always assume that $n(a,t)$ is described by the following Gurtin-MacCamy equation [19]:

$$\begin{cases} \dfrac{\partial n(a,t)}{\partial t} + \dfrac{\partial n(a,t)}{\partial a} = -\mu(a)n(a,t), \\ n(a,0) = n_0(a), \\ n(0,t) = R_0^d \Phi(Q(t)) \displaystyle\int_0^{a^+} \beta(a)n(a,t)da. \end{cases} \tag{2.1}$$

In (2.1), $n_0(a)$ is a given function. Furthermore, a^+ represents the maximum life span of the individuals; $\mu(a)$ is the age-specific death rate, which satisfies $\mu(\cdot) \in L^1_{loc}(0,a^+), \mu(a) \geq 0$ in $(0,a^+)$ and

$$\int_0^{a^+} \mu(\sigma) = +\infty;$$

$\beta(a)$ is the age-specific per capita birth rate, with $\beta(\cdot) \in L^\infty(0,a^+), \beta(a) \geq 0$ in $(0,a^+)$ and $\beta(a)$ normalized to satisfy

$$\int_0^{a^+} \beta(\sigma)\pi(\sigma)d\sigma = 1. \tag{2.2}$$

In the expression above $\pi(\sigma)$ is the survival probability which gives the probability at birth of surviving to age σ. The probability of survival is defined as $\pi(\sigma) = e^{-\int_0^\sigma \mu(\tau)d\tau}, \sigma \in (0,a^+)$. $Q(t)$ is a weighted average of the population density defined as

$$Q(t) = \int_0^{a^+} r(\sigma)n(\sigma,t)d\sigma,$$

where $r(\sigma)$ is a weight kernel and $r(\cdot) \in L^\infty(0,a^+), r(\sigma) \geq 0, \sigma \in (0,a^+)$. Furthermore, $\Phi(x)$ is a function describing density-dependence of births and satisfying the following properties:

(P1) $\Phi(x)$ is continuously differentiable;

(P2) $\Phi(x)$ is strictly decreasing;

(P3) $\Phi(0) = 1, \lim_{x\to+\infty} \Phi(x) = 0$.

Finally, R_0^d is the demographic basic reproduction number.

Now we are able to formulate our model. In order to formulate an age-structured epidemic model with two competing strains, we need to introduce some additional notation. We consider an age-structured SI epidemic model so that the population $n(a,t)$ is divided into three classes: susceptible, infected with strain one, and infected with strain two. Let $s(a,t), i_1(a,t)$ and $i_2(a,t)$ denote the associated density functions with these respective epidemiological age-structured classes, then we have

$$n(a,t) = s(a,t) + i_1(a,t) + i_2(a,t).$$

We assume that all newborns are susceptible and the extra-mortality at age a due to the infection of strain j is $\gamma_j(a), j = 1,2$. We take the transmission rate $\lambda_j(a,t), j = 1,2$ in the separable inter-cohort constitutive form for the force of infection generated by $i_j(a,t)$:

$$\lambda_j(a,t) = K_j(a)\int_0^{a^+} q_j(\sigma)i_j(\sigma,t)d\sigma, \quad j = 1,2,$$

where $q_j(a)$ is the age-specific infectiousness for strain j, $K_j(a)$ is the age-specific susceptibility of susceptible individuals, and $q_j(a), K_j(a) \geq 0$ in $[0,a^+], q_j(\cdot), K_j(\cdot) \in L^\infty(0,a^+)$.

Based on the above assumption, the joint dynamics of the age-structured epidemiological model are governed by the following partial differential equations

$$\begin{cases} \dfrac{\partial s(a,t)}{\partial t} + \dfrac{\partial s(a,t)}{\partial a} = -(\mu(a) + \lambda_1(a,t) + \lambda_2(a,t))s(a,t), \\ \dfrac{\partial i_1(a,t)}{\partial t} + \dfrac{\partial i_1(a,t)}{\partial a} = \lambda_1(a,t)s(a,t) - (\mu(a) + \gamma_1(a))i_1(a,t), \\ \dfrac{\partial i_2(a,t)}{\partial t} + \dfrac{\partial i_2(a,t)}{\partial a} = \lambda_2(a,t)s(a,t) - (\mu(a) + \gamma_2(a))i_2(a,t), \end{cases} \tag{2.3}$$

with initial and boundary conditions

$$\begin{aligned} &s(0,t) = R_0^d\Phi(Q(t))\int_0^{a^+} \beta(a)n(a,t)da, \quad i_1(0,t) = i_2(0,t) = 0; \\ &s(a,0) = s_0(a), \quad i_1(a,0) = i_{10}(a), \quad i_2(a,0) = i_{20}(a). \end{aligned} \tag{2.4}$$

In the following sections we mainly analyze the dynamics of system (2.3) to investigate how the age-structure affects the interactions between the two competing strains. In order to study this question, we consider in the next section the dynamical properties of the age-independent model.

3 The age-independent case

In this section we present some results on model (2.3) for the age independent case. Here we always assume that $a^+ = \infty$. If the parameters $\mu(a), \beta(a), r(a), K_i(a), q_i(a), \gamma_i(a), j = 1, 2$ in the model (2.3) do not depend on age, i.e,

$$\mu(a) \equiv \mu, \beta(a) \equiv \beta, r(a) \equiv r, K_i(a) \equiv K_i, q_i(a) \equiv q_i, \gamma_i(a) \equiv \gamma_i, j = 1, 2,$$

then condition (2.2) implies that $\beta = \mu$, and without the disease, i.e., $i_1(a,t) = i_2(a,t) = 0$, the corresponding equation for the total population size $N(t)$, where

$$N(t) = \int_0^{a^+} n(a,t)da,$$

can be obtained by integrating (2.1) from $a = 0$ to a^+, using the initial condition, and $n(a^+, t) = 0$. The last condition simply means that no individual in the population can survive to age a^+. Then the total population $N(t)$ develops according to the equation

$$\frac{dN(t)}{dt} = R_0^d \Phi(rN(t))\beta N(t) - \mu N(t). \tag{3.1}$$

It follows from the properties (P1)-(P3) that if $R_0^d \leq 1$ all solutions of the system (3.1) converge towards zero, and if $R_0^d > 1$ all solutions of (3.1) converge towards $\frac{1}{r}\Phi^{-1}(\frac{1}{R_0^d})$. In fact, $\frac{1}{r}\Phi^{-1}(\frac{1}{R_0^d})$ is the environmental carrying capacity for the population.

Let

$$S(t) = \int_0^{a^+} s(a,t)da, I_1(t) = \int_0^{a^+} i_1(a,t)da, I_2(t) = \int_0^{a^+} i_2(a,t)da.$$

Then the corresponding equations for $S(t), I_1(t)$ and $I_2(t)$ can also be obtained by integrating (2.3) from $a = 0$ to a^+, and using the conditions (2.4) and $s(a^+,t) = i_1(a^+,t) = i_2(a^+,t) = 0$. The resulting equations are

$$\begin{cases} \dfrac{dS(t)}{dt} = R_0^d \Phi(rN(t))\beta N(t) - \mu S(t) - K_1 q_1 I_1(t)S(t) - K_2 q_2 I_2(t)S(t), \\ \dfrac{dI_1(t)}{dt} = K_1 q_1 S(t)I_1(t) - (\mu + \gamma_1)I_1(t), \\ \dfrac{dI_2(t)}{dt} = K_2 q_2 S(t)I_2(t) - (\mu + \gamma_2)I_2(t). \end{cases} \tag{3.2}$$

It is easy to see that system (3.2) is the standard SI epidemic model with two strains. The system (3.2) has been analyzed in [11]. The reproduction number of system (3.2) for strain $j, j = 1, 2$ is established in paper [11], which can be expressed as

$$\tilde{\mathcal{R}}_j = \frac{K_j q_j}{\mu + \gamma_j}\frac{1}{r}\Phi^{-1}(\frac{1}{R_0^d}).$$

By the results in [11], we can obtain the following theorem.

Theorem 3.1. *The disease-free equilibrium* $E_0 = (\frac{1}{r}\Phi^{-1}(\frac{1}{R_0^d}), 0, 0)$ *of system* (3.2) *is globally asymptotically stable if* $\tilde{\mathcal{R}}_1 \le 1$ *and* $\tilde{\mathcal{R}}_2 \le 1$*. If* $\tilde{\mathcal{R}}_j > 1$ *for at least one* $j \in \{1, 2\}$*, then the strain with the larger reproduction number uniformly persist and the other one dies out. Coexistence of the two strains is no possible except for the degenerate case* $\tilde{\mathcal{R}}_1 = \tilde{\mathcal{R}}_2$*.*

Theorem 3.1 implies that if the system (2.3) does not depend on the age structure the competitive exclusion principle holds, i.e, two competing strains can not coexist. In the following sections we will prove that the coexistence of two competing strains is possible if the system depends on age structure. Accordingly, this indicates that the age-structure of the host is one of the mechanisms which can lead to the coexistence of two competing strains.

4 Boundary equilibria and local analysis

We now study the dynamics of the model (2.3) to investigate the impact of the age structure on strain competition and coexistence. We start with some notations and some results on the subsystem of model (2.3). Let

$$L_+^1(0, a^+) = \{\phi \in L^1(0, a^+) : \phi(a) \in R_+^1 \text{ for almost all } a \in (0, a^+)\};$$

$$\tilde{a} = \min\{a : \textstyle\int_a^{a^+} \beta(\sigma) d\sigma = 0\};$$

$$\beta_{\max} = ess \sup_{a \in (0, a^+)} \beta(a);$$

$$\beta_{\min} = ess \inf_{a \in (0, a^+)} \beta(a);$$

$$q_{j\min} = ess \inf_{a \in (0, a^+)} q_j(a);$$

$$K_{j\min} = ess \inf_{a \in (0, a^+)} K_j(a);$$

$$r_{\min} = ess \inf_{a \in (0, a^+)} r(a);$$

$$x \vee y := \max\{x, y\}, x, y \in R;$$

$$x \wedge y := \min\{x, y\}, x, y \in R.$$

The relation $\varphi \le \phi, \varphi, \phi \in L_+^1(0, a^+)$ indicates that $\varphi(a) \le \phi(a)$ for almost all $a \in (0, a^+)$. The biological interpretation of the variables requires us to consider the phase space of the model (2.3) to be the space

$$X_+ := \{(s_0(\cdot), i_{10}(\cdot), i_{20}(\cdot)) : s_0(\cdot) \in L_+^1(0, a^+), i_{j0}(\cdot) \in L_+^1(0, a^+), j = 1, 2\}.$$

By standard methods [20], it is possible to prove existence and uniqueness of solutions to system (2.3). Moreover, it is easy to show that all the solutions remain nonnegative and bounded for $t > 0$. Furthermore, in the following two sections, we always assume that

(H1) There exists $\tilde{a}_0 < \tilde{a}$ such that $\beta(a) > 0$ if $a \in (\tilde{a}_0, \tilde{a})$;

(H2) $K_j(a) > 0, q_j(a) > 0, j = 1, 2$ for all $a \in [0, a^+)$

The dynamics of the Gurtin-MacCamy system has been extensively studied by many authors [19, 21, 22]. In the paper, in order to ensure that the system (2.1) is dissipative we assume the following:

(H3) $r_{\min} > 0$ and there exists a constant $M > 0$ such that $\Phi(r_{\min}x)\beta_{\max}x < M$ for all $x \in [0, +\infty)$.

