(1) (Due 9/4) Prove that if $W$ is a subspace of a vector space $V$ and $w_1, w_2, ..., w_n$ are in $W$, then $a_1w_1 + a_2w_2 + ... + a_nw_n \in W$ for any scalars $a_1, a_2, ..., a_n$.

(2) (Due 9/11) Let $\{u, v, w\}$ be a basis for a vector space $V$. Prove that $\{u - w, v - w, w\}$ is also a basis for $V$.

(3) (Due 9/18) Let $T: \mathbb{R} \to \mathbb{R}$ be a linear transformation. Prove that there exists $m \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $T(x) = mx$.

(4) (Due 9/25) Let $V$ and $W$ be vector spaces, and let $T$ and $U$ be non-zero linear transformations from $V$ to $W$. If $\text{Range}(T) \cap \text{Range}(U) = \{0\}$, prove that $\{T, U\}$ is a linearly independent subset of $\mathcal{L}(V, W)$.

(5) (Due 10/2) Let $T: F^n \to F^m$ be a function. Prove that $T$ is a linear transformation if and only if there exists $a_{ij} \in F$ for $1 \leq i \leq m$, $1 \leq j \leq n$, such that for all $(x_1, ..., x_n) \in F^n$,

$$T(x_1, ..., x_n) = (a_{11}x_1 + ... + a_{1n}x_n, ..., a_{m1}x_1 + ... + a_{mn}x_n).$$

(6) (Due 10/9) Let $A, B \in \mathcal{M}_{n \times n}(F)$ be similar matrices. Prove that there exists a linear transformation $T: F^n \to F^m$ and basis $\beta_1, \beta_2$ of $F^n$ such that $[T]_{\beta_1} = A$ and $[T]_{\beta_2} = B$.

(7) (Due 10/23) Let $A$ be an $n \times n$ matrix which is not invertible. Prove that there exists a non-zero $n \times n$ matrix $B$ such that $BA$ is equal to the zero matrix.
(8) (Due 10/30) Let \( \delta : \mathcal{M}_{2 \times 2}(F) \to F \) be a function such that \( \delta(I_2) = 1 \) and \( \delta \) satisfies the following conditions with respect to elementary row operations:

(a) If \( A \xrightarrow{r_i \leftrightarrow r_j} B \), then \( \delta(B) = -\delta(A) \).

(b) If \( A \xrightarrow{r_i \rightarrow cr_i} B \), then \( \delta(B) = c\delta(A) \).

(c) If \( A \xrightarrow{r_i \rightarrow r_i + cr_j} B \), then \( \delta(B) = \delta(A) \).

Prove that \( \delta(A) = \det(A) \) for all \( A \in \mathcal{M}_{2 \times 2}(F) \).