1 Introduction

One of the main themes of geometric group theory is to study a (finitely generated) group $G$ in terms of the geometric properties of the Cayley graph of $G$. These “geometric properties” come in the form of quasi-isometry invariants. Our goal this semester is to look at some specific quasi-isometry invariants such as Dehn functions, hyperbolicity, growth functions, and amenability and try to understand what algorithmic, algebraic, and analytic properties of groups they are capturing.

We begin by giving necessary definitions and establishing the notation that we will use throughout.

**Word metric and Cayley graphs** Let $G$ be a group generated by $S \subseteq G$; for convenience, we will always assume that our generating sets are symmetric, that is $S = S^{-1}$. A word in $S$ is a finite concatenation of elements of $S$. For such a word $W$, let $\|W\|$ denote its length. If two words $W$ and $U$ are letter for letter equivalent, we write $W \equiv U$, and if $W$ and $U$ represent the same element of the group $G$, we write $W =_G U$. For an element $g \in G$, let $|g|_S$ denote the length of the shortest word in $S$ which represents $g$ in the group $G$. Given $g, h \in G$, let $d_S(g, h) = |g^{-1}h|_S$. $d_S$ is called the word metric on $G$ with respect to $S$.

We let $\Gamma(G, S)$ denote the Cayley graph of $G$ with respect to $S$. This is the graph whose vertex set is $G$ and there is an oriented edge labeled by $s \in S$ between any two vertices of the form $g$ and $gs$. We typically identify the edges labeled by $s$ and $s^{-1}$ with the same endpoints and consider these as the same edge with opposite orientations. $\text{Lab}(e)$ denotes the label of the edge $e$; similarly, for a (combinatorial) path $p$, $\text{Lab}(p)$ will denote the concatenation of the labels of the edges of $p$. Also for such a path $p$, we let $p_-$ and $p_+$ denote the initial and the terminal vertex of $p$ respectively, and $\ell(p)$ will denote the number of edges of $p$. The metric obtained on the vertices of $\Gamma(G, S)$ by the shortest path metric is clearly equivalent to the word metric $d_S$; identifying each edge with the unit interval $[0, 1]$ in the natural way allows us to extend this metric to all of $\Gamma(G, S)$.

**Metric spaces** Throughout these notes, we denote a metric space by $X$ and its metric by $d$ (or $d_X$ if necessary). For $x \in X$ and $n \geq 0$, let $B_n(x) = \{y \in X \mid d(x, y) \leq n\}$, that is the closed ball
of radius $n$ centered at $x$. For a subset $A \subseteq X$, we usually denote the closed $n$-neighborhood of $A$ by $A^{+n} = \{ x \in X \mid d(x, A) \leq n \}$.

A path in $X$ is a continuous map $p: [a, b] \to X$ for some $[a, b] \subseteq \mathbb{R}$. We will often abuse notation by using $p$ to refer to both the function and its image in $X$. As above, we let $p_- = p(a)$ and $p_+ = p(b)$. Similarly, a ray is a continuous map $p: [a, \infty) \to X$, and a bi-infinite path is a continuous map $p: (-\infty, \infty) \to X$.

A geodesic is a path $p$ which is also an isometry onto its image (see below for the definition of an isometry). Equivalently, A path $p$ is a geodesic if $\ell(p) = d(p_-, p_+)$ where $\ell(p)$ denotes the length of $p$, and is defined as

$$\ell(p) = \sup_{a \leq t_1 \leq \ldots \leq t_n \leq b} \sum_{i=1}^{n-1} d(p(t_i), p(t_{i+1}))$$

where the supremum is taken over all $n \geq 1$ and all possible choices of $t_1, \ldots, t_n$.

Geodesic rays and bi-infinite geodesics are defined similarly. $X$ is called a geodesic metric space if for all $x, y \in X$, there exists a geodesic path $p$ such that $p_- = x$ and $p_+ = y$. Note that geodesic metric spaces are clearly path connected. For $x, y$ in a geodesic metric space $X$, we let $[x, y]$ denote a geodesic from $x$ to $y$.

We will usually assume throughout these notes that $X$ is a geodesic metric space. However, most statements and proofs will also work under the weaker assumption that $X$ is a length space, that is a path connected space such that for any $x, y \in X$, $d(x, y) = \inf \{ \ell(p) \mid p_- = x, p_+ = y \}$.

Let $X$ and $Y$ be metric spaces and $f: X \to Y$. If $f$ is onto and for all $x_1, x_2 \in X$, $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$, then $f$ is called isometry. If $f$ is onto and there is a constant $\lambda \geq 1$ such that for all $x_1, x_2 \in X$,

$$\frac{1}{\lambda} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2).$$

Then $f$ is called a bi-lipschitz equivalence. In this case, we say that $X$ and $Y$ are bi-lipschitz equivalent and write $X \sim_{lip} Y$. Now suppose there are constant $\lambda \geq 1$, $C \geq 0$, and $\varepsilon \geq 0$ such that $f(X)$ is $\varepsilon$-quasi-dense in $Y$, i.e. $f(X)^{\varepsilon} = Y$, and for all $x_1, x_2 \in X$,

$$\frac{1}{\lambda} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + C.$$

Then $f$ is called a quasi-isometry, or a $(\lambda, c, \varepsilon)$-quasi-isometry if we need to keep track of the constants. In this case we say $X$ and $Y$ are quasi-isometric and write $X \sim_{qi} Y$. Note that unlike isometries and bi-lipschitz equivalences, quasi-isometries are not required to be continuous.

If the condition that $f$ is onto (or $f(X)$ is quasi-dense) is dropped from the above definitions, then $f$ is called an isometric embedding, bi-lipschitz embedding, or a quasi-isometric embedding respectively.

**Exercise 1.1.** Show that $\sim_{lip}$ and $\sim_{qi}$ are both equivalence relations on metric spaces.

**Exercise 1.2.** Let $X$ and $Y$ be bounded metric spaces. Prove that $X \sim_{qi} Y$.

**Exercise 1.3.** Suppose $S \subseteq G$ and $T \subseteq G$ are two finite generating sets of $G$. Show that $(G, d_S) \sim_{lip} (G, d_T)$, and hence $\Gamma(G, S) \sim_{qi} \Gamma(G, T)$.

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$^1$In general this may be infinite, but we will usually only consider rectifiable paths, that is paths $p$ for which $\ell(p) < \infty$. 

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It follows from this exercise that any finitely generated group is canonically associated to a \( \sim_{qi} \)-equivalence class of metric spaces. We will often abuse notation by considering the group \( G \) itself as a metric space, but it should be understood that the metric on \( G \) is only well-defined up to quasi-isometry.

If \( P \) is a property of metric spaces such that whenever \( X \sim_{qi} Y \), \( X \) has \( P \) if and only if \( Y \) has \( P \), then \( P \) is called a quasi-isometry invariant. If \( G \) is a finitely generated group, then for any two choices of finite generating sets the corresponding Cayley graphs will have exactly the same quasi-isometry invariants. Hence these invariants are inherent properties of the group \( G \).

**Group actions** Let \( G \) be a group acting on a metric space \( X \). We will always assume that such actions are by isometries, that is for all \( x, y \in X \) and \( g \in G \),

\[
d(x, y) = d(gx, gy).
\]

There is a natural correspondence between actions of \( G \) on \( X \) and homomorphisms \( \rho: G \to Isom(X) \), where \( Isom(X) \) denotes the group of all isometries of \( X \). We say the action is faithful if the corresponding homomorphism is injective. This is equivalent to saying for all \( g \in G \), there exists \( x \in X \) such that \( gx \neq x \). The action is called free if for all \( x \in X \), \( Stab_G(x) = \{1\} \), where \( Stab_G(x) = \{g \in G \mid gx = x\} \); equivalently, for all \( x \in X \) and for all \( g \in G \), \( gx \neq x \). The action is called proper\(^2\) if for any bounded subset \( B \subseteq X \), \( \{g \in G \mid gB \cap B \neq \emptyset\} \) is finite. The action is called cobounded if \( X/G \) is bounded, or equivalently there exists a bounded subset \( B \subseteq X \) such that

\[
X = \bigcup_{g \in G} gB.
\]

The following lemma is fundamental to geometric group theory. It was first proved by Efremovic.

**Lemma 1.4** (Milnor-Svarč Lemma). Let \( G \) be a group acting properly and coboundedly on a geodesic metric space \( X \). Then \( G \) has a finite generating set \( S \) and

\[
\Gamma(G, S) \sim_{qi} X.
\]

**Proof.** Fix a point \( o \in X \). Since the action of \( G \) is cobounded, there exists a constant \( K \) such that for all \( x \in X \), there exists \( g \in G \) such that \( d(x, go) \leq K \). Let \( S = \{g \in G \mid d(o, go) \leq 2K + 1\} \). By properness, the set \( S \) is finite. Note that if \( s_1, s_2 \in S \), then \( d(o, s_1s_2o) \leq d(o, s_1o) + d(s_1o, s_1s_2o) = d(o, s_1o) + d(o, s_2o) \leq 2(2K + 1) \). Similarly, it is easy to show by induction that for all \( g \in \langle S \rangle \), 
\[
d(o, go) \leq |g|_S(2K + 1).
\]

Now fix \( g \in G \), and let \( p \) be a geodesic from \( o \) to \( go \). Choose points \( o = x_0, x_1, ..., x_n = go \) on \( p \) such that \( d(x_i, x_{i+1}) = 1 \) for \( 0 \leq i \leq n - 2 \) and \( d(x_{n-1}, x_n) \leq 1 \). For each \( 1 \leq i \leq n - 1 \), choose \( h_i \in G \) such that \( d(x_i, h_io) \leq K \), and set \( h_0 = 1 \) and \( h_n = g \). By the triangle inequality, \( d(o, h_i^{-1}h_{i+1}o) = d(h_io, h_{i+1}o) \leq 2K + 1 \) for all \( 0 \leq i \leq n - 1 \). Hence \( h_i^{-1}h_{i+1} \in S \). Furthermore,

\[
h_1(h_1^{-1}h_2)(h_2^{-1}h_3)...(h_{n-1}^{-1}h_n) = h_n = g
\]

\(^2\)This is the metric version of properness. There is also a topological version, where bounded is replaced by compact.
Thus \( g \in \langle S \rangle \), and since \( g \) is arbitrary we get that \( S \) generates \( G \). Furthermore, \(|g|_S \leq n\) and by our choice of \( x_i, n - 1 < d(o,go) \leq n\). Let \( f: G \to X \) be the function defined by \( f(g) = go \). Then we have shown that

\[
|g|_S - 1 \leq d(o,go) \leq (2K + 1)|g|_S.
\]

Furthermore, our choice of \( K \) implies that \( f(G) \) is \( K \)-quasi-dense in \( X \). It follows easily that the map \( f \) is a quasi-isometry from \( G \) with the metric \( d_S \) to \( X \).

\[
\text{Corollary 1.5. 1. If } G \text{ is finitely generated and } H \text{ is a finite index subgroup of } G, \text{ then } H \text{ is finitely generated and } G \sim_{qi} H.
\]

2. If \( N \lhd G \) is a finite normal subgroup of \( G \) and \( G/N \) is finitely generated, then \( G \) is finitely generated and \( G \sim_{qi} G/N \).

3. If \( M \) is a closed Riemannian manifold with universal cover \( \tilde{M} \), then \( \pi_1(M) \) is finitely generated and \( \pi_1(M) \sim_{qi} \tilde{M} \).

4. If \( G \) is a connected Lie group with a left-invariant Riemannian metric and \( \Gamma \leq G \) is a uniform lattice in \( G \), then \( \Gamma \) is finitely generated and \( \Gamma \sim_{qi} G \). It follows that any two uniform lattices in the same connected Lie group are quasi-isometric to each other.

One important example that follows from Corollary 1.5(3) is if \( S_g \) is a closed, orientable surface of \( g \) with \( g \geq 2 \), then \( S_g \) can be equipped with a hyperbolic Riemannian metric which allows us to identify the universal cover \( \tilde{S}_g \) with the hyperbolic plane \( \mathbb{H}^2 \). Thus, \( \pi_1(S_g) \sim_{qi} \mathbb{H}^2 \).