Then we have the following theorem:

Theorem 4.1. *1) If $R_0^d < 1$, then $\lim_{t\to+\infty} n(\cdot, t, n_0(a)) = 0$ for each $n_0(\cdot) \in L^1_+(0, a^+)$.*

2) Assume $R_0^d > 1$. Then

i) if $\int_0^{\tilde{a}} n_0(a)da = 0$ we have

$$\lim_{t\to+\infty} n(\cdot, t, n_0(a)) = 0;$$

ii) if (H1),(H3) hold, then there exists a constant $\epsilon > 0$, which does not depend on the initial conditions, such that every solution $n(a, t)$ with initial condition $n_0(\cdot) \in \Gamma_0$ satisfies

$$\liminf_{t\to+\infty} \int_0^{a^+} n(a, t, n_0(\cdot))da \geq \epsilon,$$

where

$$\Gamma_0 := \{n_0(\cdot) \in L^1_+(0, a^+) : \int_0^{\tilde{a}} n_0(a)da > 0\}.$$

The proof of Theorem 4.1 can be found in the Appendix. In the case $R_0^d < 1$, Theorem 4.1 tells us that the global behavior of system (2.1) is clear: the population dies out. So in the following we always assume that $R_0^d > 1$. When $R_0^d > 1$, it is easy to see that system (2.1) has a unique nontrivial equilibrium, given by

$$n^*(a) = \frac{\pi(a)}{\int_0^{a^+} r(\sigma)\pi(\sigma)d\sigma}\Phi^{-1}(\frac{1}{R_0^d}).$$

In order to make the mathematics tractable, we make the following additional assumption:

(H4) If $R_0^d > 1$, then the unique nontrivial steady state $n^*(a)$ of system (2.1) is globally asymptotically stable in Γ_0.

Now let us consider the dynamical properties of a reduced system of system (2.3) in which only one strain is present. The reduced system is governed by the equations:

$$\begin{cases} \dfrac{\partial s(a,t)}{\partial t} + \dfrac{\partial s(a,t)}{\partial a} = -(\mu(a) + \lambda_j(a,t))s(a,t), \\ \dfrac{\partial i_j(a,t)}{\partial t} + \dfrac{\partial i_j(a,t)}{\partial a} = \lambda_j(a,t)s(a,t) - (\mu(a) + \gamma_j(a))i_j(a,t), \\ s(0,t) = R_0^d \Phi(Q(t)) \displaystyle\int_0^{a^+} \beta(a)(s(a,t) + i_j(a,t))da, \\ i_j(0,t) = 0, s(a,0) = s_0(a), i_j(a,0) = i_{j0}(a). \end{cases} \tag{4.1}$$

The existence and local stability of endemic equilibria for the subsystem (4.1) has been analyzed in paper [23] where $\gamma_j(a)$ is constant, but the analysis method can be used to analyze the subsystem (4.1). As in paper [23], we define the basic reproduction number for strain $j, j = 1, 2$ as

$$\mathcal{R}_j = \int_0^{a+} K_j(a)n^*(a) \int_a^{a^+} q_j(\sigma)e^{-\int_a^\sigma (\mu(\theta)+\gamma_j(\theta))d\theta} d\sigma da, j = 1, 2.$$

Then we have

Theorem 4.2. *Let $R_0^d > 1$ and (H1)-(H4) hold. If $\mathcal{R}_j > 1$, the disease free equilibrium $\tilde{E}_0 := (n^*(a), 0)$ of system (4.1) is unstable, and*

1) if $\int_0^{\tilde{a}} s_0(a) + i_{j0}(a)da = 0$, then as $t \to +\infty$ we have

$$s(\cdot, t, (s_0(\cdot), i_{j0}(\cdot))) \to 0, \quad i_j(\cdot, t, (s_0(\cdot), i_{j0}(\cdot))) \to 0;$$

2) if $\int_0^{\tilde{a}} s_0(a) + i_{j0}(a)da > 0$ and $i_{j0}(a) \equiv 0$ for almost all $a \in (0, a^+)$, then as $t \to +\infty$ we have

$$s(\cdot, t, (s_0(\cdot), i_{j0}(\cdot))) \to n^*(\cdot), i_j(\cdot, t, (s_0(\cdot), i_{j0}(\cdot))) \to 0;$$

3) otherwise, $(s_0(\cdot), i_{j0}(\cdot)) \in \Gamma_j$ and there exists a constant $\delta > 0$ such that every solution $(s(a,t), i_j(a,t))$ with initial condition $(s_0(\cdot), i_{j0}(\cdot)) \in \Gamma_j$ satisfies

$$\liminf_{t\to\infty} \int_0^{a^+} i_j(a,t)da \geq \delta,$$

where

$$\Gamma_j := \{(s_0(\cdot), i_{j0}(\cdot)) \in L_+^1(0, a^+) \times L_+^1(0, a^+) : \\ \int_0^{a^+} i_{j0}(a)da > 0, \int_0^{\tilde{a}} (s_0(a) + i_{j0}(a))da > 0\}.$$

The first and second conclusions can be proved by using a similar way as in the proof of Theorem 4.1. Similar to the proof of Theorem 4.1 we can also prove the third conclusion of Theorem

4.2. Here, we omit the detailed proofs of the theorem, and they are left to the reader. It follows from the results in paper [23] that the system (4.1) has at least one positive equilibrium if $\mathcal{R}_j > 1$. In this paper we always assume that

(H5) If $\mathcal{R}_j > 1$, the system (4.1) has a unique positive equilibrium, denoted by $\tilde{E}_0 := (s_j^*(a), i_j^*(a))$, which is globally asymptotically stable in Γ_j.

It is easy to see that if $R_0^d > 1$ and $\mathcal{R}_j > 1, j = 1, 2$ the system (2.3) has four boundary equilibria which we label as $E_{00} = (0, 0, 0), E_0 = (n^*(a), 0, 0), E_1 = (s_1^*(a), i_1^*(a), 0), E_2 = (s_2^*(a), 0, i_2^*(a))$. In order to study the local stabilities of the boundary equilibria, let us introduce the invasion reproduction number for each of the strains. The invasion reproduction number of the strain $j, j = 1, 2$ measures the ability of the strain j to invade an equilibrium of the strain $k, i = 1, 2, k \neq j$. We define the invasion reproduction of the strain j as

$$\mathscr{R}_k^j = \int_0^{a+} K_j(a) s_k^*(a) \int_a^{a^+} q_j(\sigma) e^{-\int_a^\sigma (\mu(\theta)+\gamma_j(\theta))d\theta} d\sigma da, \quad j, k = 1, 2, j \neq k.$$

Then we have

Theorem 4.3. *Assume $R_0^d > 1, \mathcal{R}_j > 1, j = 1, 2$ and (H5) holds. Then*

(1) E_{00}, E_0 are always unstable;

(2) E_1 is locally stable if $\mathscr{R}_1^2 < 1$, and E_1 is unstable if $\mathscr{R}_1^2 > 1$;

(3) E_2 is locally stable if $\mathscr{R}_2^1 < 1$, and E_2 is unstable if $\mathscr{R}_2^1 > 1$.

Proof. The proof of the first conclusion is relatively simple, and it is left to the reader. In the following we need to prove the second and third conclusions. Here we only pick the second conclusion to prove, and the third conclusion can be proved in a similar way.

Linearizing the system (2.3) about E_1, by defining the perturbation variables

$$x_1(a, t) = s(a, t) - s_1^*(a), x_2(a, t) = i_1(a, t) - i_1^*(a), x_3(a, t) = i_2(a, t),$$

we obtain the following system

$$\left\{\begin{array}{rcl}
\dfrac{\partial x_1(a,t)}{\partial t}+\dfrac{\partial x_1(a,t)}{\partial a} & = & -\big(\mu(a)+K_1(a)\displaystyle\int_0^{a^+} q_1(\sigma)i_1^*(\sigma)d\sigma\big)x_1(a,t)+ \\
 & & \qquad s_1^*(a)\displaystyle\sum_{j=1}^{2}K_j(a)\int_0^{a^+} q_j(\sigma)x_{j+1}(\sigma,t)d\sigma, \\
\dfrac{\partial x_2(a,t)}{\partial t}+\dfrac{\partial x_2(a,t)}{\partial a} & = & K_1(a)\displaystyle\int_0^{a^+} q_1(\sigma)x_2(\sigma,t)d\sigma s_1^*(a)+K_1(a)\times \\
 & & \displaystyle\int_0^{a^+} q_1(\sigma)i_1^*(\sigma)d\sigma x_1(a,t)-(\mu(a)+\gamma_1(a))x_2(a,t), \\
\dfrac{\partial x_3(a,t)}{\partial t}+\dfrac{\partial x_3(a,t)}{\partial a} & = & K_2(a)\displaystyle\int_0^{a^+} q_2(\sigma)x_3(\sigma,t)d\sigma s_1^*(a)-(\mu(a)+\gamma_2(a))x_3(a,t), \\
x_1(0,t) & = & R_0^d\Phi(Q_1^*)\displaystyle\int_0^{a^+}\beta(\sigma)(x_1(\sigma,t)+x_2(\sigma,t)+x_3(\sigma,t))d\sigma \\
 & & +R_0^d\Phi'(Q_1^*)\displaystyle\int_0^{a^+}\beta(\sigma)(s_1^*(\sigma)+i_1^*(\sigma))d\sigma\times \\
 & & \displaystyle\int_0^{a^+} r(\sigma)(x_1(\sigma,t)+x_2(\sigma,t)+x_3(\sigma,t))d\sigma, \\
x_2(0,t) & = & 0, \\
x_3(0,t) & = & 0.
\end{array}\right. \tag{4.2}$$

where $Q_1^*=\int_0^{a^+} r(\sigma)(s_1^*(\sigma)+i_1^*(\sigma))d\sigma$.