In geometric group theory, we are often concerned with connections between the algebra and the geometry of a group \( G \). When \( G = \pi_1(M) \) for a Riemannian manifold \( M \), then the algebra of \( G \) is determined by the topology of \( M \) while the geometry of \( G \) is determined up to quasi-isometry by the geometry of \( \tilde{M} \). The geometry/topology of \( \tilde{M} \) is determined locally, and in some ways globally, by the geometry/topology of \( M \). These connections lead to a fruitful interplay between geometry, topology and (geometric) group theory.

**Definition 1.6.** Finitely generated groups \( G_1 \) and \( G_2 \) are **commensurable** if each \( G_i \) contains a finite-index subgroup \( H_i \) such that \( H_1 \cong H_2 \). If each \( G_i \) contains a finite-index subgroup \( H_i \) and each \( H_i \) contains a finite normal subgroup \( N_i \) such that \( H_1/N_1 \cong H_2/N_2 \), then \( G_1 \) and \( G_2 \) are called **weakly commensurable**.

**Examples 1.7.** 1. Any two finite groups are commensurable.

2. If \( G \) is any finitely generated group and \( K \) is any finite group, then \( G \) and \( G \times K \) are commensurable.

3. For all \( n,m \geq 2 \), \( F_n \) is commensurable with \( F_m \).

4. For all \( g,g' \geq 2 \), \( \pi_1(S_g) \) is commensurable with \( \pi_1(S_{g'}) \).

It is easy to see that (weak) commensurability is an equivalence relation on finitely generated groups. From Corollary 1.5, any two (weakly) commensurable groups are quasi-isometric. The above examples allow us to see some group theoretic properties which are not quasi-isometry invariants. For example, from (1), many algebraic properties such as being abelian, nilpotent,
solvable, or even simple are not quasi-isometry invariants. Similarly, from (2) it follows any group theoretic property $P$ which passes to subgroups and for which there exists a finite group which does not have $P$ is not a quas-isometry invariant, even for infinite groups. However, some of these properties only fail to be quasi-isometry invariants because of this issue with finite groups. The following definition captures this notion:

**Definition 1.8.** Let $P$ be a property of groups. A group $G$ is said to be **virtually $P$** if $G$ has a finite index subgroup $H$ such that $H$ has the property $P$.

It turns out that being virtually abelian and virtually nilpotent are both quasi-isometry invariants. This is a consequence of Gromov’s polynomial growth theorem which we will discuss later. On the other hand, being virtually solvable is not a quasi-isometry invariant, though some important subclasses of solvable groups are preserved under quasi-isometries and this is still an active area of research (see, for example, [10, 13]).

From examples (3) and (4) above we can also see that the rank of a groups, that is the minimal size of a finite generating set is not quasi-isometric invariant. In addition the ordinary group homology/cohomology is not a quasi-isometry invariant, though there do exist “coarse” versions of homology and cohomology which do provide quasi-isometry invariants (see [29]).

## 2 Dehn functions and algorithmic problems

### 2.1 Group presentations

Given a set $S$, we denote the free group on $S$ by $F(S)$. Recall that the elements of this group are equivalence classes of words in $S$, where words two words are equivalent if you can obtain one from the other by adding or removing subwords of the form $ss^{-1}$ finitely many times. Equivalently, $F(S)$ can be defined as the unique group (up to isomorphism) such that for any group $G$ and any function $f: S \to G$, there is a unique homomorphism $\tilde{f}: F(S) \to G$ extending $f$. If $S = \{s_1, ..., s_n\}$, we typically denote $F(S)$ by $F_n$.

Given a subset $R \subseteq G$, where $G$ is a a group, the **normal closure** of $R$, denoted $\langle\langle R \rangle\rangle$, is defined as the intersection of all normal subgroups of $G$ which contain $R$. Equivalently,

$$\langle\langle R \rangle\rangle = \{f_1^{-1}r_1f_1r_2f_2^{-1}r_2f_2 ... f_k^{-1}r_kf_k \mid k \geq 0, f_i \in G, r_i \in R^\pm\}.$$  

Given a set $S$ and $R \subseteq F(S)$, we say that

$$\langle S | R \rangle$$

is a **presentation** of the group $G$ if $G \cong F(S)/\langle\langle R \rangle\rangle$. In this case $S$ is called the set of **generators** and $R$ is called the set of **relations** of the presentation. The presentation is called **finite** if both $S$ and $R$ are finite sets, and $G$ is called **finitely presentable** if $G$ has a finite presentation. For convenience we will always assume that our set of relations is symmetric, that is $r \in R$ implies $r^{-1} \in R$. We will also abuse notation and write $G = \langle S | R \rangle$ to indicate that $\langle S | R \rangle$ is a presentation for the group $G$.

**Exercise 2.1.** Identify the groups given by the following presentations:
1. \( \langle a \mid a^n = 1 \rangle \).
2. \( \langle a_1, \ldots, a_n \mid \emptyset \rangle \).
3. \( \langle a, b \mid [a, b] = 1 \rangle \).
4. \( \langle a, b \mid a^n = b^m = 1 \rangle \).

Given a group presentation \( \langle S \mid R \rangle \) for a group \( G \), there is an associated CW-complex \( Y \) with \( \pi_1(Y) \cong G \), called the presentation complex. This \( Y \) contains a single vertex \( v \), one edge (labeled by \( s \)) with both ends glued to \( v \) for each \( s \in S \), and one 2-cell \( \Pi \) for each \( r \in R \), glued to the 1-skeleton of \( Y \) such that \( \partial \Pi \) is labeled by \( r \).

The universal cover \( \tilde{Y} \) is called the Cayley complex associated to \( \langle S \mid R \rangle \). Note that the 1-skeleton of the Cayley complex can be naturally identified with \( \Gamma(G, S) \), and the 2-skeleton of \( \tilde{Y} \) is obtained by gluing, for each \( g \in G \) and \( r \in R \), a 2-cell with boundary a loop based at \( g \) and labeled by \( r \).

### 2.2 Van Kampen Diagrams

Suppose \( \langle S \mid R \rangle \) is a presentation for a group \( G \) and \( W \) is a word in \( S \). Then \( W =_G 1 \) if and only if there exist \( r_1, \ldots, r_k \in R \) and \( f_1, \ldots, f_k \in F(S) \) such that

\[
W =_{F(S)} f_1^{-1} r_1 f_1 \cdots f_k^{-1} r_k f_k.
\]  

(2)

We now show how this can be encoded geometrically. Let \( \Delta \) be a finite, connected, simply connected, planar 2-complex in which every edge is oriented and labeled by an element of \( S \). If \( e \) is an edge of \( \Delta \) with label \( s \) and \( \bar{e} \) is the same edge with the opposite orientation, then \( \text{Lab}(\bar{e}) = s^{-1} \).

Labels of paths in \( \Delta \) are defined the same as in Cayley graphs. If \( \Pi \) is a 2-cell of \( \Delta \), then \( \text{Lab}(\partial \Pi) \) is the word obtained by choosing a base point \( v \in \partial \Pi \) and reading the label of the path \( \partial \Pi \) starting and ending at \( v \). Note that a different choice of basepoint results in a cyclic permutation of the word \( \text{Lab}(\partial \Pi) \), so we consider \( \text{Lab}(\partial \Pi) \) as being defined only up to cyclic permutations. \( \text{Lab}(\partial \Delta) \) is defined similarly. \( \Delta \) is called a van Kampen diagram over the presentation \( \langle S \mid R \rangle \) if for every 2-cell \( \Pi \) of \( \Delta \), (a cyclic permutation of) \( \text{Lab}(\partial \Pi) \) belongs to \( R \). In this case it can be shown by a reasonably straightforward induction on the number of 2-cells of \( \Delta \) that \( \text{Lab}(\partial \Delta) =_G 1 \). It turns out the converse is also true.

**Exercise 2.2.** Suppose \( G \) is a group with presentation \( \langle S \mid R \rangle \) and \( \Delta \) is a van Kampen diagram over \( \langle S \mid R \rangle \). Prove that \( \text{Lab}(\partial \Delta) =_G 1 \).

**Lemma 2.3** (van Kampen Lemma). Suppose \( \langle S \mid R \rangle \) is a presentation for a group \( G \) and \( W \) is a word in \( S \). Then \( W =_G 1 \) if and only if there exists a van Kampen diagram \( \Delta \) over the presentation \( \langle S \mid R \rangle \) such that \( \text{Lab}(\partial \Delta) \equiv W \).

**Proof.** If \( W \) is the boundary label of a van Kampen diagram, then \( W =_G 1 \) be the previous exercise. Now suppose that \( W =_G 1 \). Then there exist \( r_1, \ldots, r_k \in R \) and \( f_1, \ldots, f_k \in F(S) \) such that

\[
W =_{F(S)} f_1^{-1} r_1 f_1 \cdots f_k^{-1} r_k f_k.
\]
Each word of the form $f_i^{-1}r_if_i$ is the label of a van Kampen diagram consisting of a path labeled by $f_i$ connected to a 2-cell with boundary label $r_i$. Glueing the initial points of each of these paths together produces a van Kampen diagram with boundary label $f_1^{-1}r_1f_1...f_k^{-1}r_kf_k$ (sometimes called a “wedge of lollipops”). Now $f_1^{-1}r_1f_1...f_k^{-1}r_kf_k$ can be transformed into the word $W$ by a finite sequence of moves consisting of adding or deleting subwords of the form $ss^{-1}$. One can check that if one of these moves is applied to a word $U$ produces $U'$ and $U$ is the boundary label of a van Kampen diagram, then there is a natural move on the diagram which produces a new van Kampen diagram with boundary label $U'$. It follows that the “wedge of lollipops” diagram can be modified by a finite sequence of moves to produce a van Kampen diagram with boundary label $W$.

Exercise 2.4. Suppose $\Delta$ is a van Kampen diagram over a presentation $\langle S \mid R \rangle$ for a group $G$, and $p$ is a closed (combinatorial) path in $\Delta$. Prove that $\text{Lab}(p) = G \triangleright 1$.

From this exercise, it follows that if you fix a vertex $v \in \Delta$, there is a well-defined, label preserving map from the 1-skeleton of $\Delta$ to $\Gamma(G, S)$ which sends $v$ to $1$.

A van Kampen diagram can be interpreted topologically as follows: Any word $W$ defines a path in the Cayley graph $\Gamma(G, S)$, and $W = G \triangleright 1$ if and only if this path is a loop. In this case, there is a homotopy which contracts this loop to a point in the Cayley complex corresponding to the presentation $\langle S \mid R \rangle$. A van Kampen diagram is a combinatorial description of such a homotopy.

### 2.3 Dehn functions

Given a van Kampen diagram $\Delta$, let $\text{Area}(\Delta)$ be the number of 2-cells of $\Delta$. For a fixed group presentation $\langle S \mid R \rangle$ and a word $W$ in $S$ such that $W = G \triangleright 1$, let

$$ \text{Area}(W) = \min \{ \text{Area}(\Delta) \mid \Delta \text{ is a van Kampen diagram over } \langle S \mid R \rangle \text{ and } \text{Lab}(\partial \Delta) \equiv W \}.$$  

equivalently, $\text{Area}(W)$ is equal to the minimal $k$ such that $W$ is equal to a product of $k$ conjugates of elements of $R$ (see (2)).

Exercise 2.5. Let $W$ be a word in $S$ such that $W \equiv W_1UW_2$ and let (some cyclic shift of) $UV^{-1} \in R$ with $V$ possibly the empty word. We say that $W'$ is obtained from $W$ by an $R$-move if $W = F(S)$ $W_1VW_2$. Given $G = \langle S \mid R \rangle$ and a word $S$ such that $W = G \triangleright 1$, prove that $\text{Area}(W)$ is equal to the minimal number of $R$ moves needed to transform $W$ into the empty word.

The Dehn function of a finitely presented group $G$, denoted $\delta_G$, is the function $\delta_G: \mathbb{N} \to \mathbb{N}$ defined by

$$ \delta_G(n) = \max_{\|W\| \leq n} \text{Area}(W) $$

Of course, this depends not only on $G$, but also on the chosen presentation of $G$. In order to make the Dehn function of $G$ independent of the presentation (as is suggested by the notation $\delta_G$), we consider this function as defined only up to the following equivalence relation: functions $f, g: \mathbb{N} \to \mathbb{N}$ are equivalent if there exist constant $A_1, B_1, C_1$ and $A_2, B_2, C_2$ such that for all $n \in \mathbb{N}$,

$$ f(n) \leq A_1g(B_1n) + C_1n \text{ and } g(n) \leq A_2f(B_2n) + C_2n.$$  

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Note that the linear term in the above equivalence is indeed necessary, since even the trivial group has the presentation $\langle s \mid s = 1 \rangle$ and $\text{Area}(s^n) = n$.

**Exercise 2.6.**

(a) Show that this is indeed an equivalence relation.

(b) Show that $f_1(n) = 1$, $f_2(n) = \log n$, and $f_3(n) = n$ are all equivalent.

(c) Show that two polynomials $p$ and $q$ are equivalent if and only if they have the same degree.

(d) Show that $2^n$ and $3^n$ are equivalent.

**Exercise 2.7.**

Prove that a finite group has at most linear Dehn function.

**Exercise 2.8.**

Prove that a finitely generated abelian group has at most quadratic Dehn function, and that the Dehn function of $\mathbb{Z}^2$ is equivalent to $n^2$.

**Examples 2.9.**

1. If $G$ is nilpotent of class $c$, then $\delta_G(c) \leq n^{c+1}$

2. If $G$ is the fundamental group of a compact, orientable surface of genus $g \geq 2$, then Dehn’s algorithm shows that $\delta_G$ is linear.

3. The Dehn function of $BS(1, 2) = \langle a, t \mid t^{-1}at = a^2 \rangle$ is equivalent to $2^n$

4. For $G = \langle a, b, c \mid a^b = c, a^c = a^2 \rangle$, $\delta_G$ is equivalent to $2^{2^2} \cdots n$, where this tower has length $\log_2(n)$.

5. $SL(2, \mathbb{Z})$ has linear Dehn function. $SL(3, \mathbb{Z})$ has exponential Dehn function. For $m \geq 5$, the Dehn function of $SL(m, \mathbb{Z})$ is quadratic. The Dehn function of $SL(4, \mathbb{Z})$ is unknown, but is conjectured to be quadratic.

If $M$ is a Riemannian manifold and $G = \pi_1(M)$, then the Dehn function $\delta_G$ is equivalent to the isoperimetric function on the universal cover $\tilde{M}$, that is the function which measures the maximal area of a disc whose boundary is a curve of length at most $n$.

**Definition 2.10.**

Give an presentation $\langle S \mid R \rangle$ for a group $G$, A **Tietze transformation** on $\langle S \mid R \rangle$ one of the following four types of operations:

1. (Add a generator) $\langle S \mid R \rangle \to \langle S \cup \{t\} \mid R \cup \{t^{-1}W\} \rangle$, where $t \notin S$ and $W$ is any word in $S$.

2. (Remove a generator) $\langle S \mid R \rangle \to \langle S \setminus \{s\} \mid R' \rangle$, where $s \in S$ and there exist a word $W$ in $S \setminus \{s\}$ such that $s =_G W$ and $R'$ is obtained from $R$ by replacing each occurrence of $s^{\pm 1}$ with $W^{\pm 1}$.

3. (Add a relation) $\langle S \mid R \rangle \to \langle S \mid R \cup \{W\} \rangle$ where $W \in \langle \langle R \rangle \rangle$ in the free group $F(S)$.

4. (Remove a relation) $\langle S \mid R \rangle \to \langle S \mid R \setminus \{U\} \rangle$ where $U \in R$ such that $U \in \langle \langle R \setminus U \rangle \rangle$ in the free group $F(S)$.

**Exercise 2.11.** Check that applying one Tietze transformation to a group presentation produces a presentation with equivalent Dehn function.

**Theorem 2.12.** [21, Proposition 2.1] Suppose $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ with both presentation finite. Then $G_1$ is isomorphic to $G_2$ if and only if there is a finite sequence of Tietze transformations which turn $\langle S_1 \mid R_1 \rangle$ into $\langle S_2 \mid R_2 \rangle$. 

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It follows from the previous theorem and exercise that the Dehn function of a finitely presented group $G$ is independent of the choice of finite presentation up to the equivalence relation given above. We next prove that the Dehn function is also a quasi-isometry invariant.

**Theorem 2.13.** Suppose $G$ is finitely presented and $H$ is finitely generated. If $G \cong_{q_i} H$, then $H$ is finitely presented and $\delta_G$ is equivalent to $\delta_H$.

**Proof.** Let $\langle S | R \rangle$ be a finite presentation for $G$ and let $T$ a finite generating set for $H$. Let $M = \max_{\langle r \rangle} \{||r|| \mid r \in R\}$. Let $f : \Gamma(H, T) \to \Gamma(G, S)$ be a $(\lambda, c, \varepsilon)$ quasi-isometry. Let $p$ be a closed (combinatorial) path in $\Gamma(H, T)$, and let $v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ denote the vertices of $p$. Let $q$ be the closed path in $\Gamma(G, S)$ formed by connecting each $f(v_i)$ to $f(v_{i+1})$ by a geodesic. Since $d_T(v_i, v_{i+1}) = 1, d_S(f(v_i), f(v_{i+1})) \leq \lambda + c$, and hence $\ell(q) \leq (\lambda + c)n$. Since $q$ is a closed path, $\text{Lab}(q) =_G 1$, so there exists a van Kampen diagram $\Delta$ with $\text{Lab}(\partial \Delta) \equiv \text{Lab}(q)$. We also choose $\Delta$ such that $\text{Area}(\Delta) \leq \delta_G((\lambda + c)n)$. We identify the 1-skeleton of $\Delta$ with its image in $\Gamma(G, S)$ under the natural map $\Delta^{(1)} \to \Gamma(G, S)$ which sends $\partial \Delta$ to $q$. Now we build a map $g : \Delta^{(1)} \to \Gamma(H, T)$ for each interior vertex $v \in \Delta$, choose a vertex $u \in \Gamma(H, T)$ such that $d_S(v, f(u)) \leq \varepsilon$, and set $g(v) = u$. Each exterior vertex $v \in \Delta$ lies on some geodesic $[f(v_i), f(v_{i+1})]$; if $v$ is closer to $f(v_i)$ we set $g(v) = v_i$, otherwise we set $g(v) = v_{i+1}$. Now if two vertices $v$ and $u$ are adjacent, then we join $g(v)$ and $g(u)$ by geodesics in $\Gamma(H, T)$. Note that for such $u$ and $v$, $d_S(f(g(u)), f(g(v))) \leq 2\varepsilon + 1$, and hence $d_T(g(u), g(v)) \leq (2\varepsilon + 1 + c)\lambda$. It follows that if $\Pi$ is a 2-cell of $\Delta$, there is a closed loop in $g(\Delta^{(1)})$ corresponding to the image of $\partial \Pi$ of length at most $(2\varepsilon + 1 + c)\lambda \ell(\partial \Pi) \leq (2\varepsilon + 1 + c)\lambda M$.

Let $R' = \{r \in F(T) \mid ||r|| \leq (2\varepsilon + 1 + c)\lambda M \text{ and } r =_H 1\}$. From above, we have that there is a van Kampen diagram $\Delta'$ whose 1-skeleton is $g(\Delta^{(1)})$ and each two cell is labeled by an element of $R'$. Hence $\Delta'$ is a van Kampen diagram over $(T | R')$ and $W \equiv \text{Lab}(\partial \Delta')$. Thus $\langle T | R' \rangle$ is a presentation for $H$, in particular $H$ is finitely presented. Furthermore,$$
\text{Area}(W) \leq \text{Area}(\Delta') = \text{Area}(\Delta) \leq \delta_G((\lambda + c)n)
$$
since $W$ is an arbitrary word of length $n$, we get that $\delta_H(n) \leq \delta_G((\lambda + c)n)$. Reversing the roles of $G$ and $H$ in the above proof will result in the reverse inequality (with possibly different constants), hence $\delta_G$ is equivalent to $\delta_H$.

**Corollary 2.14.** Finite presentability is a quasi-isometry invariant.

### 2.4 Algorithmic problems

The following algorithmic problems were introduced by Max Dehn in 1911.

**Word Problem:** Given a presentation $\langle S | R \rangle$ of a group $G$, find an algorithm such that for any word $W$ in $S$, the algorithm determines whether or not $W =_G 1$.

**Conjugacy Problem:** Given a presentation $\langle S | R \rangle$ of a group $G$, find an algorithm such that for any two words $W$ and $U$ in $S$, the algorithm determines whether or not $W$ and $U$ represent conjugate elements of the group $G$.

**Isomorphism Problem:** Find an algorithm which accepts as input two group presentations and determines whether or not they represent isomorphic groups.
Exercise 2.15. Describe an algorithm which solves the word problem for the standard presentation of $\mathbb{Z}^n$, that is $\langle a_1, \ldots, a_n \mid [a_i, a_j], 1 \leq i < j \leq n \rangle$.

Note that the word problem is equivalent to deciding whether a given element of $F(S)$ belongs to the normal subgroup $\langle \langle R \rangle \rangle$. It is not hard to see that there is an algorithm for listing all elements of the normal subgroup $\langle \langle R \rangle \rangle$, but by itself this algorithm will not be able to certify that a given element $g \notin \langle \langle R \rangle \rangle$.

The following is a classical result in computability and group theory, first proved by Novikov in 1955; another proof was given by Boone in 1958.

Theorem 2.16 (Novikov-Boone). There exists a finite group presentation for which the word problem is undecidable.

Note that the word problem can be viewed as a special case of the conjugacy problem, since $W =_1 G$ if and only if $W$ is conjugate to 1 in $G$. It follows that any group with undecidable word problem will also have undecidable conjugacy problem. There do, however, exist group presentations with decidable word problem but undecidable conjugacy problems.

Similarly, the isomorphism problem is undecidable in general, though as with the other algorithmic problems it can solved in certain special cases, that is if one only considers presentations which represent groups belonging to a specific class of groups.

Proving the existence of a group with undecidable word problem is quite difficult, but once it is known that such a group exists many other algorithmic questions about groups can be reduced to the word problem and hence proved to be undecidable in general. For more details, see the Adian-Rabin theorem.

Given a van Kampen diagram $\Delta$, we define the type of $\Delta$ by the ordered pair of natural numbers $(\text{Area}(\Delta), \ell(\partial \Delta))$.

Exercise 2.17. Show that for a finite presentation $\langle S \mid R \rangle$ and a fixed type $(k, n)$, there are only finitely many van Kampen diagrams over $\langle S \mid R \rangle$ of type $(k, n)$.

A function $f : \mathbb{N} \to \mathbb{N}$ is called recursive is there exists an algorithm which computes $f(n)$ for all $n \in \mathbb{N}$.

Theorem 2.18. Let $G$ be a finitely presented group. The following are equivalent.

1. $\delta_G$ is recursive.

2. There exists a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $\delta_G(n) \leq f(n)$.

3. The word problem in $G$ is solvable.

Proof. Fix a finite presentation $\langle S \mid R \rangle$ for the group $G$.

(1) $\implies$ (2)

Trivial.

(2) $\implies$ (3)
Let $W$ be a word in $S$ with $W = G_1$ and $\|W\| = n$. By assumption, there exists a van Kampen diagram $\Delta$ with $\text{Lab}(\partial \Delta) \equiv W$ and $\text{Area}(\Delta) \leq \delta_G(n) \leq f(n)$. However, by the previous exercise there are only finitely many van Kampen diagrams of type $(k, n)$ with $1 \leq k \leq f(n)$. Hence one can list all of these diagrams; if a some diagram in this list has boundary label $W$, then $W = G_1$, otherwise $W \neq G_1$.

(3) $\implies$ (2).

Fix $n \in \mathbb{N}$, and let $R_n$ be the set of words $W$ in $S$ such that $\|W\| \leq n$ and $W = G_1$. This set can be explicitly computed by applying the algorithm which solves the word problem in $G$ to each word of length at most $n$. Now for each $W \in R_n$, we can compute $\text{Area}(W)$ by listing all van Kampen diagrams of type $(1, \|W\|)$, then type $(2, \|W\|)$ etc. Since we know $W = G_1$, there must be some $k$ such that this list produces a van Kampen diagram with boundary label $W$ and area $k$; if $k$ is the smallest natural number for which such a diagram occurs, then $\text{Area}(W) = k$. Hence we can compute the area of each of the the finitely many words in $R_n$, and by definition $\delta_G(n)$ is the maximum of these areas.