Let

$$x_1(a,t)=x_1^0(a)e^{\lambda t},\quad x_2(a,t)=x_2^0(a)e^{\lambda t},\quad x_3(a,t)=x_3^0(a)e^{\lambda t}, \tag{4.3}$$

where $x_1^0(a), x_2^0(a), x_3^0(a)$ are to be determined. Substituting (4.3) into (4.2), we obtain

$$
\begin{cases}
\lambda x_1^0(a) + \dfrac{\partial x_1^0(a)}{\partial a} = -\big(\mu(a) + K_1(a)\displaystyle\int_0^{a^+} q_1(\sigma) i_1^*(\sigma) d\sigma\big) x_1^0(a) + \\
\qquad\qquad\qquad\qquad s_1^*(a)\displaystyle\sum_{j=1}^{2} K_j(a)\int_0^{a^+} q_j(\sigma) x_{j+1}^0(\sigma) d\sigma, \\
x_1^0(0) = R_0^d \Phi(Q_1^*)\displaystyle\int_0^{a^+} \beta(\sigma)(x_1^0(\sigma) + x_2^0(\sigma) + x_3^0(\sigma)) d\sigma + R_0^d \Phi'(Q_1^*) \times \\
\qquad \displaystyle\int_0^{a^+} \beta(\sigma)(s_1^*(\sigma) + i_1^*(\sigma)) d\sigma \int_0^{a^+} r(\sigma)(x_1^0(\sigma) + x_2^0(\sigma) + x_3^0(\sigma)) d\sigma,
\end{cases}
\tag{4.4a}
$$

$$
\begin{cases}
\lambda x_2^0(a) + \dfrac{\partial x_2^0(a)}{\partial a} = K_1(a)\displaystyle\int_0^{a^+} q_1(\sigma) x_2^0(\sigma) d\sigma s_1^*(a) + K_1(a) \times \\
\qquad\qquad\qquad \displaystyle\int_0^{a^+} q_1(\sigma) i_1^*(\sigma) d\sigma x_1^0(a) - (\mu(a) + \gamma_1(a)) x_2^0(a), \\
x_2^0(0) = 0,
\end{cases}
\tag{4.4b}
$$

$$
\begin{cases}
\lambda x_3^0(a) + \dfrac{\partial x_3^0(a)}{\partial a} = K_2(a)\displaystyle\int_0^{a^+} q_2(\sigma) x_3^0(\sigma) d\sigma s_1^*(a) - (\mu(a) + \gamma_2(a)) x_3^0(a), \\
x_3^0(0) = 0.
\end{cases}
\tag{4.4c}
$$

Integrating the first equation of (4.4c) from 0 to a yields

$$
x_3^0(a) = \int_0^a K_2(\sigma) s_1(\sigma) e^{-\int_\sigma^a (\lambda + \mu(\theta) + \gamma_2(\theta)) d\theta} d\sigma \int_0^{a^+} q_2(a) x_3^0(a) da. \tag{4.5}
$$

Multiplying both sides of (4.5) by $q_2(a)$, and integrating, we obtain

$$
\int_0^{a^+} q_2(a) x_3^0(a) da = \int_0^{a^+} q_2(a) \int_0^a K_2(\sigma) s_1^*(\sigma) e^{-\int_\sigma^a (\lambda + \mu(\theta) + \gamma_2(\theta)) d\theta} d\sigma da \int_0^{a^+} q_2(a) x_3^0(a) da.
$$

This leads to the following characteristic equation:

$$
1 = \int_0^{a^+} q_2(a) \int_0^a K_2(\sigma) s_1^*(\sigma) e^{-\int_\sigma^a (\lambda + \mu(\theta) + \gamma_2(\theta)) d\theta} d\sigma da. \tag{4.6}
$$

Let

$$
\begin{aligned}
\mathscr{H}(\lambda) &:= \int_0^{a^+} q_2(a) \int_0^a K_2(\sigma) s_1^*(\sigma) e^{-\int_\sigma^a (\lambda + \mu(\theta) + \gamma_2(\theta)) d\theta} d\sigma da \\
&= \int_0^{a+} K_2(a) s_1^*(a) \int_a^{a^+} q_2(\sigma) e^{-\int_a^\sigma (\lambda + \mu(\theta) + \gamma_j(\theta)) d\theta} d\sigma da.
\end{aligned}
$$

Then $\mathscr{H}(\lambda)$ is a continuously differentiable function with $\lim\limits_{\lambda \to +\infty} \mathscr{H}(\lambda) = 0$, $\lim\limits_{\lambda \to -\infty} \mathscr{H}(\lambda) = +\infty$. Furthermore, we have $\mathscr{H}'(\lambda) < 0$. It then follows that $\mathscr{H}(\lambda)$ is a decreasing function. So the equation (4.6) has a unique real root λ^*. Noting that

$$
\mathscr{H}(0) = \mathscr{R}_1^2,
$$

we have $\lambda^* < 0$ if $\mathscr{R}_1^2 < 1$, and $\lambda^* > 0$ if $\mathscr{R}_1^2 > 1$. Let $\lambda = \xi + \eta i$ be an arbitrary complex root to equation (3.1). Then

$$1 = \mathscr{H}(\lambda) = |\mathscr{H}(\xi + \eta i)| \leq \mathscr{H}(\xi),$$

which implies that $\lambda^* > \xi$. Thus, all the roots of the equation (4.6) have negative real part if $\mathscr{R}_1^2 < 1$. In addition, if $\mathscr{R}_1^2 > 1$ we can easily see that $1 = \mathscr{H}(\lambda)$ has a positive real root and E_1 is unstable.

However, if $\mathscr{R}_1^2 < 1$ the stability of E_1 is completely determined by the equations (4.4a) and (4.4b) under the condition that $x_3(a) \equiv 0$. Assumption (H5) implies that E_1 is locally stable if $x_3(a) \equiv 0$. Thus if $\mathscr{R}_1^2 < 1$ then E_1 is locally stable. This completes the proof of Theorem 4.3.

□

5 Persistence of the pathogens

In this section we present our main result: sufficient conditions for the persistence of two competing strains. One consequence of the persistence of the strains is that they coexist. Our principal result in this section can be stated as follows.

Theorem 5.1. *Assume $R_0^d > 1, \mathcal{R}_j > 1, j = 1, 2$ and (H1)-(H5) hold. If $\mathscr{R}_1^2 > 1$ and $\mathscr{R}_2^1 > 1$, then there exists a constant $\varepsilon > 0$ such that every solution $(s(a,t), i_1(a,t), i_2(a,t))$ with initial condition $(s_0(\cdot), i_{10}(\cdot), i_{20}(\cdot)) \in \Gamma$ satisfies*

$$\liminf_{t\to\infty} \int_0^{a^+} i_1(a,t)da \geq \varepsilon, \liminf_{t\to\infty} \int_0^{a^+} i_2(a,t)da \geq \varepsilon,$$

where

$$\begin{aligned}\Gamma := \{&(s_0(\cdot), i_{10}(\cdot), i_{20}(\cdot)) \in X_+ : \\ &\int_0^{a^+} i_{10}(a)da > 0, \int_0^{a^+} i_{20}(a)da > 0, \int_0^{\tilde{a}} s_0(a)da > 0\}.\end{aligned}$$

In order to prove Theorem 5.1, we need the following lemmas:

Lemma 5.2. *Assume (H1) holds, then the system* (2.3) *is dissipative.*

Proof. Since $n(a,t) = s(a,t) + i_1(a,t) + i_2(a,t)$, it follows from system (4.1) that $n(a,t)$ satisfies the following differential equation:

$$\begin{cases} \dfrac{\partial n(a,t)}{\partial t} + \dfrac{\partial n(a,t)}{\partial a} = -\mu(a)n(a,t) - \displaystyle\sum_{j=1}^{2} \gamma_j(a)i_j(a,t), \\ n(0,t) = R_0^d \Phi(Q(t)) \displaystyle\int_0^{a^+} \beta(a)n(a,t)da, \\ n(a,0) = n_0(a), \end{cases} \tag{5.1}$$

where $n_0(a) = s_0(a) + \sum_{j=1}^{2} i_{j0}(a)$. Integrating the system (5.1) along the characteristic lines we get

$$n(a,t) = \begin{cases} R_0^d \Phi(Q(t-a)) \int_0^{a^+} \beta(\sigma) n(\sigma, t-a) d\sigma e^{-\int_0^a (\mu(\theta) + \sum_{j=1}^{2} \gamma_j(\theta) \frac{i_j(\theta, \theta+t-a)}{n(\theta, \theta+t-a)}) d\theta}, & t > a, \\ n_0(a-t) e^{-\int_{a-t}^a (\mu(\theta) + \sum_{j=1}^{2} \gamma_j(\theta) \frac{i_j(\theta, \theta+t-a)}{n(\theta, \theta+t-a)}) d\theta}, & t < a. \end{cases}$$

Since $\Phi(x)$ is strictly decreasing, it follows that

$$n(a,t) \le \begin{cases} R_0^d \Phi(r_{\min} \int_0^{a^+} n(\sigma, t-a) d\sigma) \beta_{\max} \int_0^{a^+} n(\sigma, t-a) d\sigma e^{-\int_0^a \mu(\theta) d\theta}, & t > a, \\ n_0(a-t) e^{-\int_{a-t}^a \mu(\theta) d\theta}, & t < a. \end{cases}$$

Moreover, (H1) implies that

$$n(a,t) \le \begin{cases} M e^{-\int_0^a \mu(\theta) d\theta}, & t > a, \\ n_0(a-t) e^{-\int_{a-t}^a \mu(\theta) d\theta}, & t < a. \end{cases} \tag{5.2}$$

It is easy to see that $n(a,t) \le M$ for almost all $a \in (0, a^+)$ and $t > a^+$. Thus the system (2.3) is dissipative. $\square$

Lemma 5.3. *Assume (H1)-(H3) hold, then Γ is positively invariant for system* (2.3).

Proof. Let $(\phi, \varphi, \psi) \in \Gamma$, and $(s(a,t), i_1(a,t), i_2(a,t))$ be the solution to system (2.3) with the initial conditions $s(\cdot, 0) = \phi, i_1(\cdot, 0) = \varphi, i_2(\cdot, 0) = \psi$. First, we prove that $\int_0^{\tilde{a}} s(a,t) da > 0$ for all $t > 0$. To this aim, let us prove that the following assertion:

Assertion: If $s(0,t) > 0$ for $t \in (t_1, t_2)$ then $s(0,t) > 0$ for $t \in (t_1 + \tilde{a}_0, t_2 + \tilde{a})$.

In fact, if $t \in (t_1 + \tilde{a}_0, t_2 + \tilde{a})$, then $\tilde{a}_0 \vee (t - t_2) < \tilde{a} \wedge (t - t_1)$, and $(\tilde{a}_0 \vee (t - t_2), \tilde{a} \wedge (t - t_1)) \subseteq (\tilde{a}_0, \tilde{a}), \{t - (\tilde{a}_0 \vee (t - t_2), \tilde{a} \wedge (t - t_1))\} \subset (t_1, t_2)$. It follows from (2.3) that we have

$$\begin{aligned} s(0,t) &= R_0^d \Phi(Q(t)) \int_0^{a^+} \beta(a) n(a,t) da \\ &> R_0^d \Phi(M \int_0^{a^+} r(\sigma) d\sigma) \int_{\tilde{a}_0 \vee (t-t_2)}^{\tilde{a} \wedge (t-t_1)} \beta(a) s(a,t) da \\ &= R_0^d \Phi(M \int_0^{a^+} r(\sigma) d\sigma) \int_{\tilde{a}_0 \vee (t-t_2)}^{\tilde{a} \wedge (t-t_1)} \beta(a) s(0, t-a) e^{-\int_0^a (\mu(\theta) + \sum_{j=1}^{2} \lambda_j(\theta, \theta+t-a)) d\theta} da \\ &> 0. \end{aligned}$$

The proof of the assertion is completed.

Because $\int_0^{\tilde{a}} \phi(a)da > 0$ and the way $\tilde{a}$ is defined, it follows that there exists a $\bar{t} \in (0, \tilde{a})$ such that $\int_{\bar{t}}^{\tilde{a}} \beta(a)\phi(a-\bar{t}) > 0$. Then we have

$$\begin{aligned} s(0,\bar{t}) &= R_0^d \Phi(Q(t)) \int_0^{a^+} \beta(a) s(a,\bar{t}) da \\ &> R_0^d \Phi(M \int_0^{a^+} r(\sigma)d\sigma) \int_{\bar{t}}^{\tilde{a}} \beta(a)\phi(a-\bar{t}) e^{-\int_{a-\bar{t}}^{a} (\mu(\theta) + \sum\limits_{j=1}^{2} \lambda_j(\theta, \theta+t-a))d\theta} da \\ &> 0. \end{aligned}$$

By the continuity of the function $s(0,t)$, we can conclude that there exists an interval (t_1, t_2) such that $\bar{t} \in (t_1, t_2)$ and $s(0,t) > 0$ for all $t \in (t_1, t_2)$. Iterating the above assertion, we have for any integer m

$$s(0,t) > 0 \;\; t \in (t_1 + m\tilde{a}_0, t_2 + m\tilde{a}). \tag{5.3}$$

We now are able to prove that $\int_0^{\tilde{a}} s(a,t) > 0$ for all $t > 0$. If $t \in [0, \bar{t}]$, then we have

$$\begin{aligned} \int_0^{\tilde{a}} s(a,t) &\geq \int_t^{\tilde{a}} \phi(a-t) e^{-\int_{a-t}^{a} (\mu(\theta) + \sum\limits_{j=1}^{2} \lambda_j(\theta, \theta+t-a))d\theta} da \\ &> \int_{\bar{t}}^{\tilde{a}} \phi(a-\bar{t}) e^{-\int_{a-\bar{t}}^{a} (\mu(\theta) + \sum\limits_{j=1}^{2} \lambda_j(\theta, \theta+t-a))d\theta} da \\ &> 0. \end{aligned}$$

If $t \in (t_1, t_2 + \tilde{a})$, then we have

$$\begin{aligned} \int_0^{\tilde{a}} s(a,t) &\geq \int_{0 \vee (t-t_2)}^{\tilde{a} \wedge (t-t_1)} s(a,t) dt \\ &\geq \int_{0 \vee (t-t_2)}^{\tilde{a} \wedge (t-t_1)} s(0, t-a) e^{-\int_0^a (\mu(\theta) + \sum\limits_{j=1}^{2} \lambda_j(\theta, \theta+t-a))d\theta} da \\ &> 0. \end{aligned}$$

Similarly, if $t \in (t_1 + \tilde{a}_0, t_2 + 2\tilde{a})$ then we have $\int_0^{\tilde{a}} s(a,t)da > 0$. Thus, iterating the above procedure yields that $\int_0^{\tilde{a}} s(a,t)da > 0$ for all $t \geq 0$.

Second, let us show that $\int_0^{a^+} i_j(a,t)da > 0, j = 1, 2$ for all $t > 0$. To this aim, we divide the proof into three steps:

Step 1. Define

$$i_{jc}(a) := i_j(a, a+c) \qquad a \in [0 \vee (-c), a^+)$$

for $c > -a^+$. Then

$$\begin{aligned} \frac{di_{jc}(a)}{da} &= \lambda_j(a, a+c) s(a, a+c) - (\mu(a) + \gamma_j(a)) i_{jc}(a) \\ &\geq -(\mu(a) + \gamma_j(a)) i_{jc}(a). \end{aligned} \tag{5.4}$$

It follows from (5.4) that if $i_{jc}(a_0) > 0$ we have $i_{jc}(a) > 0$, i.e., $i_j(a, a+c) > 0$, for all $a > a_0$.

Step 2. Since $\int_0^{\tilde{a}} s(a,0)da > 0$, there exists $\bar{a} < \tilde{a}_0 + \frac{\tilde{a}-\tilde{a}_0}{2}$ such that $\int_{\bar{a}}^{(\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2})} s(a,0)da > 0$. In this step, we prove that $\frac{di_{jc}(a)}{da}|_{a=0} > 0$ for all $c \in (\tilde{a}_0 - \bar{a}, \frac{\tilde{a}+\tilde{a}_0}{2} - \bar{a})$.

For ease of presentation, we assume that $\bar{a} < \tilde{a}_0$, and if $\bar{a} \geq \tilde{a}_0$ we can analyze in a similar way. Integrating the system (2.3) along the characteristic lines yields

$$i_j(a,t) = \begin{cases} \int_0^a \lambda_j(\theta, \theta+t-a)s(\theta,\theta+t-a)e^{-\int_\theta^a(\mu(\xi)+\gamma_j(\xi))d\xi}d\theta, & t > a, \\ i_{j0}(a-t)e^{-\int_{a-t}^a(\mu(\theta)+\gamma_j(\theta))d\theta} + & \\ \qquad \int_{a-t}^a \lambda_j(\theta,\theta+t-a)s(\theta,\theta+t-a)e^{-\int_\theta^a(\mu(\xi)+\gamma_j(\xi))d\xi}d\theta, & t < a. \end{cases} \tag{5.5}$$

Let $c \in (\tilde{a}_0 - \bar{a}, \frac{\tilde{a}+\tilde{a}_0}{2} - \bar{a})$, and consider two cases: $\int_{\bar{a}}^{\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2}} i_{j0}(a)da > 0$ and $\int_{\bar{a}}^{\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2}} i_{j0}(a)da = 0$.

Case 1) $\int_{\bar{a}}^{(\bar{a}+\frac{\tilde{a}-\tilde{a}_0)}{2}} i_{j0}(a)da > 0$. Then it follows from (5.5) that

$$\begin{aligned} \int_{c+\bar{a}}^{c+\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2}} i_j(a,c)da &> \int_{c+\bar{a}}^{c+\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2}} i_{j0}(a-c)e^{-\int_{a-c}^a(\mu(\theta)+\gamma_j(\theta))d\theta}da \\ &= \int_{\bar{a}}^{\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2}} i_{j0}(a)e^{-\int_a^{a+c}(\mu(\theta)+\gamma_j(\theta))d\theta}da \\ &> 0. \end{aligned}$$

Consequently, we have $(c+\bar{a}, c+\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2}) \subset (\tilde{a}_0, \tilde{a})$ and for all $c \in (\tilde{a}_0 - \bar{a}, \frac{\tilde{a}+\tilde{a}_0}{2} - \bar{a})$

$$\begin{aligned} \frac{di_{jc}(a)}{da}|_{a=0} &= \lambda_j(0,c)n(0,c) \\ &= K_j(0)\int_0^{a^+} q_j(\sigma)i_j(\sigma,c)d\sigma n(0,\sigma) \\ &\geq K_j(0)\int_{c+\bar{a}}^{c+\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2}} q_j(\sigma)i_j(\sigma,c)d\sigma R_0^d\Phi(M\int_0^{a^+} r(\sigma))\int_{c+\bar{a}}^{c+\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2}} \beta(\sigma)i_j(\sigma,c)d\sigma \\ &\geq K_j(0)q_{j\,\min}\beta_{\min}R_0^d\Phi(M\int_0^{a^+} r(\sigma))(\int_{c+\bar{a}}^{c+\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2}} i_j(\sigma,c)d\sigma)^2 \\ &> 0. \end{aligned} \tag{5.6}$$

Case 2) $\int_{\bar{a}}^{(\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2})} i_{j0}(a)da = 0$, i.e., $\int_{\bar{a}}^{(\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2})} \phi(a)da > 0$. It follows from (5.5) that

$$\begin{aligned} &\int_{c+\bar{a}}^{c+\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2}} i_j(a,c)da \\ &> \int_{c+\bar{a}}^{c+\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2}} \int_{a-c}^a \lambda_j(\theta,\theta+c-a)s(\theta,\theta+c-a)e^{-\int_\theta^a(\mu(\xi)+\gamma_j(\xi))d\xi}d\theta da \\ &\geq \int_{c+\bar{a}}^{c+\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2}} K_{j\,\min}q_{j\,\min}s(a,c)\int_{a-c}^a\int_0^{a^+} i_j(\sigma,\theta+c-a)d\sigma e^{-\int_\theta^a(\mu(\xi)+\gamma_j(\xi))d\xi}d\theta da. \end{aligned}$$