Corollary 2.19. Solvability of the word problem is a quasi-isometry invariant.

The Dehn function can be interpreted as a measure of the geometric complexity of the word problem in a group $G$. In particular, the Dehn will give an upper bound on the time complexity of the word problem in $G$. In general, however, this upper bound is far from being sharp as can be seen in the following example:

Example 2.20. Recall that for $G = BS(1,2) = \langle a, t | t^{-1}at = a^2 \rangle$, $\delta_G \sim 2^n$. However, the time complexity of the word problem in $G$ is at most polynomial in $n$. Indeed, $G$ is isomorphic to the subgroup of $GL(2, \mathbb{Q})$ generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so the time complexity of the word problem in $G$ is at most the time complexity of multiplying $2 \times 2$ matrices.

3 Hyperbolic groups

3.1 Hyperbolic metric spaces

Before we define hyperbolic groups, we need to defined hyperbolic metric spaces and study some basic properties of their geometry. In particular, we need to show that hyperbolicity is invariant under quasi-isometry in order for hyperbolicity to be well-defined in the world of groups.

The following is the mostly commonly cited definition of hyperbolicity and is attributed to Rips.

Definition 3.1 (Slim Triangles or the Rips Condition). Let $\delta \geq 0$. We say that a geodesic metric space $X$ is $\delta$–hyperbolic if for any geodesic triangle $T$ in $X$ with sides $p, q, r$ and any point $a \in p$, there exists $b \in q \cup r$ such that $d(a, b) \leq \delta$. We say $X$ is hyperbolic if it is $\delta$–hyperbolic for some $\delta \geq 0$.

A triangle $T$ which satisfies the conditions in this definition is called $\delta$-slim.
Exercise 3.2. Let \( X \) be a \( \delta \)-hyperbolic geodesic metric space and \( P = p_1p_2...p_n \) a geodesic \( n \)-gon in \( X \) for \( n \geq 3 \). Let \( a \) be a point on \( p_i \) for some \( 1 \leq i \leq n \). Prove that there exists \( j \neq i \) and \( b \in p_j \) such that \( d(a,b) \leq (n-2)\delta \). (In fact, \( n - 2 \) can be replaced by \( \log_2(n) \)).

Examples 3.3.  
1. If \( X \) is a bounded metric space, then \( X \) is \( \delta \)-hyperbolic for \( \delta = \text{diam}(X) \).

2. \( \mathbb{R} \) with the standard metric is \( 0 \)-hyperbolic.

3. Is \( X \) is a simplicial tree, that is a connected graph with no cycles equipped with the combinatorial metric, then \( X \) is \( 0 \)-hyperbolic (equivalently, every triangle is a tripod).

4. Generalizing the previous two examples, a \( 0 \)-hyperbolic geodesic metric space is called a \( \mathbb{R} \)-tree. Some more examples of \( \mathbb{R} \)-trees:

   (a) \( X = \{(x,y) \mid x \in [0,1], y = 0\} \cup \{(x,y) \mid x \in \mathbb{Q}, y \in [0,1]\} \) with the metric 
   \[ d((x_1,y_1),(x_2,y_2)) = |y_1| + |x_2-x_1| + |y_2| \text{ when } x_1 \neq x_2 \] 
   and 
   \[ d((x_1,y_1),(x_2,y_2)) = |y_2-y_1| \text{ otherwise}. \]

   (b) \( X = \mathbb{R}^2 \) with the following metric: If the line containing \((x_1,y_1)\) and \((x_2,y_2)\) passes through the origin, then 
   \[ d((x_1,y_1),(x_2,y_2)) \] 
   is the usual Euclidean distance. Otherwise, 
   \[ d((x_1,y_1),(x_2,y_2)) = \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}. \]

5. \( \mathbb{R}^n \) with the Euclidean metric is not hyperbolic for any \( n \geq 2 \).

6. The classical hyperbolic space \( \mathbb{H}^2 \) is \( \delta \)-hyperbolic. Recall that a triangle \( T \) in \( \mathbb{H}^2 \) with angles \( \alpha, \beta, \) and \( \gamma \) has area \( \pi - \alpha - \beta - \gamma \). For a point \( x \) on \( T \), consider the largest semi-circle contained in \( T \) and centered at \( x \). This semi-circle has area at most the area of \( T \) which is at most \( \pi \); this provides a bound on the radius of the semi-circle, which can be explicitly computed to show that \( \mathbb{H}^2 \) is \( \delta \)-hyperbolic for \( \delta = 4\log \varphi \), where \( \varphi = \frac{1+\sqrt{5}}{2} \) is the golden ratio.

7. From the previous example, it follows that \( \mathbb{H}^n \) is \( \delta \)-hyperbolic for all \( n \geq 2 \).

8. If \((X,d)\) is any metric space, then we can define a new metric \( d \) on \( X \) by 
   \[ d(x,y) = \log(1 + d(x,y)). \] 
   Then \((X,d)\) is \( 2\log \) \( 2 \) \( \)-hyperbolic.

9. \( O(n,1), U(n,1), SP(n,1) \) are all hyperbolic when given left-invariant Riemannian metrics.

Definition 3.4. Let \( X \) be a metric space and \( A, B \) closed subsets of \( X \). The Hausdorff distance between \( A \) and \( B \) is the infimum of all \( \varepsilon \) such that \( A \subseteq B^{+\varepsilon} \) and \( B \subseteq A^{+\varepsilon} \). We denote this distance by 
\[ d_{\text{Haus}}(A,B). \]

Exercise 3.5. Verify that \( d_{\text{Haus}} \) is a metric on the set of closed subsets of \( X \).

Definition 3.6. Suppose \( X \) is a metric space and \( p: [a,b] \to X \) is a \((\lambda,C)\)-quasi-isometry onto its image in \( X \). Then \( p \) is called a \((\lambda,C)\) quasi-geodesic.

If \( p \) is continuous (and hence a path), then we say that \( p \) is a \((\lambda,C)\) quasi-geodesic path if for any subpath \( q \) of \( p \), 
\[ \ell(q) \leq \lambda d(q_-,q_+) + C \]
Note that, in general, quasi-geodesic are not required to be continuous.

Given these definitions, being a quasi-geodesic path depends only on the image of $p$ while being a quasi-geodesic depends on the chosen parameterization. However, any quasi-geodesic path is in fact a quasi-geodesic when it is parameterized by arc length, that is when $p: [a, b] \to X$ is such that for all $a \leq s < t \leq b$, $\ell(p|_{s,t}) = |t - s|$. Furthermore, the next lemma shows that every quasi-geodesic is close to a quasi-geodesic path.

**Lemma 3.7.** Let $p: [a, b] \to X$ be a $(\lambda, c)$-quasi-geodesic. Then there exists $\lambda', c'$, and $D$ depending only on $\lambda$ and $c$ and there exits a $(\lambda', c')$ quasi-geodesic path $p'$ with the same endpoints as $p$ such $d_{Hau}(p, p') < D$

**Sketch.** Define $p'(t) = p(t)$ for all $t \in \mathbb{Z} \cap [a, b]$. Now “connect the dots” by geodesics. Verifying that $p'$ satisfies the above conditions is straightforward. □

**Exercise 3.8.** Let $p: [a, b] \to X$ be a geodesic and let $f: X \to Y$ be a $(\lambda, c)$ quasi-isometric embedding. Prove that $f \circ p$ is a $(\lambda, c)$ quasi-geodesic.

**Definition 3.9.** Suppose $X$ is a metric space and $p$ is a path in $X$. $p$ is called a $k$-local geodesic if every subpath of $p$ of length $\leq k$ is a geodesic.

**Remark 3.10.** Quasi-geodesic rays, bi-infinite quasi-geodesics, local geodesic rays, and bi-infinite local geodesics are all similarly defined in the obvious ways.

We will assume for the rest of this section that $X$ is a geodesic and $\delta$-hyperbolic metric space.

**Lemma 3.11.** Let $p$ be a (rectifiable) path in $X$ from $x$ to $y$. Then for any geodesic $[x, y]$ and any point $a \in [x, y]$, there exists $b \in p$ such that

\[ d(a, b) \leq \delta \log_2(\ell(p)) + 1 \]

**Proof.** We assume that $p$ is parameterized such that $p: [0, 1] \to X$ and for all $0 \leq i < j \leq 1$, $\ell(p|_{i,j}) = \frac{1}{j-i} \ell(p)$. Choose $N$ such that $N^2 \leq \ell(p) \leq 2N+1$. Let $z_1 = p\left(\frac{1}{2}\right)$. Let $T_1$ be a triangle with sides $[x, y]$, $[x, z_1]$, and $[z_1, y]$. Since $T_1$ is $\delta$-slim, there exists a point $b_1 \in [x, z_1] \cup [z_1, y]$ with $d(a, b_1) \leq \delta$. If $b_1 \in [x, z_1]$, let $z_2 = p\left(\frac{1}{3}\right)$ and $T_2 = [x, z_1][x, z_2][z_1, z_2]$; if $b_1 \in [z_1, y]$, let $z_2 = p\left(\frac{2}{3}\right)$ and $T_2 = [z_1, y][z_1, z_2][z_2, y]$. We apply slinness to $T_2$ and $b_1$ to find a point $b_2$ on one of the other two sides of $T_2$ that is $\delta$ close to $b_1$. We then define $z_3$ as the midpoint of the subpath of $p$ that is “above” the side of $T_2$ containing $b_2$ and $T_3$ as the triangle which contains the side of $T_2$ that contains $b_2$ and geodesics connecting the endpoints of this side to $z_3$. Continue this process inductively until we obtain $b_N$.

Note that by construction, for each $1 \leq i \leq N - 1$, $d(b_i, b_{i+1}) \leq \delta$, and hence $d(a, b_N) \leq N\delta \leq \delta \log_2(\ell(p))$. Furthermore, $b_N$ belongs to a geodesic $q$ with endpoints on $p$ such that $\ell(q) \leq \ell(p)$. Let $b \in p$ be the closest endpoint of $q$ to $b_N$, hence $d(b, b_N) \leq \frac{1}{2} \ell(q) \leq 1$. Therefor,

\[ d(a, b) \leq d(a, b_N) + d(b_N, b) \leq \delta \log_2(\ell(p)) + 1. \] □
Theorem 3.12 (Morse Lemma). Let \( X \) be a \( \delta \)-hyperbolic metric space. Then for any \( \lambda \geq 1 \), \( C \geq 0 \), there exists \( K = K(\delta, \lambda, C) \) such that for any geodesic \( p \) and any \((\lambda, C)\)-quasi-geodesic \( q \) with \( p_- = q_- \) and \( p_+ = q_+ \), \( d_{\text{Haus}}(p, q) \leq K \).

Proof. Note that, by Lemma 3.7, we can assume that \( q \) is a quasi-geodesic path. Let \( D = \sup_{x \in p} \{d(x, q)\} \); our first goal will be to bound \( D \) in terms of \( \delta, \lambda \) and \( C \). Since \( p \) and \( q \) are compact, there is point \( x_0 \in p \) which realizes this supremum. In particular, the the interior of \( B_D(x_0) \) does not intersect \( q \). Now choose \( y \in [p_-, x_0] \) such that \( d(x_0, y) = 2D \), or if no such \( y \) exists then set \( y = p_- \). Choose \( z \in [x_0, p_+] \) similarly. By definition of \( D \), there exists some \( y', z' \in q \) such that \( d(y, y') \leq D \) and \( d(z, z') \leq D \). By the triangle inequality, \[ d(y', z') \leq d(y', y) + d(y, z) + d(z, z') \leq 6D \]

If \( q' \) is the subpath of \( q \) joining \( y' \) to \( z' \), then since \( q \) is a \((\lambda, C)\)-quasi-geodesic, \( \ell(q') \leq 6\lambda D + C \).

Let \( c = [y, y']q'[z', z] \), and note that \( \ell(c) \leq 6\lambda D + C + 2D \) and \( d(x_0, c) = D \). By Lemma 3.11, \( d(x_0, c) \leq \delta \| \log_2(\ell(c)) \| + 1 \), and combinging this with the previous estimates gives \[ D \leq \delta \| \log_2(6\lambda D + 2D + C) \| + 1. \]

This equation implies that \( D \) must be bounded in terms of \( \delta, \lambda \) and \( C \).

It remains to show that \( q \) is contained in a bounded neighborhood of \( p \). Suppose \( q = q_1q_2q_3 \) such that \( q_3 \) is a maximal subpath of \( q \) which lies outside \( p^\perp D \). Now every point of \( p \) is within \( D \) of some point on either \( q_1 \) or \( q_3 \); by connectedness of \( p \), there must exist some \( x \in p \) and \( y \in q_1 \), \( z \in q_3 \) such that \( d(x, y) \leq D \) and \( d(x, z) \leq D \). In particular, this means that \( \ell(q_2) \leq \lambda(2D) + C \).

It follows that \( q \) is contained in the \( 2\lambda D + D + C \) neighborhood of \( p \).

Corollary 3.13. Let \( X \) be a \( \delta \)-hyperbolic metric space. Then for any \( \lambda \geq 1 \), \( C \geq 0 \), there exists \( \kappa = \kappa(\delta, \lambda, C) \) such that for any \((\lambda, C)\)-quasi-geodesics \( p \) and \( q \) with \( p_- = q_- \) and \( p_+ = q_+ \), \( d_{\text{Haus}}(p, q) \leq \kappa \).

Exercise 3.14. Prove that there exist \( \lambda \geq 1 \) and \( C \geq 0 \) such that for any \( K \geq 0 \), there exists a \((\lambda, C)\)-quasi-geodesic \( q \) in \( \mathbb{R}^2 \) such that \( d_{\text{Haus}}(q, [q_-, q_+]) \geq K \).

Exercise 3.15. Let \( X \) be a geodesic metric space. Prove \( X \) is hyperbolic if and only if for all \( \lambda \geq 1 \), \( C \geq 0 \) that there exists \( \delta' \) such that for any triangle \( T \) in \( X \) whose sides are \((\lambda, C)\)-quasi-geodesics is \( \delta' \)-slim.

Proposition 3.16. Let \( X \) be a hyperbolic metric space. Suppose \( Y \) is a geodesic metric space and \( f : Y \to X \) is a quasi-isometric embedding. Then \( Y \) is hyperbolic.

Proof. Let \( f : Y \to X \) be a \((\lambda, c)\) quasi-isometric embedding and let \( p_i : [a_i, b_i] \to Y \) be geodesics for \( 1 \leq i \leq 3 \) such that \( T = p_1p_2p_3 \) is a geodesic triangle in \( Y \). By Exercise 3.8 \( f(T) = f(p_1)f(p_2)f(p_3) \) is a \((\lambda, c)\) quasi-geodesic triangle in \( X \), and hence \( f(T) \) is \( \delta' \)-slim for some \( \delta' \) depending only on \( \delta, \lambda, \) and \( c \) by Exercise 3.15.

Now let \( x \) be a point on \( p_1 \) and let \( x' = f(x) \in f(p_1) \). Then there exists some \( y' \in f(p_2) \cup f(p_3) \) such that \( d_X(x', y') \leq \delta' \). Without loss of generality, suppose \( y' \in f(p_2) \) and \( y' = f(p_2(s)) \) for some \( a_2 \leq s \leq b_2 \). Let \( y = p_2(s) \). Then \( d_Y(x, y) \leq \lambda \delta' + \lambda c \), hence \( T \) is \((\lambda \delta' + \lambda c)\)-slim.
Corollary 3.17. Suppose $X$ and $Y$ are geodesic metric spaces and $X \sim_{qi} Y$. Then $X$ is hyperbolic if and only if $Y$ is hyperbolic.

Definition 3.18. A finitely generated group $G$ is hyperbolic if for some (equivalently, any) finite generating set $S$, $\Gamma(G,S)$ is a hyperbolic metric space.

Remark 3.19. By the Milnor-Svarc Lemma, a group $G$ is hyperbolic if and only if $G$ admits a proper, cobounded action on a geodesic hyperbolic metric space.

Examples 3.20. 1. Finite groups are hyperbolic.

2. $\mathbb{Z}$ is hyperbolic. More generally, any group which is virtually $\mathbb{Z}$, such as $\mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z}$.

3. $F_n$ is hyperbolic for any $n \geq 1$.

4. $SL(2,\mathbb{Z})$ is virtually free and hence hyperbolic.

5. If $M$ is a closed hyperbolic manifold, then $\pi_1(M)$ is hyperbolic. In particular, if $S$ is an orientable surface of genus $g$, then $\pi_1(S)$ is hyperbolic if and only if $g \geq 2$.

6. $\mathbb{Z}^n$ is hyperbolic if and only if $n = 1$.

Definition 3.21. A group is called elementary if it contains a cyclic subgroup of finite index

There are three types of elementary groups: every elementary group is either finite, finite-by-$\mathbb{Z}$, or finite-by-$D_\infty$, where $D_\infty$ is the infinite dihedral group. All elementary groups are hyperbolic, but there will be a number of results which only hold for the non-elementary hyperbolic groups.

Lemma 3.22. Suppose $p$ is a $k$-local geodesic in $X$ from $x$ to $y$ for $k > 8\delta$. Then

1. $p \subseteq [x,y]^{+2\delta}$.

2. $[x,y] \subseteq p^{+3\delta}$.

3. $p$ is a $(\lambda,c)$-quasi-geodesic path for $\lambda = \frac{k+4\delta}{k+4\delta}$ and $c = 2\delta$.

Proof. (1) Choose a point $a \in p$ which maximizes the distance to $[x,y]$. Choose $b,c \in p$ such that $a$ is the midpoint of the subpath of $p$ from $b$ to $c$ and $8\delta < d(b,c) \leq k$. (if such points do not exist, we use the endpoints of $p$ instead and an obvious modification of the following argument will work). Choose $b', c'$ as the points on $[x,y]$ closest to $b$ and $c$ respectively, and consider the quadrilateral $(b',b,c,c')$. $a$ must be $2\delta$ from one of the other sides of this quadrilateral by hyperbolicity. If $a$ is within $2\delta$ of a point on $[b',b]$ or $[c,c']$, it would contradict our choice of $a$ as the point which maximizes the distance to $[x,y]$. Hence $a$ is within $2\delta$ of a point on $[b',c'] \subseteq [x,y]$.

(2) Now let $a \in [x,y]$. Since $p$ is connected, there exists some $b \in p$ such that $d(b,[x,a]) \leq 2\delta$ and $d(b,[a,y]) \leq 2\delta$. Applying hyperbolicity to the triangle spanned by $b$ and the two points which realize these inequalities produces the desired result.

(3) We subdivide $p$ into subpaths $p = p_1p_2...p_{n+1}$ such that $\ell(p_i) = k' = \frac{k}{2} + 2\delta$ for $1 \leq i \leq n$ and $0 \leq \ell(p_{n+1}) = \eta < k'$. Note that

$$\ell(p) = nk' + \eta$$

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Now let $a_i = (p_i)_-$, and let $a_i'$ be a point on $[x, y]$ with $d(a_i, a_i') \leq 2\delta$. We first need to show that each $a_i'$ is “between” $a_{i-1}'$ and $a_{i+1}'$ on $[x, y]$, which will imply that $x = a_1', a_2'...a_{n+1}', y$ forms a monotone sequence along $[x, y]$.

Let $x_0 \in p_{i-1}$ with $d(a_{i-1}, x_0) = 2\delta$ and $y_0 \in p_i$ with $d(a_{i+1}, y_0) = 2\delta$. Note that $d(x_0, y_0) = 2k' - 4\delta = k$, hence a geodesic $[x_0, y_0]$ can be chosen as a subpath of $p$. Consider the triangle $T$ with endpoints $a_{i-1}, a_{i+1}'$, and $x_0$. By hyperbolicity, $T \subseteq B_{3\delta}(a_{i-1})$. Since $d(a_{i-1}, a_i) = k' > 6\delta$, $T$ does not intersect $B_{3\delta}(a_i)$. Similarly, a triangle with endpoints $a_{i+1}, a_{i+1}'$, and $y_0$ will not intersect $B_{3\delta}(a_i)$. Now we apply hyperbolicity to the quadrilateral with vertices $a_{i-1}', x_0, y_0, a_{i+1}$ and the point $a_i$, we get a point $a_i'' \in [a_{i-1}', a_{i+1}']$ with $d(a_i, a_i'') \leq 2\delta$. By hyperbolicity of the triangle $(a_i, a_i', a_i''), d(a_i, z) \leq 3\delta$ for any point $z$ which is between $a_i'$ and $a_i''$. In particular, neither $a_{i-1}'$ nor $a_{i+1}'$ is between $a_i'$ and $a_i''$, and since $a_i'' \in [a_{i-1}', a_{i+1}']$, we must also have $a_i' \in [a_{i-1}', a_{i+1}']$.

Since $x = a_1', a_2'...a_{n+1}', y$ forms a monotone sequence along $[x, y]$, we get that

$$d(x, y) = \sum_{i=1}^{n} d(a_i', a_{i+1}') + d(a_{n+1}', y)$$

Now for each $1 \leq i \leq n$, $d(a_i', a_{i+1}') \geq k' - 4\delta$, and $d(a_{n+1}', y) \geq \eta - 2\delta$. Hence,

$$d(x, y) \geq nk' - 4\delta n + \eta - 2\delta = \ell(p) - 4\delta n - 2\delta$$

Finally, since $n \leq \frac{\ell(p)}{k'}$,

$$d(x, y) \geq \left(\frac{k' - 4\delta}{k'}\right)\ell(p) - 2\delta.$$ 

Finally, it only remains to note that every subpath of $p$ is again a $k$-local geodesic to which the above proof applies. □

**Corollary 3.23.** Suppose $p$ is a $k$-local geodesic in $X$ for $k > 8\delta$. Then either $p$ is constant or $p_- \neq p_+$.

### 3.2 Algorithmic and isoperimetric characterizations of hyperbolic groups

Given a group presentation $\langle S | R \rangle$ and a word $W$ in $S$, Dehn’s algorithm is the following procedure: First freely reduce $W$; if this produces the empty word, the algorithm stops. Now if $W$ is freely reduced and non-empty, search $W$ for subwords $U$ such that $U$ is also a subword of relation (or a cyclic shift of a relation) $r \in R$ and $\|U\| > \frac{1}{2}\|r\|$. If no such subword exists, the algorithm stops. If such a $U$ exists, then there is a (possibly empty) word $V$ (the complement of $U$ in $r$) such that $UV^{-1} = G_1$ and $\|V\| < \|U\|$. In this case, the algorithm replaces $U$ with $V$ and repeats.

If the presentation $\langle S | R \rangle$ is finite, then Dehn’s algorithm terminates after finitely many steps for any word $W$.

**Definition 3.24.** Let $\langle S | R \rangle$ be a finite presentation for a group $G$. Dehn’s algorithm solves the word problem for $\langle S | R \rangle$ if for any non-empty word $W$ for which Dehn’s algorithm stops, $W \neq G_1$.

**Exercise 3.25.** Find a group presentation for which Dehn’s algorithm does not solve the word problem.
Exercise 3.26. Suppose \( \langle S|R \rangle \) is a finite presentation for a group \( G \) for which Dehn’s algorithm solves the word problem. Prove that \( G \) has linear Dehn function.

**Theorem 3.27.** For any finitely generated group \( G \), the following are equivalent.

1. \( G \) is hyperbolic.
2. \( G \) has a finite presentation \( \langle S|R \rangle \) for which Dehn’s algorithm solves the word problem.
3. \( G \) is finitely presented and has linear Dehn function.
4. \( G \) is finitely presented and has subquadratic Dehn function.

Exercise 3.28. Suppose \( G \) and \( H \) are hyperbolic. Prove that \( G \ast H \) is hyperbolic.

**Proof.** (1) \( \Rightarrow \) (2)

Let \( S \) be a finite, symmetric generating set of \( G \), and let \( \delta \) be the hyperbolicity constant of \( \Gamma(G,S) \). Let \( R = \{ U \mid U \) is a word in \( S \) and \( \|U\| \leq 16\delta, U =_G 1 \} \). We will show that \( \langle S|R \rangle \) is a presentation for \( G \) for which Dehn’s algorithm solves the word problem.