From the fact $\int_0^{a^+} i_j(\sigma,0)d\sigma > 0$ and the continuity of solutions, we have

$$f(a) := \int_{a-c}^{a}\int_0^{a^+} i_j(\sigma,\theta+c-a)d\sigma e^{-\int_\theta^a(\mu(\xi)+\gamma_j(\xi))d\xi}d\theta > 0$$

for all $a > c$. Integrating the first equation of system (2.3) along the characteristic lines, we get the form

$$s(a,c) = \phi(a-c)e^{-\int_{a-c}^a(\mu(\xi)+\sum\limits_{j=1}^{2}\lambda_j(\xi,\xi+c-a))d\xi}$$

for $a \in (c+\bar{a}, c+\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2})$. Since $\int_{\bar{a}}^{(\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2})} \phi(a)da > 0$, it follows that $\int_{c+\bar{a}}^{c+\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2}} s(a,c)da > 0$. Then we have

$$\begin{aligned}
&\int_{c+\bar{a}}^{c+\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2}} i_j(a,c)da \\
&\geq K_{j\min}q_{j\min}\int_{c+\bar{a}}^{c+\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2}} s(a,c)\int_{a-c}^{a}\int_0^{a^+} i_j(\sigma,\theta+c-a)d\sigma e^{-\int_\theta^a(\mu(\xi)+\gamma_j(\xi))d\xi}d\theta da. \\
&> 0.
\end{aligned}$$

Using a similar way as in the proof of the Case 1), we can conclude that $(c+\bar{a}, c+\bar{a}+\frac{\tilde{a}-\tilde{a}_0}{2}) \subset (\tilde{a}_0,\tilde{a})$ and $\frac{i_{jc}(a)}{da}|_{a=0} > 0$ for all $c \in (\tilde{a}_0-\bar{a}, \frac{\tilde{a}+\tilde{a}_0}{2}-\bar{a})$.

Step 3. In this step, we prove the following assertion:

$$\text{If } \frac{di_{jc}(a)}{da}|_{a=0} > 0 \text{ for } c \in (t_3,t_4), \text{ then } \frac{di_{jc}(a)}{da}|_{a=0} > 0 \text{ for } c \in (t_3+\tilde{a}_0, t_4+\tilde{a}). \tag{5.7}$$

In fact, if $\frac{di_{jc}(a)}{da}|_{a=0} > 0$ for $c \in (t_3,t_4)$ then it follows from (5.4) that $i_j(a,c) > 0$ for all $a \in (0\vee(c-t_4), (c-t_3)\wedge a^+)$ and $c \in (t_3, t_4+a^+)$. Let $c \in (t_3+\tilde{a}_0, t_4+\tilde{a})$, then $\tilde{a}_0\vee(c-t_4) < \tilde{a}\wedge(c-t_3)$, and $(\tilde{a}_0\vee(c-t_4), \tilde{a}\wedge(c-t_3)) \subseteq (\tilde{a}_0,\tilde{a})$, $(\tilde{a}_0\vee(c-t_4), \tilde{a}\wedge(c-t_3)) \subset (0\vee(c-t_4), (c-t_3)\wedge a^+)$. Thus we have $i_j(a,c) > 0$ for all $a \in (\tilde{a}_0\vee(c-t_4), \tilde{a}\wedge(c-t_3))$. It follows from (5.4) and the fact that $i_{jc}(0) = 0$ that

$$\begin{aligned}
\frac{di_{jc}(a)}{da}|_{a=0} &= \lambda_j(0,c)s(0,c) \\
&= K_j(0)\int_0^{a^+} q_j(\sigma)i_j(\sigma,c)d\sigma s(0,c) \\
&\geq K_j(0)\int_0^{a^+} q_j(\sigma)i_j(\sigma,c)d\sigma s(0,c) \\
&\geq K_j(0)\int_{\tilde{a}_0\vee(c-t_4)}^{\tilde{a}\wedge(c-t_3)} q_j(\sigma)i_j(\sigma,c)d\sigma R_0^d\Phi(M\int_0^{a^+} r(\sigma))\int_{\tilde{a}_0\vee(c-t_4)}^{\tilde{a}\wedge(c-t_3)} \beta(\sigma)i_j(\sigma,c)d\sigma \\
&\geq K_j(0)q_{j\min}\beta_{\min}R_0^d\Phi(M\int_0^{a^+} r(\sigma))(\int_{\tilde{a}_0\vee(c-t_4)}^{\tilde{a}\wedge(c-t_3)} i_j(\sigma,c)d\sigma)^2 \\
&> 0.
\end{aligned} \tag{5.8}$$

Thus, the statement (5.7) is proved. It then follows from (5.4) that $i_j(a,c) > 0$ for all $a \in (0 \vee (c-(t_4+\tilde{a})), (c-(t_3+\tilde{a}_0)) \wedge a^+)$ and $c \in (t_3+\tilde{a}_0, t_4+\tilde{a}+a^+)$.

Since there exists $\bar{a} < \tilde{a}_0 + \frac{\tilde{a}-\tilde{a}_0}{2}$ such that $\frac{di_{jc}(a)}{da}|_{a=0} > 0$ for all $c \in (\tilde{a}_0 - \bar{a}, \frac{\tilde{a}+\tilde{a}_0}{2} - \bar{a})$, iterating (5.7) we have

$$\frac{di_{jc}(a)}{da}|_{a=0} > 0 \text{ for all } c \in (\tilde{a}_0 - \bar{a} + n\tilde{a}_0, \frac{\tilde{a}+\tilde{a}_0}{2} - \bar{a} + n\tilde{a}). \tag{5.9}$$

If there exists T such that $\int_0^{a^+} i_j(a,T)da = 0$, then it is easy to see that $i_j(a,t) = 0$ is also a solution of (2.3) for $t > T$. Thus we have $\frac{di_{jc}(a)}{da}|_{a=0} = 0$ for all $t > T$. This contradicts (5.9) due to the uniqueness of solutions to (2.3). This contradiction implies that $\int_0^{a^+} i_j(a,t) > 0$ for all $t > 0$. This completes the proof Lemma 5.3. $\square$

Proof. (Theorem 5.1*):* Define

$$\begin{aligned}
B_0 &:= \{(s(\cdot), i_1(\cdot), i_2(\cdot)) \in X_+ : \\
&\qquad \int_0^{\tilde{a}} s(a)da > 0, i_j(a) \equiv 0, j = 1,2 \text{ for almost all } a \in (0, a^+)\}; \\
B_{11} &:= \{(s(\cdot), i_1(\cdot), i_2(\cdot)) \in X_+ : \\
&\qquad \int_0^{\tilde{a}} (s(a) + i_1(a))da = 0, i_2(a) \equiv 0 \text{ for almost all } a \in (0, a^+)\}; \\
B_{12} &:= \{(s(\cdot), i_1(\cdot), i_2(\cdot)) \in X_+ : \textstyle\int_0^{\tilde{a}} (s(a) + i_1(a))da > 0, \\
&\qquad \int_0^{a^+} i_1(a)da > 0, i_2(a) \equiv 0 \text{ for almost all } a \in (0, a^+)\}; \\
B_{21} &:= \{(s(\cdot), i_1(\cdot), i_2(\cdot)) \in X_+ : \\
&\qquad \int_0^{\tilde{a}} (s(a) + i_2(a))da = 0, i_1(a) \equiv 0 \text{ for almost all } a \in (0, a^+)\}; \\
B_{22} &:= \{(s(\cdot), i_1(\cdot), i_2(\cdot)) \in X_+ : \textstyle\int_0^{\tilde{a}} (s(a) + i_2(a))da > 0, \\
&\qquad \int_0^{a^+} i_2(a)da > 0, i_1(a) \equiv 0 \text{ for almost all } a \in (0, a^+)\}; \\
\partial\Gamma &:= B_0 \cup B_{11} \cup B_{12} \cup B_{21} \cup B_{22}, \\
\mathscr{X} &:= \Gamma \cup \partial\Gamma.
\end{aligned}$$

In order to show the theorem, it suffices to show that $\partial\Gamma$ repeals uniformly the solutions of Γ.

We can easily see that B_0 and $B_{ij}, i, j = 1, 2$ are positively invariant for system (2.3). Thus $\partial\Gamma$ is positively invariant for system (2.3). It follows from Lemma 5.3 that Γ is positively invariant for system (2.3). Similar approach as in Proposition 3.16 by Webb [20] can result in the fact that the dynamical system is asymptotically smooth. From Theorem 4.1, Theorem 4.3 and the assumptions (H4) and (H5), we have the following conclusions:

(1) E_0 is a global attractor in B_0 for system (2.3);

(2) E_{00} is a global attractor in $B_{11} \cup B_{21}$ for system (2.3);

(3) E_1 is a global attractor in B_{21} for system (2.3);

(4) E_2 is a global attractor in B_{22} for system (2.3).

Thus we have

$$\begin{aligned}\tilde{A}_\partial &:= \bigcup_{(s_0(\cdot),i_{10}(\cdot),i_{20}(\cdot))\in\partial\Gamma} \omega((s_0(\cdot), i_{10}(\cdot), i_{20}(\cdot)))\\ &= \{E_{00}, E_0, E_1, E_2\}.\end{aligned}$$

By the above conclusions, it follows that $\tilde{A}_\partial$ is isolated and has an acyclic covering $M = \{E_{00}, E_0, E_1, E_2\}$. It follows from Lemma 5.2 that the system (2.3) is dissipative and the orbit of any bounded set is bounded. By Theorem 4.2 in [24], in order to show that $\partial\Gamma$ repeals uniformly the solutions of Γ we only need to show that $W^s(E_{00}) \cap \Gamma = \emptyset, W^s(E_0) \cap \Gamma = \emptyset, W^s(E_1) \cap \Gamma = \emptyset, W^s(E_2) \cap \Gamma = \emptyset$ if $R_0^d > 1, \mathscr{R}_1^2 > 1, \mathscr{R}_2^1 > 1$.