Let \( W \) be a non-empty word in \( S \) such that \( W =_G 1 \). Let \( p \) be the path in \( \Gamma(G,S) \) with \( p_\sim = 1 \) and \( \text{Lab}(p) \equiv W \). By assumption, \( p \) is not constant and \( p_\sim = p_+ \), so by Corollary 3.23 \( p \) is not a \( k \)-local geodesic for any \( k > 8\delta \). This means that \( p \) contains a subpath \( q \) with \( \ell(q) \leq 8\delta \) such that \( q \) is not a geodesic. Let \( r \) be a geodesic from \( q_- \) to \( q_+ \). Then \( \ell(r) < \ell(q) \leq 8\delta \), so \( qr^{-1} \) is a closed loop with \( \ell(qr^{-1}) \leq 16\delta \). This means that \( \text{Lab}(qr^{-1}) \in R \), and since \( \ell(r) < \ell(q) \), Dehn’s algorithm will not stop on \( W \).

Thus we have shown that every word in \( S \) which is equal to 1 in \( G \) by be reduced to the empty word via Dehn’s algorithm using only relations from \( R \). Therefore, \( \langle S|R \rangle \) is a finite presentation for \( G \) and Dehn’s algorithm solves the word problem for \( \langle S|R \rangle \).

(2) \( \Rightarrow \) (3) by Exercise 3.26

(3) \( \Rightarrow \) (4) is trivial.

We will need a few auxillary results before proving the final implication. First, however, I would like to highlight the following consequence of the above theorem, which is a purely algebraic consequence of the geometric assumption of hyperbolicity.

**Corollary 3.29.** If \( G \) is a hyperbolic group, then \( G \) is finitely presented.

In fact, this corollary is a special case of a more general finiteness phenomenon for hyperbolic groups which we will see later when we introduce the Rips complex.

Now, we return to the proof of Theorem 3.27.

Given a polygon \( P = p_1p_2...p_n \) in \( X \), we say \( P \) is \( t \)-slim if for any point \( a \in p_i \), there exists \( j \neq i \) and a point \( b \in p_j \) such that \( d(a,b) \leq t \). We define the thickness of \( P \), denoted \( t(P) \), as the minimal constant \( t \) such that \( P \) is \( t \)-slim. Clearly, if \( X \) is non-hyperbolic it will have triangles of arbitrarily large thickness. We will show that in this case there are polygons or arbitrarily large
thickness \( t \) whose perimeter length is linear in \( t \). Next we will show that the area of a polygon \( P \) is bounded below by a quadratic function of the thickness of \( P \). These results together will finish the proof of Theorem 3.27.

For the next two lemmas I am following the proofs from [25].

**Lemma 3.30** (Thick polygons with linear perimeter). Suppose a geodesic metric space \( X \) is not hyperbolic. Then for all \( t_0 \geq 0 \), there exists \( t \geq t_0 \) such that \( X \) contains a polygon of thickness \( t \) whose perimeter length is at most \( 4t \).

*Proof.* Let \( T = pqr \) be a geodesic triangle in \( X \) with \( x = p_+ = q_- \), \( y = q_+ = r_- \), and \( z = p_+ = r_+ \). Let \( a \in p \) a point such that \( d(a, q) = d(a, r) = t \geq t_0 \), and \( b \in q, c \in r \) such that \( d(a, b) = t = d(a, c) \).

Let \( e \in [b, y] \) such that \( d(d, e) = 7t \) (or \( e = y \) if no such point exists). Let \( f \) be the point of \([y, c]\) which is closest to \( e \).

**Case 1:** \( d(e, f) \geq 4t \). In this case we analyze the triangle with vertices \( b, c, \) and \( y \). Choose a point \( o \in [b, y] \) which maximizes \( d(o, [y, c]) \). Note that by our assumption, \( d(o, [y, c]) \geq 4t \geq \frac{1}{2}d(b, c) \) (hence this is an example of a wide triangle). Let \( D = d(o, [y, c]) \), and let \( g \in [o, y] \) with \( d(o, g) = \frac{3d}{2} \) and \( i \in [o, b] \) with \( d(o, i) = \frac{3D}{2} \) (as usual, choose \( g \) and \( i \) to be the endpoints if needed). By definition of \( o \), there exist \( h, j \in [y, c] \) with \( d(g, h) \leq D \) and \( d(i, j) \leq D \). In case \( i = b \), we set \( j = c \), and if \( g = y \), then \( h = g = y \).

Now we show the quadrilateral \( Q \) with vertices \([g, h, i, j] \) is \( \frac{D}{2} \)-thick. Indeed, \( d(o, [h, j]) \leq d(o, [y, c]) = 2D \). Since \( d(o, g) = \frac{3D}{2} \) and \( d(g, h) \leq D \), \( d(o, [g, h]) \geq \frac{D}{2} \). Similarly, if \( i \neq b \), \( d(o, [i, j]) \geq \frac{D}{2} \). For the case \( i = b \), observe that \( d(b, c) \leq \frac{D}{2} \), and since \( d(o, c) \geq D \) we must have \( d(o, [b, c]) \geq \frac{D}{2} \). Hence, \( \frac{D}{2} \leq \frac{D}{2} \leq t(Q) \).

Finally, \( d(g, h) \) and \( d(i, j) \) are both bounded by \( D \), \( d(g, i) \leq 3D \), and hence the triangle inequality gives \( d(h, j) \leq 5D \). Therefore the length of the perimeter of \( Q \) is at most \( 10D \leq 20t(Q) \).

**Case 2:** \( d(e, f) \leq 4t \). First, we are going to show that for any \( k \in [e, f] \), \( d(a, k) \geq t \). First note that \( d(f, c) \geq d(b, c) - d(b, e) - d(e, f) \geq 7t - 2t - 4t = t \).

Now, the following two inequalities can be extracted via applying the triangle inequality to the relevant sequences of points, which can be easily traced out if the right picture is drawn.

\[
d(x, z) \leq d(x, b) + d(b, c) + d(c, z) \leq d(x, b) + d(c, z) + 2t. \tag{3}
\]

\[
d(x, e) + d(z, f) \leq d(x, a) + d(a, k) + d(k, e) + d(z, a) + d(a, k) + d(k, f)
\]

Substituting \( d(b, e) + d(c, f) \geq 7t + t, d(e, f) \leq 4t \), and \( d(x, z) = d(x, a) + d(a, z) \) into the above equation gives

\[
d(x, b) + d(z, c) + 8t \leq d(x, z) + 2d(a, k) + 4t
\]

Summing this with (3) produces \( d(a, k) \geq t \), as desired.

We now continue constructing the desired polygon. Let \( g \in [x, a] \) and \( i \in [a, z] \) with \( d(g, a) = d(a, i) = 3t \) (as usual, we may need to choose the end points, and the proof is easily modified to work in this case). Furthermore, we can assume that there are points \( h \in [x, b] \) and \( j \in [z, c] \) such that \( d(g, h) \leq 2t \) and \( d(i, j) \leq 2t \). If these points do not exist, then we will get a wide triangle, and we can then proceed as in Case 1.
Now the inequalities \( d(a, g) = 3t \) and \( d(g, h) \leq 2t \) imply that \( d(a, \{ g, h \}) \geq t \). Similarly, \( d(a, \{ i, j \}) \geq t \). It follows that the hexagon \( H \) with vertices \( \{ g, h, e, f, j, i \} \) is at least \( t \)-thick, since the distance from \( a \) to any other side of \( H \) is at least \( t \).

It only remains to estimate the perimeter of \( H \). I will leave this as an exercise, but using known lengths and estimating the rest with the triangle inequality will produce a bound on the perimeter of \( 46t \).

Exercise 3.31. Show the hexagon \( H \) constructed in the above proof as perimeter \( \leq 46t \).

The following lemma can be proved in the context of general geodesic metric spaces. However, we will restrict our attention to the case of Cayley graphs of finitely presented groups. This restriction is purely for convenience of notation, there are no essential differences in the following proofs for general geodesic metric space once a suitable notion of area is defined.

Given a polygon \( P = p_1...p_n \) in a Cayley graph \( \Gamma(G, S) \), let \( \text{Lab}(P) = \text{Lab}(p_1)...\text{Lab}(p_n) \). Also, we slightly modify our notion of thickness for such polygons by only measuring distance between points which are vertices of the Cayley graph. This change decreases the thickness of a polygon by at most 1, so it clearly does not affect our previous result.

Lemma 3.32 (Thick polygons have quadratic area). Let \( G \) be a group given by a finite presentation \( \langle S|R \rangle \), and let \( M = \max_{r \in R} \{ \|r\| \} \). Let \( P \) be a polygon in \( \Gamma(G, S) \) of thickness \( t \) with \( W \equiv \text{Lab}(P) \). Then \( \text{Area}(W) \geq \frac{4}{M^4}t^2 \).

Proof. By definition of thickness, there exists some side \( p \) of \( P \) and a vertex \( a \in p \) such that \( d(a, P \setminus p) \geq t \). Let \( q \) be the remaining sides of \( P \), so \( P = pq \).

We now fill the closed loop \( pq \) with a van Kampen diagram \( \Delta \). We now set \( x_0 = y_0 = a \) and inductively define a sequence of simple closed paths, \( z_i = x_iy_i \) for \( 0 \leq i \leq \frac{2t}{M} + 1 \) which satisfy the following properties:

1. \( x_i \) is a subpath of \( p \) containing \( x_{i-1} \).
2. For every vertex \( b \in y_i \), \( d(a, b) \leq \frac{M_i}{2} \).
3. The subdiagram \( \Delta_i \) bound by \( z_i = x_iy_i \) contains the maximal area over all simple closed paths which satisfy the first two properties.

Increasing \( x_i \) is necessary, we can assume that each \( y_i \) has no edges in common with \( p \). Furthermore, if \( i \leq \frac{2t}{M} \) then \( y_i \) does not intersect \( q \), since if \( b \in y_i \), \( d(a, b) \leq \frac{M_i}{2} < t \). Suppose \( b \) is a vertex of both \( y_{i-1} \) and \( y_i \). Then \( b \) is a vertex of \( \partial \Delta_i \), and since \( b \) does not belong to the boundary of \( \Delta \), there must exist some \( 2 \)-cell \( \Pi \) such that \( b \in \partial \Pi \) but \( \Pi \) does not belong to \( \Delta_i \). But since \( b \in y_{i-1} \), \( d(a, b) \leq \frac{M(i-1)}{2} \), and the definition of \( M \) gives that for any vertex \( c \in \partial \Pi \), \( d(b, c) \leq \frac{M}{2} \). Hence \( \Delta_i \) could be enlarged by adding \( \Pi \) without violating the first two conditions, which contradicts the third condition of the definition of \( \Delta_i \). Thus the vertices of \( y_i \) and \( y_{i-1} \) are disjoint.

It follows that every edge of \( y_i \) belongs to the boundary of a \( 2 \)-cell which is contained in \( \Delta_i \) but not in \( \Delta_{i-1} \). Let \( m_i \) be the number of such faces, and note that \( m_i \) is at least \( \frac{t(y_i)}{M} \). Since \( y_i \) and \( y_{i-1} \) are disjoint, \( x_i \) must contain at least 2 more vertices then \( x_{i-1} \), one on each end. Thus,
\[ \ell(x_i) \geq 2i, \text{ and since each } x_i \text{ is a subpath of a geodesic, } d((x_i)_-, (x_i)_+) \geq 2i. \]  
\[ y_i \text{ has the same endpoints as } x_i, \text{ so } \ell(y_i) \geq 2i, \text{ which implies that } m_i \geq \frac{2i}{M}. \]  
Finally, we get

\[ \text{Area}(\Delta) \geq \sum_{i=1}^{\frac{2i}{M} + 1} m_i \geq \sum_{i=1}^{\frac{2i}{M} + 1} \frac{2i}{M} \geq \frac{4t^2}{M^3}. \]

Combining the previous two lemmas gives the following corollary, which finishes the proof of Theorem 3.27 (in particular, it shows that (4) \( \implies \) (1) part the proof of Theorem 3.27).

**Corollary 3.33.** Let \( G \) be a finitely presented group which is not hyperbolic. Then the Dehn function of \( G \) is at least quadratic.