Since $R_0^d > 1$, we can choose $\eta_1 > 0$ small enough such that

$$R_0^d \Phi(3\eta_1 \int_0^{a^+} r(\sigma)d\sigma)e^{-\eta_1 \sum\limits_{j=1}^{2} \int_0^{a^+} K_j(a)da \int_0^{a^+} q_j(a)da} > 1. \tag{5.10}$$

Assume that $W^s(E_{00}) \cap \Gamma \neq \emptyset$. Then there exists a positive solution $(\hat{s}(a,t), \hat{i}_1(a,t), \hat{i}_2(a,t))$ with the initial conditions $\hat{s}(\cdot, 0) = \psi_1, \hat{i}_1(\cdot, 0) = \varphi_1, \hat{i}_2(\cdot, 0) = \phi_1, (\psi_1, \varphi_1, \phi_1) \in \Gamma$ such that $(\hat{s}(a,t), \hat{i}_1(a,t), \hat{i}_2(a,t)) \to E_{00}$ as $t \to +\infty$. Thus there exists $T_1 > 0$ such that for $t > T_1$ we have $0 < \hat{s}(\cdot, t) < \eta_1, 0 < \hat{i}_1(\cdot, t) < \eta_1, 0 < \hat{i}_2(a, t) < \eta_1$, $(\hat{s}(\cdot, T_1), \hat{i}_1(\cdot, T_1), \hat{i}_2(\cdot, T_1)) \in \Gamma$, and

$$\begin{cases} \dfrac{\partial \hat{s}(a,t)}{\partial t} + \dfrac{\partial \hat{s}(a,t)}{\partial a} > -(\mu(a) + \eta_1 \sum\limits_{j=1}^{2} K_j(a) \displaystyle\int_0^{a^+} q_j(\sigma)d\sigma)\hat{s}(a,t), \\ \hat{s}(a,0) = \psi_1(a), \\ \hat{s}(0,t) > R_0^d \Phi(3\eta_1 \displaystyle\int_0^{a^+} r(\sigma)d\sigma) \int_0^{a^+} \beta(a)\hat{s}(a,t)da. \end{cases}$$

Consider the following auxiliary system

$$\begin{cases} \dfrac{\partial \hbar(a,t)}{\partial t} + \dfrac{\partial \hbar(a,t)}{\partial a} = -(\mu(a) + \eta_1 \sum\limits_{j=1}^{2} K_j(a) \displaystyle\int_0^{a^+} q_j(\sigma)d\sigma)\hbar(a,t), \\ \hbar(a,T) = \hat{s}(a,T_1), \\ \hbar(0,t) = R_0^d \Phi(3\eta_1 \displaystyle\int_0^{a^+} r(\sigma)d\sigma) \int_0^{a^+} \beta(a)\hbar(a,t)da. \end{cases} \tag{5.11}$$

By the comparison principle we have $\hat{s}(\cdot,t) \geq \hbar(\cdot,t)$ for all $t > T_1$. By the theory of linear systems, it is easy to see that $\hbar(\cdot,t) \to +\infty$ as t tends to infinity if (5.10) holds. This contradicts $\hat{s}(\cdot,t) \to 0$ as $t \to +\infty$. This contradiction implies that we have $W^s(E_{00}) \cap \Gamma = \emptyset$.

Now we show that $W^s(E_0) \cap \Gamma = \emptyset$. Since $\mathscr{R}_1^2 > 1, \mathscr{R}_2^1 > 1$ imply $\mathcal{R}_j > 1, j = 1,2$, we can choose $\eta_2 > 0$ small enough such that

$$\int_0^{a+} K_1(a)(n^*(a) - \eta_2) \int_a^{a^+} q_1(\sigma) e^{-\int_a^\sigma (\mu(\theta)+\gamma_1(\theta))d\theta} d\sigma da > 1,$$
$$\int_0^{a+} K_2(a)(n^*(a) - \eta_2) \int_a^{a^+} q_2(\sigma) e^{-\int_a^\sigma (\mu(\theta)+\gamma_2(\theta))d\theta} d\sigma da > 1.$$

Assume that $W^s(E_0) \cap \Gamma \neq \emptyset$. Then there exists a positive solution $(\tilde{s}(a,t), \tilde{i}_1(a,t)), \tilde{i}_2(a,t))$ with initial conditions $\tilde{s}(\cdot,0) = \psi_2, \tilde{i}_1(\cdot,0) = \varphi_2, \tilde{i}_2(\cdot,0) = \phi_2, (\psi_2,\varphi_2,\phi_2) \in \Gamma$ such that $(\tilde{s}(a,t), \tilde{i}_1(a,t), \tilde{i}_2(a,t)) \to E_0$ as $t \to +\infty$. Thus there exists $T_2 > 0$ such that for $t > T_2$ we have

$$n^*(\cdot) - \eta_2 < \tilde{s}(\cdot,t) < n^*(\cdot) + \eta_2, 0 < \tilde{i}_1(\cdot,t) < \eta_2, 0 < \tilde{i}_2(a,t) < \eta_2,$$

for $(\tilde{s}(\cdot,T_2), \tilde{i}_1(\cdot,T_2), \tilde{i}_2(a,T_2)) \in \Gamma$, and

$$\begin{cases} \dfrac{\partial \tilde{i}_1(a,t)}{\partial t} + \dfrac{\partial \tilde{i}_1(a,t)}{\partial a} > K_1(a) \displaystyle\int_0^{a^+} q_1(\sigma)\tilde{i}_1(\sigma,t)d\sigma(n^*(a) - \eta_2) - (\mu(a) + \gamma_1(a))\tilde{i}_1(a,t), \\ \dfrac{\partial \tilde{i}_2(a,t)}{\partial t} + \dfrac{\partial \tilde{i}_2(a,t)}{\partial a} > K_2(a) \displaystyle\int_0^{a^+} q_2(\sigma)\tilde{i}_2(\sigma,t)d\sigma(n^*(a) - \eta_2) - (\mu(a) + \gamma_2(a))\tilde{i}_2(a,t), \\ \tilde{i}_1(a,0) = \varphi_2(a), \\ \tilde{i}_2(a,0) = \phi_2(a), \\ \tilde{i}_1(0,t) = 0 \\ \tilde{i}_2(0,t) = 0. \end{cases}$$

Consider the auxiliary system

$$\begin{cases} \dfrac{\partial \ell_1(a,t)}{\partial t} + \dfrac{\partial \ell_1(a,t)}{\partial a} = K_1(a) \displaystyle\int_0^{a^+} q_1(\sigma)\ell_1(\sigma,t)d\sigma(n^*(a) - \eta_2) - (\mu(a) + \gamma_1(a))\ell_1(a,t), \\ \dfrac{\partial \ell_2(a,t)}{\partial t} + \dfrac{\partial \ell_2(a,t)}{\partial a} = K_2(a) \displaystyle\int_0^{a^+} q_2(\sigma)\ell_2(\sigma,t)d\sigma(n^*(a) - \eta_2) - (\mu(a) + \gamma_2(a))\ell_2(a,t), \\ \ell_1(a,T_2) = \tilde{i}_1(a,T_2), \\ \ell_2(a,T_2) = \tilde{i}_2(a,T_2), \\ \ell_1(0,t) = 0, \\ \ell_2(0,t) = 0. \end{cases} \tag{5.12}$$

By the comparison principle, we can prove that $\tilde{i}_j(\cdot,t) \geq \ell_j(\cdot,t), j = 1,2$ for all $t > T_2$. From the theory of linear systems it is easy to see that $\ell_j(\cdot,t) \to +\infty, j = 1,2$ as t tends to infinity if $\int_0^{a+} K_j(a)(n^*(a) - \eta_2) \int_a^{a^+} q_j(\sigma) e^{-\int_a^\sigma (\mu(\theta)+\gamma_j(\theta))d\theta} d\sigma da > 1, j = 1,2$. This contradicts $\tilde{i}_j(\cdot,t) \to 0, j = 1,2$ as $t \to +\infty$. This contradiction implies that we have $W^s(E_0) \cap \Gamma = \emptyset$.

In the following, we show that $W^s(E_1) \cap \Gamma = \emptyset, W^s(E_2) \cap \Gamma = \emptyset$. Here we only show that $W^s(E_1) \cap \Gamma = \emptyset$ if $\mathscr{R}_1^2 > 1$. If $\mathscr{R}_2^1 > 1$, $W^s(E_2) \cap \Gamma = \emptyset$ can be proved in a similar way. Since $\mathscr{R}_1^2 > 1$, we can choose $\eta_3 > 0$ small enough such that

$$\int_0^{a+} K_2(a)(s_1^*(a) - \eta_3) \int_0^{a^+} q_2(\sigma) e^{-\int_a^\sigma (\mu(\theta)+\gamma_2(\theta))d\theta} d\sigma da > 1.$$

Assume that $W^s(E_1) \cap \Gamma \neq \emptyset$. Then there exists a positive solution $(\check{s}(a,t), \check{i}_1(a,t), \check{i}_2(a,t))$ with initial conditions $\check{s}(\cdot,0) = \psi_3, \check{i}_1(\cdot,0) = \varphi_3, \check{i}_2(\cdot,0) = \phi_3, (\psi_3,\varphi_3,\phi_3) \in \Gamma$ such that $(\check{s}(a,t), \check{i}_1(a,t), \check{i}_2(a,t)) \to E_1$ as $t \to +\infty$. Thus there exists a $T_3 > 0$ such that for $t > T_3$ we have

$$\begin{aligned}
&s_1^*(\cdot) - \eta_3 < \check{s}(\cdot,t) < s_1^*(\cdot) + \eta_3,\\
&i_1^*(\cdot) - \eta_3 < \check{i}_1(\cdot,t) < i_1^*(\cdot) + \eta_3,\\
&0 < \check{i}_2(a,t) < \eta_3,\\
&(\check{s}(\cdot,T_3), \check{i}_1(\cdot,T_3), \check{i}_2(\cdot,T_3)) \in \Gamma,
\end{aligned}$$

and

$$\begin{cases}
\dfrac{\partial \check{i}_2(a,t)}{\partial t} + \dfrac{\partial \check{i}_2(a,t)}{\partial a} > K_2(a) \displaystyle\int_0^{a^+} q_2(\sigma) \check{i}_2(\sigma,t) d\sigma (s_1^*(a) - \eta_3) - (\mu(a) + \gamma_2(a)) \check{i}_2(a,t),\\
\check{i}_2(a,0) = \varphi_2(a),\\
\check{i}_2(0,t) = 0.
\end{cases}$$

Consider the auxiliary system

$$\begin{cases}
\dfrac{\partial \ell_2(a,t)}{\partial t} + \dfrac{\partial \ell_2(a,t)}{\partial a} = K_2(a) \displaystyle\int_0^{a^+} q_2(\sigma) \ell_2(\sigma,t) d\sigma (s_1^*(a) - \eta_3) - (\mu(a) + \gamma_2(a)) \ell_2(a,t),\\
\ell_2(a,T_3) = \check{i}_2(a,T_3),\\
\ell_2(0,t) = 0.
\end{cases} \tag{5.13}$$

By the comparison principle, we can prove that $\check{i}_2(\cdot,t) \geq \ell_2(\cdot,t)$ for all $t > T_3$. From the theory of linear systems it is easy to see that $\int_0^{a^+} \ell_2(a,t)da \to +\infty$ as t tends to infinity if $\int_0^{a+} K_2(a)(s_1^*(a) - \eta_3) \int_0^{a^+} q_2(\sigma) e^{-\int_a^\sigma (\mu(\theta)+\gamma_2(\theta))d\theta} d\sigma da > 1$. This contradicts $\check{i}_2(\cdot,t) \to 0$ as $t \to +\infty$. This contradiction implies that we have $W^s(E_1) \cap \Gamma = \emptyset$.

Since $W^s(E_{00}) \cap \Gamma = \emptyset, W^s(E_0) \cap \Gamma = \emptyset, W^s(E_j) \cap \Gamma = \emptyset, j = 1,2$, and $\{E_{00}, E_0, E_1, E_2\}$ are acyclic in $\partial\Gamma$, by Theorem 4.2 in [24] we are able to conclude that the system (2.3) is uniformly persistent with respect to $(\Gamma, \partial\Gamma)$. This completes the proof of Theorem 5.1. □

6 Numerical results

In this section, we present some numerical results that support and extend the analytical results. Backward Euler and the linearized finite difference method are used to discretize the PDEs, and the integral is evaluated using trapezoidal rule.

For illustration purposes, we choose $a^+ = 20, R_0^d = 20$. The functions $\mu(a), \beta(a), r(a), \gamma_j(a), j = 1, 2$ are assumed to be constant. These parameters are chosen as follows:

$$\mu(a) \equiv 0.6, \beta(a) \equiv 0.6, r(a) \equiv 1, \gamma_j(a) \equiv 0, q_j(a) \equiv 1, j = 1, 2, a \in [0, 20].$$

The functions $\Phi(x), K_1(a), K_2(a)$ are chosen to have the following forms:

$$\Phi(x) = \tfrac{1}{1+0.3x};$$

$$K_1(a) = \tfrac{m_1}{1+k_1 a};$$

$$K_2(a) = m_2 + k_2 a.$$

The simulation results are shown in Figure 1. Figure 1 (a) is for the case $k_1 = 0, k_2 = 0, m_1 = 0.1, m_2 = 0.06$. In this case system (2.3) does not depend on the age structure, and it follows from Theorem 3.1 that the strain with the larger reproduction number, i.e., strain 1, uniformly persists, and the other strain dies out. Figure 1 (b) is for the case $m_1 = 0.1, k_1 = 1.0, m_2 = 0.06, k_2 = 0.06$. In this case, we can obtain that $\mathscr{R}_2^1 = 1.2921, \mathscr{R}_1^2 = 1.0406$. Since both invasion numbers are larger than one, we expect persistence of the strains. Indeed, as Figure 1 (b) demonstrates the two competing strains coexist in a unique positive equilibrium which as simulations seem to suggest seems globally attracting. Figure 1 (c) is for the case $m_1 = 0.1, k_1 = 1.0, m_2 = 0.03, k_2 = 0.06$. In the case, the system (2.3) has four boundary equilibria E_{00}, E_0, E_1, E_2. Straight forward computation yields that $\mathscr{R}_1^2 = 0.628, \mathscr{R}_2^1 = 1.8304$. It follows from Theorem 4.3 that E_2 is locally stable under assumption (H4). Figure 1 (c) is showing that E_2 is indeed asymptotically stable. Figure 1 (d) is for the case $m_1 = 0.1, k_1 = 1.0, m_2 = 0.09, k_2 = 0.06$. In this case, we have that $\mathscr{R}_1^2 = 1.4586, \mathscr{R}_2^1 = 0.9431$ and the system (2.3) also has four boundary equilibria. In Figure 1 (d), we can easily see that the boundary equilibrium E_1 is asymptotically stable. This simulation results have either confirm or extended our analytical results.

7 Discussion

In this paper, we have studied an age-structured epidemic model with two competing strains. The main focus of the paper is the proof of persistence of the two strains when both invasion numbers are greater than one. In our case the coexistence of the strains is induced by the host age-structure. Although the presence of disease-induced mortality into the age-structured model makes

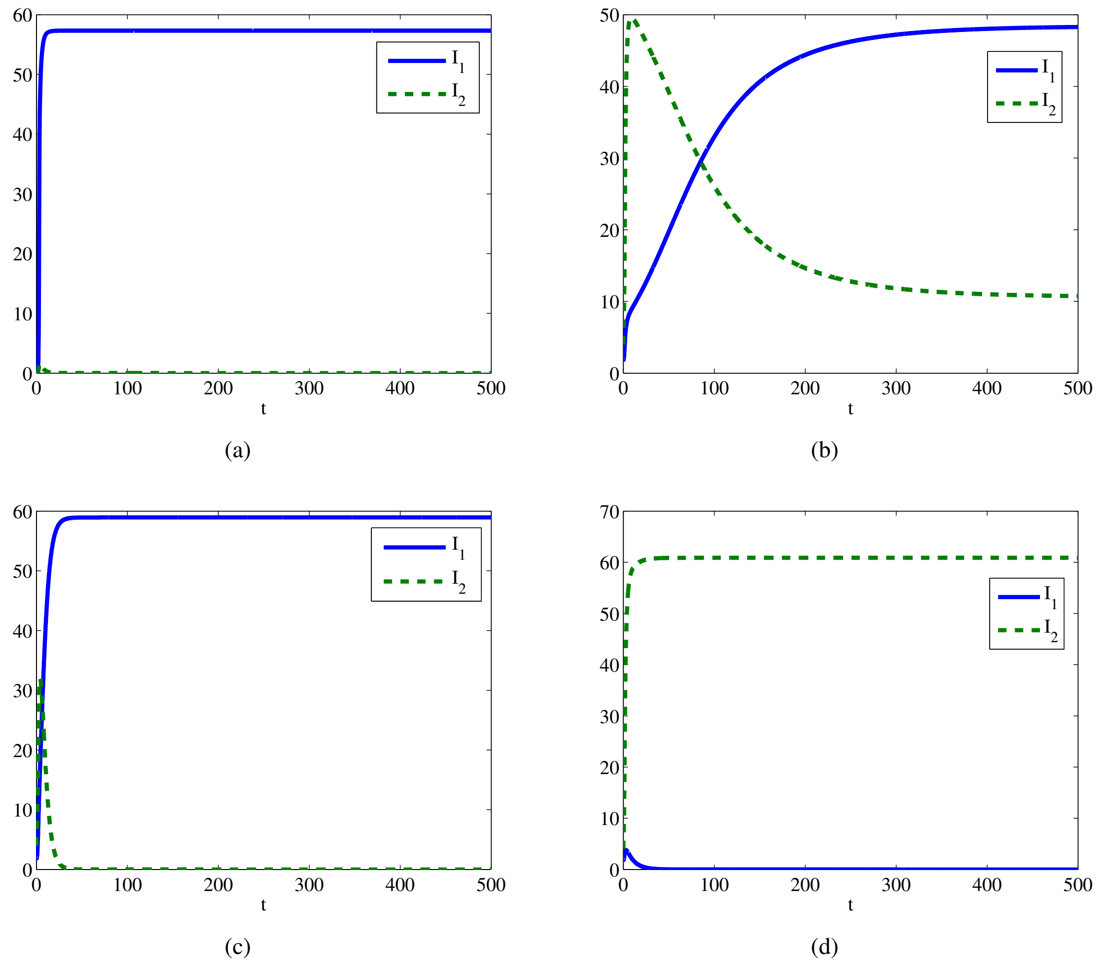

Figure 1: Numerical solutions of the system (2.3). I_1 and I_2 represent the number of the infected cases with strain one and two, respectively. The values of of $m_j, k_j, j = 1, 2$ used in the four plots are: (a) $m_1 = 0.1, m_2 = 0.06, k_1 = k_2 = 0$ (age independent case); (b) $m_1 = 0.1, m_2 = 0.06, k_1 = 1.0, k_2 = 0.06$; (c) $m_1 = 0.1, m_2 = 0.03, k_1 = 1.0, k_2 = 0.06$; and (d) $m_1 = 0.1, m_2 = 0.09, k_1 = 1.0, k_2 = 0.06$.

the analysis of the resulting system very difficulty, we are able to carry out systematic analysis of the model and establish rigorously uniform persistence of both strains.

The basic reproduction numbers $\mathcal{R}_j$ for both strains are defined in section 3, and Theorem 4.2 shows the uniform persistence of the single strain j in the case when $\mathcal{R}_j > 1$. We have also obtained the explicit expression of the invasion reproduction numbers $\mathscr{R}_k^j, j, k = 1, 2, j \neq k$ for both strains. Local stabilities of the boundary equilibria are discussed in Theorem 4.3. By the theory of uniform persistence of infinite dimensional dynamical systems the coexistence of two competing strains is rigorously proved under the conditions that both invasion reproduction numbers are lager than one. However, results for the corresponding age-independent model are summarized in section 3 and show that the two competing strains can not coexist. This indicates that age structure leads to the coexistence of the strains. In section 6 numerical simulations are further conducted to confirm and extend the analytical results.