**Exercise 3.34.** Suppose \( G \) has a presentation \( \langle S \mid R \rangle \) with sublinear Dehn function. Prove that \( R = \emptyset \), so in fact \( G \) is the free group on \( S \).

### 3.3 More properties of hyperbolic groups

In addition to having solvable word problem, the other two classical algorithmic questions of Dehn are solvable for hyperbolic groups.

**Theorem 3.35.** If \( G \) is a hyperbolic group, then the conjugacy problem is solvable in \( G \).

**Theorem 3.36** (Sela,...). The isomorphism problem is solvable for presentations of hyperbolic groups.

Hyperbolicity also implies a number of algebraic properties of the group. In particular, there are strong restrictions on subgroups of hyperbolic groups:

**Theorem 3.37** (Strong Tits alternative). Let \( G \) be a hyperbolic group and let \( H \leq G \). Then either \( H \) is virtually cyclic or \( H \) contains a subgroup isomorphic to \( F_2 \).

It is also important to note that subgroups of hyperbolic groups, even finitely presented subgroups, are not necessarily hyperbolic.

The above dichotomy implies, for example, that \( G \) contains no subgroups isomorphic to \( \mathbb{Z}^2 \). It also implies that \( G \) cannot be decomposed as a direct product of two infinite subgroups.

**Theorem 3.38.** If \( G \) is hyperbolic and \( G = A \times B \), then either \( A \) is finite or \( B \) is finite.

Moreover, all infinite normal subgroups of hyperbolic groups must fall into the second case in this dichotomy.

**Theorem 3.39.** If \( G \) is hyperbolic and non-elementary then every infinite normal subgroup of \( G \) contains a subgroup isomorphic to \( F_2 \). In particular, \( |Z(G)| < \infty \).
Furthermore non-elementary hyperbolic groups have lots of these normal subgroups and hence are very far from being simple. A group $G$ is called *SQ-universal* if every countable group can be embedded in some subgroup of $G$. It is a classical theorem of HNN and $F_2$ is SQ-universal. Olshanskii extended this to all non-elementary hyperbolic groups.

**Theorem 3.40.**[26] Every non-elementary hyperbolic group is SQ-universal.

Since there are uncountably many finitely generated groups and every finitely generated group has only countably many finitely generated subgroups, this theorem implies the following.

**Corollary 3.41.** Every non-elementary hyperbolic group has uncountably many normal subgroups.

**Theorem 3.42.** If $G$ is a hyperbolic group, then $G$ contains only finitely many conjugacy classes of elements of finite order. More generally, $G$ contains only finitely many conjugacy classes of finitie subgroups.

Finally, hyperbolicity also has implications for the homology of a group. This is based on the following result of Rips.

**Proposition 3.43.** Let $G$ be generated by a finite set $S$ such that $\Gamma(G, S)$ is $\delta$-hyperbolic. Then for all $d \geq 4\delta + 1$, the Rips complex $P_d(X)$ is contractible. In particular, if $G$ is torsion-free, then the quotient $P_d(X)/G$ is a finite $K(G, 1)$.

**Corollary 3.44.** If $G$ is a torsion-free hyperbolic group, then there exists $N \in \mathbb{N}$ such that $H_n(G) = \{0\}$ for all $n > N$.

If $G$ is a hyperbolic group with torsion, then the action on the Rips complex is still sufficient to prove the following, see [44, Remark 7.3.2].

**Corollary 3.45.** If $G$ is any hyperbolic group, then $H_n(G)$ is finitely generated for all $n \geq 1$.

### 4 Growth in groups

#### 4.1 Basic properties and examples

**Definition 4.1.** Let $G$ be a group generated by a finite set $S$. Then the *growth function* of $G$, $\gamma_G: \mathbb{N} \rightarrow \mathbb{N}$, is defined by

$$\gamma_G(n) = |\{g \in G||g|_S \leq n\}$$

**Exercise 4.2.** Let $G$ be a group generated by a finite set $S$. Let $x, y \in G$. Show that for all $n \geq 0$, $|B_n(x)| = |B_n(y)|$ where these balls are taken in the metric space $(G, d_S)$.

We consider these functions up to the following equivalence relation: Given $f, g: \mathbb{N} \rightarrow \mathbb{N}$, we say that $f \preceq g$ if there exist constants $A$ and $B$ both $\geq 1$, such that for all $n \in \mathbb{N}$,

$$f(n) \leq Ag(Bn).$$

We say that $f \sim g$ if $f \preceq g$ and $g \preceq f$. 

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Exercise 4.3.  1. Check that this is an equivalence relation.

2. Check that the equivalence class of a polynomial only depends on its degree.

3. Check that for any \( a, b > 1, a^n \sim b^n \).

Up to this equivalence it is straightforward to check that \( \gamma_G \) is independent of the choice of finite generating set of \( G \). Moreover,

**Lemma 4.4.** If \( G \sim_{qi} H \), then \( \gamma_G \sim \gamma_H \).

**Proof.** Let \( S \) and \( T \) be finite generating sets for \( G \) and \( H \) respectively and let \( f : (G, d_S) \to (H, d_T) \) be a \((\lambda, c)\)-quasi-isometric embedding. Fix \( x \in G \) and \( y = f(x) \in H \).

First observe that for any \( p, q \in G \), if \( f(p) = f(q) \) then \( d(p, q) \leq \lambda c \). Hence \( f \) is at most \( m \)-to-one, where \( m = |\{ g \in G | ||g||_S \leq \lambda c \} | \).

Now for any point \( p \in B_n(x) \), \( d(f(p), y) \leq \lambda n + c \). Hence \( f(B_n(x)) \subseteq B_{\lambda n + c}(y) \). Since \( f \) is at most \( m \)-to-one, we get

\[
|B_n(x)| \leq m |B_{\lambda n + c}(y)|
\]

And hence \( \gamma_G(n) \leq m \gamma_H(\lambda n + c) \leq m \gamma_H((\lambda + c)n) \). By symmetry of the quasi-isometry relation the result follows.

\[\Box\]

**Corollary 4.5.** If \( G \) and \( H \) are finitely generated groups and \( G \) is quasi-isometrically embedded in \( H \), then \( \gamma_G \preceq \gamma_H \).

Note that the embedding in this corollary is geometric and not necessarily algebraic; that is, \((G, d_S)\) may be quasi-isometrically embedded in \( H \) as a metric space but \( G \) may not be isomorphic to a subgroup of \( H \).

**Remark 4.6.** If \( M \) is a Riemannian manifold, then \( \gamma_{\pi_1(M)}(n) \sim \text{Vol}_{\tilde{M}}(n) \), where \( \text{Vol}_{\tilde{M}}(n) \) is the volume of a ball of radius \( n \) in \( \tilde{M} \).

If \( \gamma_G(n) \sim 2^n \), we say that \( G \) has exponential growth; otherwise, \( G \) has sub-exponential growth (see Lemma 4.12). If \( \gamma_G(n) \sim n^k \) for some \( k \in \mathbb{N} \), we say \( G \) has polynomial growth. If \( n^k \preceq \gamma_G(n) \) for all \( k \in \mathbb{N} \), we say that \( G \) has super-polynomial growth.

**Exercise 4.7.** Compute the growth functions for \( F_n \) and \( \mathbb{Z}^n \).

**Corollary 4.8.** \( \mathbb{Z}^n \sim_{qi} \mathbb{Z}^m \) if and only if \( n = m \).

Recall that in the case of (non-abelian) free groups, \( F_n \sim_{qi} F_m \) for all \( n, m \geq 2 \).

**Exercise 4.9.** The (integral) **Heisenberg group** is the group of matrices of the form

\[
\begin{pmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix}
|\ a, b, c \in \mathbb{Z}
\]

or equivalently the group given by the presentation \( \langle a, b, c | ac = ca, bc = cb, c = [a, b] \rangle \). Show that the Heisenberg group has growth equivalent to \( n^4 \).
Exercise 4.10. Show that if $H$ is finitely generated and $H \leq G$, then $\gamma_H \leq \gamma_G$.

Exercise 4.11. Show that if $H$ is a quotient of $G$, then $\gamma_H \leq \gamma_G$.

The above exercises imply the following.

**Lemma 4.12.** For any finitely generated group $G$, $\gamma_G(n) \leq 2^n$.

**Example 4.13.** Suppose $G$ is a hyperbolic group. If $G$ is elementary, then $G \simeq \mathbb{Z}$ and hence $\gamma_G(n) \sim \gamma_{\mathbb{Z}}(n) \sim 2n + 1$. If $G$ is non-elementary, then by the Tit’s alternative for hyperbolic groups $G$ contains a subgroup isomorphic to $F_2$, so by the above exercises $\gamma_G(n) \sim 2^n$.

### 4.2 Nilpotent groups

We begin by recalling the definition and some basic properties of nilpotent groups. Given a group $G$ and $g, h \in G$, the commutator of $g$ and $h$ is $[g, h] := g^{-1}h^{-1}gh$. Given subgroups $H_1$ and $H_2$ of $G$, let $[H_1, H_2]$ be the subgroup generated by $\{[g, h] \mid g \in H_1, h \in H_2\}$.

**Definition 4.14.** Given a group $G$, let $\gamma_1(G) = G$ and let

$$\gamma_i(G) = [\gamma_{i-1}(G), G]$$

This produces a sequence of normal subgroups $G \triangleright \gamma_1(G) \triangleright ...$ called the **lower central series**.

**Definition 4.15.** A group $G$ is **nilpotent** if $\gamma_{k+1}(G) = \{1\}$ for some $k \geq 1$. The nilpotency class of $G$ is the minimal such $k$.

Note that every nilpotent group is solvable and a group is abelian if and only if it is nilpotent of class 1.

It is straightforward to check that the identities $[xy, z] = [x, [y, z]][y, z]x, z$ and $[x, yz] = [x, y][y, x, z][xz]$ hold in any group (or perhaps some slight variation of these).

Suppose $G$ is generated by a finite set $S$. Applying this identity inductively gives that for any $g, h \in G$, $[g, h]$ is equal to a product of commutators of elements of $S$ and an element of $\gamma_2(G)$. A similar argument shows that any element of $\gamma_2(G)$ is equal to a product of commutators of the form $[[s_1, s_2], s_3]$ with each $s_i \in S$ and an element of $\gamma_3(G)$. Continuing inductively, we get a similar statement for each $\gamma_i(G)$. Now if $G$ is nilpotent of class $k$, then $\gamma_{k+1} = \{1\}$ and hence the above argument shows that $\gamma_k(G)$ will be generated by a finite set of $k$-fold commutators of elements of $S$. Hence $\gamma_{k-1}(G)$ will also be finitely generated, continuing in this way we get the following.

**Lemma 4.16.** If $G$ is nilpotent and finitely generated, then each $\gamma_i(G)$ is finitely generated.

It is relatively straightforward to prove that a nilpotent group has at most polynomial growth. Later we will compute the precise degree of growth of a nilpotent group, but this is a little more complicated, and we will only give a sketch of the proof.

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3The primary reference I am using for this section is [9].
4these are wrong, but something similar works.
**Proposition 4.17.** Let $G$ be a finitely generated nilpotent group. Then for some $d \in \mathbb{N}$, $\gamma_G(n) \leq n^d$.

**Proof.** We proceed by induction on the nilpotency degree of $G$. Suppose $G$ is nilpotent of class $k$ and the proposition holds for all groups of nilpotency class at most $k - 1$. Let $G$ be generated by a finite set $S = \{s_1, ..., s_m\}$. Let $T$ be a finite generating set for $[G, G]$ which contains all $i$–fold commutators of elements of $S$ for all $1 \leq i \leq k$. Let $g \in G$ such that $|g|_S \leq n$, and let $W$ be a word in $S$ such that $W =_G g$. Our goal will be show that

$$g = s_m^{k_m} \cdots s_1^{k_1} b$$

Where each $k_i$ is equal to the number of times $s_i$ occurs in the word $W$ minus the number of occurrences of $s_i^{-1}$ and $b \in [G, G]$ with $|b|_T \leq n^k$. Since $[G, G]$ is nilpotent of class $k - 1$ we get a polynomial bound on the number of such $b$ by induction and hence a polynomial bound on the number of $g$ with $|g|_S \leq n$.