Finally, there are still many interesting and challenging mathematical questions that remain open for system (2.3). For example, we could not present results on the global dynamics of system (2.3). It seems very difficult to analyze the global dynamics for that system. However, if we neglect the disease-induced death rates, then system (2.3) can be reduced to a competitive system. We may use techniques from monotone theory developed in [25, 30] to provide the global behavior of the reduced model. Numerical simulations suggest that if $\mathscr{R}_1 > 1, \mathscr{R}_2 > 1$ and the invasion numbers are greater than one, the reduced system (2.3) has a unique positive equilibrium which is globally asymptotically stable. We conclude the discussion by formulating the following conjecture:

Conjecture: *Assume $\mathscr{R}_1 > 1, \mathscr{R}_2 > 1$ and the invasion reproduction numbers are greater than one. Then system* (2.3) *with $\gamma_1(a) = \gamma_2(a) = 0$ has a unique coexistence equilibrium which is globally asymptotically stable.*

Acknowledgements

Zhipeng Qiu greatly appreciates Prof. Iannelli and Prof. Pugliese for their encouragement and many good suggestions. This work is supported by Istituto Nazionale di Alta Matematica (INdAM)-Rome and Italian Ministry of University and Research (MUR) within the China-Italy Executive Cooperation Programme of the Italian Ministry of Foreign Affairs (MAE), the NSFC grants (No. 10801074, 10911130372 10971178, 10911120387), and the US NSF under grant DMS-0817789.

Appendix: Proof of Theorem 4.1

Proof. Using a similar approach we can easily prove the second conclusion of Theorem 4.1 2). Here we only present the proofs for Theorem 4.1 1) and the first conclusion of Theorem 4.1 2). Let $\tilde{n}(a,t)$ be the solution of the following system

$$\begin{cases} \dfrac{\partial \tilde{n}(a,t)}{\partial t} + \dfrac{\partial \tilde{n}(a,t)}{\partial a} = -\mu(a)\tilde{n}(a,t), \\ \tilde{n}(a,0) = n_0(a), \\ \tilde{n}(0,t) = R_0^d \Phi(0) \displaystyle\int_0^{a^+} \beta(a)\tilde{n}(a,t)da. \end{cases} \tag{7.1}$$

We claim that $n(a,t) \leq \tilde{n}(a,t)$ for all $t \geq 0$ and almost all $a \in (0,a^+)$.

In order to prove the claim, let us integrate the system (2.1) and (7.1) along the characteristic lines. Then we get

$$n(a,t) = \begin{cases} n(0,t-a)e^{-\int_0^a \mu(\theta)d\theta}, & t > a, \\ n_0(a-t)e^{-\int_{a-t}^a \mu(\theta)d\theta}, & t < a. \end{cases} \tag{7.2}$$

and

$$\tilde{n}(a,t) = \begin{cases} \tilde{n}(0,t-a)e^{-\int_0^a \mu(\theta)d\theta}, & t > a, \\ n_0(a-t)e^{-\int_{a-t}^a \mu(\theta)d\theta}, & t < a. \end{cases} \tag{7.3}$$

We now show that $n(0,t) \leq \tilde{n}(0,t)$. Suppose the contrary, that is the inequality does not hold. Then we can define t_0 such that

$$t_0 := \inf\{t > 0 : n(0,t) > \tilde{n}(0,t)\}$$

since $n(0,0) < \tilde{n}(0,0)$ and $n(0,t), \tilde{n}(0,t)$ are both continuous functions with respect to t. Consequently, we have

$$R_0^d \Phi(Q(t_0)) \int_0^{a^+} \beta(a)n(a,t_0)da = R_0 \Phi(0) \int_0^{a^+} \beta(a)\tilde{n}(a,t_0)da.$$

Since $\Phi(Q(t_0)) \leq \Phi(0)$ it follows that there exist $\bar{a} \in (0,t_0)$ such that $n(\bar{a},t_0) \geq \tilde{n}(\bar{a},t_0)$. It follows from (7.2) and (7.3) that we have $n(0,t_0-\bar{a}) \geq \tilde{n}(0,t_0-\bar{a})$. This contradicts the definition of t_0. This contradiction implies that $n(0,t) \leq \tilde{n}(0,t)$. From (7.2) and (7.3) we get $n(a,t) \leq \tilde{n}(a,t)$ for all $t > a > 0$. When $a > t > 0$, we can easily see that $n(a,t) = \tilde{n}(a,t)$. Thus $n(a,t) \leq \tilde{n}(a,t)$ for all $t \geq 0$ and almost all $a \in (0,a^+)$.

If $R_0^d < 1$, it is easy to see that $\lim\limits_{t \to +\infty} \tilde{n}(\cdot,t,n_0(a)) = 0$ for all $n_0(a) \in L_+^1(0,a^+)$. From the above claim it also follows that we have $\lim\limits_{t \to +\infty} n(\cdot,t,n_0(a)) = 0$ for each $n_0(\cdot) \in L_+^1(0,a^+)$. This complete the proof of Theorem 4.1 1).

Assume $R_0^d > 1$ and $n_0(a) \in L^1_+(0, a^+)$. If $\int_0^{\tilde{a}} n_0(a)da = 0$, it then follows from Theorem 5.1 of paper [25] that we have

$$\lim_{t\to+\infty} \tilde{n}(\cdot, t, n_0(a)) = 0.$$

Similarly, we have

$$\lim_{t\to+\infty} n(\cdot, t, n_0(a)) = 0.$$

This completes the proof of Theorem 4.1.

□

References

[1] F. Carrat, J. Luong, H. Lao, A. Salle, C. Lajaunie, H .Wackernagel, A small-world-like model for comparing interventions aimed at preventing and controlling influenza pandemics. BMC Med., 2006, 4: 26. doi:10.1186/1741-7015-4-26.

[2] S. N. Busenberg, M. Iannelli, H. R. Thieme, Global behavior of an age-structured epdiemic model, SIAM J. Math. Anal., 1991, 22:1065-1080.

[3] H. Inaba, Mathematical analysis of an age-structured SIR epidemic model with vertical transmission, Dis. Con. Dyn. Sys. B, 2006,6:69-96.

[4] P. Zhang, Z. Feng, F. Milner, A schistosomiasis model with an age-structure in human hosts and its application to treatment strategies, Math.Biosci.,2007, 205: 83-107.

[5] C. Castillo-Chavez , Z. Feng, Global stability of an age-structure model for TB and its applications to optimal vaccination strategies, Math. Biosci., 1998, 151: 135-154.

[6] J. S. Oxford, Influenza A pandemics of the 20th century with special reference to 1918: virology, pathology and epidemiology, Rev. Med. Vir., 2000, 10:119-133.

[7] Centers for Disease Control and Prevention, West Nile Virus: Fact Sheet, http://www.cdc.gov/ncidod/dvbid/westnile/wnv factSheet.htm.

[8] Z. Feng, J. X. Velasco-Hernandes, Competitive exclusion in a vector-host model for the dengue fever, J. Math. Biol., 1997, 35: 523-544.

[9] M. Martcheva, M. Iannelli, X. Z. Li, Subthreshold coexistence of strains: the impact of vaccination and mutation, Math. Biosci. Eng., 2007, 4:287-317.

[10] S. A. Levin, Community equilibria and stability, and an extension of the competitive exclusion principle, Am. Naturalist, 1970, 104: 413-423.

[11] H. J. Bremermann, H. R. Thieme, A competitive exclusion principle for pathogen virulence, J. Math. Biol., 1989, 27:179-190.

[12] M. Iannelli, M. Martcheva, X. Z. Li, Strain replacement in an epidemic model with super-infection and perfect vaccination, Math. Biosci., 2005, 195:23-46.

[13] M. Martcheva, S.S. Pilyugin, The role of coinfection in multidisease dynamics, SIAM J. Appl. Math., 2006, 66:843-872.

[14] M. Martcheva, S.S. Pilyugin, R.D. Holt, Subthreshold and superthreshold coexistence of pathogen variants: The impact of host age-structure, Math. Biosci. 2007, 207 (1): 58-77.

[15] M. Martcheva, H. Thieme, Progression age enhanced backward bifurcation in an epidemic model with super-infection, J. Math. Biol. , 2003, 46 (5): 385-424.

[16] R. May, M. Nowak, Coinfection and the evolution of parasite virulence, Proc. R. Soc. B, 1995, 261: 209-215.

[17] C. Castillo-Chavez, H. Hethcote, V. Andreasen, S. Levin, W. M. Liu, Epidemiological models with age structure, proportionate mixing, and cross-immunity, J. Math. Biol., 1989, 27:233-258.

[18] V. Andreasen, A. Pugliese, Viral coexistence can be induced by density dependent host mortality, J. Theoret. Biol., 1995, 177:159-165.

[19] M.E. Gurtin, R.C. MacCamy, Nonlinear age-dependent population dynamics, Archiv. Rat. Mech. Anal., 1974, 54:281-300.

[20] G. Webb, Theory of Nonlinear Age-Dependent Population Dynamics, Marcel Dekker, New York, 1985.

[21] D. Breda, M. Iannelli, S. Maset, R. Vermiglio, Stability analysis of the Gurtin-MacCamy model, SIAM J. Numer. Anal., 2008, 46:980-995.

[22] M. Iannelli, Mathematical theory of age-structured population dynamics, Applied Mathematics Monograph C.N.R. 7, Giardini Ed., Pisa 1995.

[23] D. Breda, D. Visetti, Existence, uniqueness and multiplicity of endemic states for an age-structured S-I epidemic model, preprint.

[24] J. K. Hale, P Waltmam, Persistence in infinite-dimensional systems, SIAM J. Math. Anal., 1989, 20: 388-395.

[25] Z. Feng, W. Huang, C. Castillo-Chavez, Global behavior of a multi-group SIS epidemic model with age structure, J. Diff. Equs., 2005, 218: 292-324.

[26] F. A. Milner, A. Pugliese, Periodic solutions: a robust numerical method for an SIR model of epidemics, J. Math. Biol., 1999, 39: 471-492.

[27] X. Z. Li, X. C. Duan, M. Ghosh, X. Y. Ruan, Pathogen coexistence induced by saturating contact rates, Nonl. Anal.: R.W.A., 2009, 10: 3298-3311.

[28] P. van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartment models of disease transmission, Math. Biosci., 2002, 180: 29-48.

[29] H.R. Thieme, Persistence under relaxed point-dissipativity with an application to an epidemic model, SIAM J. Math. Anal., 1993, 24: 407-435.

[30] H. L. Smith, Monotone dynamical systems: an introduction to theory of competitive and cooperative systems, Math. Surveys Monogr., 41, AMS, Providence, RI, 1995.