We start by commuting each occurrence of $s_1$ in the word $W$ to the front. When we commute $s_1$ by an element $s_1$, we add a commutator $s_1s_1 = s_1s_i[s_1, s_1]$. When we commute an occurrence of $s_1$ with one of these commutators we add an element of $\gamma_2(G)$, and so on. Eventually, we get a word of the form $s_m^{k_m} U$ where $U$ is a word in $S \setminus \{s_1\}$ and elements belonging to the terms of the lower central series. Then we repreat this for all other letters in the original word until we get the desired normal form.

Note that we have to commute at most $n^2$ generators with generators. But then we commute at most $n^3$ generators with commutators. In the end we get that $b$ is a product of at most $n^2 + ... + n^k$ elements, and each of these elements is at worst a $k$-fold commutator of elements of $S$. Hence $|b|_T \leq n^k$. If $[G, G]$ has growth bounded above by $n^l$, then $\gamma_G(n) \leq n^{m+kl}$ \qed

Note that from the definition of $\gamma_G(n)$, $[\gamma_i(G), \gamma_i(G)] \leq \gamma_{i+1}(G)$, hence $\gamma_i(G)/\gamma_{i+1}(G)$ is an abelian group. If $G$ is finitely generated and nilpotent of class $k$, then each of these quotients is a finitely generated abelian group and hence isomorphic to $\mathbb{Z}^{m_i} \times A_i$ where each $m_i \geq 0$ and each $A_i$ is a finite abelian group. Define the **homogeneous dimension** of $G$ by

$$d(G) = \sum_{i=1}^{k} im_i$$

**Remark 4.18.** For nilpotent groups, the numbers $m_i$ above are quasi-isometry invariants.

**Theorem 4.19.** If $G$ is nilpotent, then $\gamma_G(n) \sim n^d$ where $d = d(G)$.

The proof again proceeds by induction on the nilpotency degree of $G$. If $G$ is nilpotent of class $k$, then let $K = \gamma_k(G)$. The $G/K$ is nilpotent of class $k - 1$ and $d(K) = d(G) - km_k$, where $m_k$ is defined as above. Hence the growth of $G/K$ is $\sim n^{d-klm_k}$ by induction. Let $S$ be a finite generating set for $G$ and consider the quotient $G/K$ with the image of $S$ as its finite generating set. Then the quotient map $\varphi: G \rightarrow G/K$ surjects $B_n^G(1)$ onto $B_n^{G/K}(1)$. If $B_n^{G/K}(1) = \{q_1, ..., q_N\}$, for each $q_i$ let $g_i \in \varphi^{-1}(q_i)$. Note that by assumption $N \sim n^{d-klm_k}$.

For the lower bound, note that every element $g$ of $B_n^G(1)$ is equal to an element of the form $g_i k$ for some $1 \leq i \leq N$ and some $k \in K$. Since $k = g_i^{-1} g$, $|k|_S \leq 2n$. Hence $|B_n^G(1)| \geq N \cdot |B_{2n}^G(1) \cap K|$
For the upper bound, $B_{2n}^G(g)$ contains all of the elements of the form $g_ik$ where $1 \leq i \leq k$ and $|k|_S \leq n$, hence $|B_{2n}^G(1)| \leq N \cdot |B_n^G(1) \cap K|$.

For both cases, we need to be able to count the number of elements in $B_n^G(1) \cap K$. Now $K$ is abelian and hence has growth $n^{m_k}$, however in order to count $B_n^G(1) \cap K$ we need to understand the distortion of $K$ in $G$.

**Definition 4.20.** Let $G$ be generated by a finite set $S$ and let $H$ be a subgroup of $G$ generated by a finite set $T$. Then the distortion of $H$ in $G$ is

$$\Delta_H^G(n) = \max\{|h|_T \mid |h|_S \leq n\}$$

It is straightforward to show that up to the equivalence relation defined on growth functions that the distortion of $H$ in $G$ is independent of the choices of finite generating sets. Hence one can assume that $T \subseteq S$, and it follows easily that $n \leq \Delta_H^G(n)$, for any finitely generated groups $G$ and $H \leq G$. In case $\Delta_H^G(n) \sim n$, we say that $H$ is undistorted in $G$. This is equivalent to the inclusion map $(H, d_T) \to (G, d_S)$ being a quasi-isometric embedding.

**Examples 4.21.**
1. If $G = BS(1, 2) = \langle a, t \mid t^{-1}at = a^2 \rangle$ and $H = \langle a \rangle$, then $\Delta_H^G(n) \sim 2^n$.
2. If $G$ is the Heisenberg group and $H = \langle c \rangle$, then $\Delta_H^G(n) \sim n^2$.
3. A fundamental property of hyperbolic groups is that every cyclic subgroup is undistorted.

We refer to [9] for a proof of the following proposition.

**Proposition 4.22.** Let $G$ be a nilpotent group of class $k$ let $K = \gamma_k(G)$. Then $\Delta^K_G(n) \sim n^k$.

This proposition together with the above argument complete the proof of Theorem 4.19.

### 4.3 Solvable and Linear groups

**Definition 4.23.** A group $G$ is called solvable if there exists a sequence of normal subgroup of $G$

$$G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_k = \{1\}$$

Such that each quotient $G_i/G_{i+1}$ is abelian. If such a sequence exists with each quotient $G_i/G_{i+1}$ cyclic, then $G$ is called polycyclic.

If $G$ is nipotent, then since each term in the lower central series is finitely generated this sequence can be refined to a sequence with cyclic quotients, hence all nilpotent groups are polycyclic. Clearly all polycyclic groups are solvable.

**Theorem 4.24** (Milnor-Wolf). If $G$ is solvable, then either $G$ has exponential growth or $G$ is virtually nilpotent.

**Definition 4.25.** A group $G$ is called linear if for some field $k$ and some $n \geq 1$, $G$ is isomorphic to a subgroup of $GL(n, k)$.
Theorem 4.26 (Tit’s alternative). If $G$ is finitely generated and linear, then either $G$ is virtually solvable or $G$ contains a subgroup isomorphic to $F_2$.

Theorem 4.27. If $G$ is either solvable or linear, then either $G$ has exponential growth or $G$ is virtually nilpotent and hence has polynomial growth.

In light of the above results, Milnor asked whether or not there exists groups whose growth is both super-polynomial and sub-exponential. These groups are called groups of intermediate growth, there existence was proven by Grigorchuk.

Theorem 4.28. Groups of intermediate growth exist.

4.4 Gromov’s polynomial growth theorem-overview

Theorem 4.29. Suppose $G$ is a finitely generated group with $\gamma_G \leq n^d$ for some $d \in \mathbb{N}$. Then $G$ is virtually nilpotent.

Since we will be frequently passing to subgroups of finite index throughout the proof, we first observe the following basic properties of this process.

Exercise 4.30. Let $G$ be a finitely generated group.

1. If $H_1$ and $H_2$ are finite index subgroups of $G$, then $H_1 \cap H_2$ is a finite index subgroup of $G$.
2. If $H_1$ is a finite index subgroup of $G$ and $H_2$ is a finite index subgroup of $H_1$, then $H_2$ is a finite index subgroup of $G$.
3. For any $i \geq 1$, $G$ has only finitely many subgroup of index $i$.
4. If $\varphi: G \to H$ is a homomorphism and $H_1$ is a finite index subgroup of $H$, then $\varphi^{-1}(H)$ is a finite index subgroup of $G$.

The main step in proving this theorem is the following:

Theorem 4.31. Suppose $G$ is a finitely generated group with $\gamma_G \leq n^d$ for some $d \in \mathbb{N}$. Then there exists a non-trivial homomorphism $\varphi: G_1 \to \mathbb{Z}$. Where $G_1$ is a finite index subgroup of $G$.

Note that passing to a subgroup of finite index is necessary in this theorem. The infinite dihedral group $D_\infty$ is generated by two elements of order two, hence there is no non-trivial map $D_\infty \to \mathbb{Z}$. However $D_\infty$ is itself virtually $\mathbb{Z}$ and hence has linear growth. Note also that $D_\infty$ is solvable of class 2.

Exercise 4.32. Prove that for a finitely generated group $G$, there exists a non-trivial homomorphism $\varphi: G \to \mathbb{Z}$ if and only if the abelianization $G/[G,G]$ is infinite.

Exercise 4.33. Let $G$ be virtually solvable. Prove that there exists a non-trivial homomorphism $\varphi: G_1 \to \mathbb{Z}$ where $G_1$ is a finite index subgroup of $G$.

Gromov’s theorem follows easily from Theorem 4.31 together with the following lemma.

Lemma 4.34. Let $G$ be finitely generated and let $\varphi: G \to \mathbb{Z}$ be a non-trivial homomorphism with kernel $K$. Then
1. If $\gamma_G(n) \leq n^d$ then $K$ is finitely generated and $\gamma_K(n) \leq n^{d-1}$.

2. If $K$ is virtually solvable then $G$ is virtually solvable.

Proof.

Proof of Gromov’s Theorem given Theorem 4.31. We induct on $d$. If $d = 0$ then $G$ is finite and the theorem is trivial. Assume now that the theorem holds for all finitely generated groups of growth $\leq d - 1$ and $G$ satisfies $\gamma_G(n) \leq n^d$. Hence $G$ has a non-trivial homomorphism to $\mathbb{Z}$ by Theorem 4.31 and by the previous lemma the kernel $K$ of this map has growth $\leq d - 1$. Hence the inductive hypothesis implies that $K$ is virtually nilpotent, so by part 3 of the lemma $G$ is virtually solvable. Hence the Milnor-Wolf Theorem implies that $G$ is virtually nilpotent.

In order to prove Theorem 4.31 Gromov constructs a space $Y$ on which $G$ acts by isometries. This space $Y$ is called an asymptotic cone, we will study the construction in the next subsection.

Theorem 4.35. If $G$ is finitely generated and $\gamma_G(n)$ is bounded by a polynomial, then there exists a geodesic metric space $Y$ together with an action of $G$ on $Y$ by isometries such that $Y$ is proper, connected, locally connected, finite-dimensional, and homogeneous.

Given such a space we can apply deep results of Gleason-Mongomery-Zippin which gave a solution to Hilbert’s 5’th problem.

Theorem 4.36. Give $Y$ as in Theorem 4.35, $\text{Isom}(Y)$ is a lie group with finitely many connected components.

Passing to the connected component of the identity of $\text{Isom}(Y)$ and the corresponding finite index subgroup $G_1$ of $G$, we get a map $G_1 \to L$, where $L$ is a connected lie group. Such a lie group has a map $ad: L \to \text{GL}(n, \mathbb{C})$, called the adjoint representation, whose kernel is the center of $L$. Hence we have an induced map $r: G_1 \to \text{GL}(n, \mathbb{C})$ There are now two cases to consider.

Case 1: The image of $G_1$ in $\text{GL}(n, \mathbb{C})$ is infinite. Since $r(G_1)$ is a quotient of $G_1$ its growth is polynomially bounded, and hence the Tit’s alternative implies that $r(G_1)$ is virtually solvable. Hence it has a finite index subgroup which maps onto $\mathbb{Z}$, so the corresponding finite index subgroup of $G_1$ has a non-trivial map to $\mathbb{Z}$.

Case 2: The image of $G_1$ in $\text{GL}(n, \mathbb{C})$ is finite. In this case we will show that there is a sequence of homomorphisms from $G_1$ onto larger and larger finite abelian groups. Since the size of these abelian groups will go to infinity, we get that $G_1$ must have infinite abelianization and hence admit a map onto $\mathbb{Z}$. In order to do this will need to study the action of $G$ on $Y$ as well as use a theorem of Jordan about finite subgroups of Lie groups.

4.5 Asymptotic cones

Definition 4.37. An ultrafilter (on $\mathbb{N}$) is a finitely additive probability measure $\mu: 2^{\mathbb{N}} \to \{0,1\}$. That is, $\mu(\mathbb{N}) = 1$ and for any disjoint sets $A$ and $B$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

An ultrafilter is called principle if for some finite $A \subseteq N$, $\mu(A) = 1$. Equivalently, $\mu$ is principle if for some $x \in \mathbb{N}$, for any $A \subseteq \mathbb{N}$ $\mu(A) = 1$ if and only if $x \in A$. 

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A non-principle ultrafilter is an ultrafilter such that $\mu(A) = 0$ for all finite $A \subseteq \mathbb{N}$.

References


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