PROOF WRITING
LINEAR ALGEBRA

(1) (Due 9/4) Prove that if W is a subspace of a vector space V and w_1, w_2,..., w_n are in W, then $a_1w_1 + a_2w_2 + ... + a_nw_n \in W$ for any scalars a_1, a_2, ..., a_n.

(2) (Due 9/11) Let $\{u, v, w\}$ be a basis for a vector space V. Prove that $\{u - w, v - w, w\}$ is also a basis for V.

(3) (Due 9/18) Let $T\colon \mathbb{R} \to \mathbb{R}$ be a linear transformation. Prove that there exists $m \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $T(x) = mx$.

(4) (Due 9/25) Let V and W be vector spaces, and let T and U be non-zero linear transformations from V to W. If $Range(T) \cap Range(U) = \{0\}$, prove that $\{T, U\}$ is a linearly independent subset of $\mathcal{L}(V, W)$.

(5) (Due 10/2) Let $T\colon F^n \to F^m$ be a function. Prove that T is a linear transformation if and only if there exists $a_{ij} \in F$ for $1 \le i \le m$, $1 \le j \le n$, such that for all $(x_1, ..., x_n) \in F^n$,

$$T(x_1, ..., x_n) = (a_{11}x_1 + ... + a_{1n}x_n, ..., a_{m1}x_1 + ... + a_{mn}x_n).$$

(6) (Due 10/9) Let $A, B \in \mathcal{M}_{n\times n}(F)$ be similar matrices. Prove that there exists a linear transformation $T\colon F^n \to F^n$ and basis β_1, β_2 of F^n such that $[T]_{\beta_1} = A$ and $[T]_{\beta_2} = B$.

(7) (Due 10/23) Let A be an $n \times n$ matrix which is not invertible. Prove that there exists a non-zero $n \times n$ matrix B such that BA is equal to the zero matrix.

(8) (Due 10/30) Let $\delta\colon \mathcal{M}_{2\times 2}(F) \to F$ be a function such that $\delta(I_2) = 1$ and δ satisfes the following conditions with respect to elementary row operations:

(a) If $A \xrightarrow{r_i \leftrightarrow r_j} B$, then $\delta(B) = -\delta(A)$.

(b) If $A \xrightarrow{r_i \to cr_i} B$, then $\delta(B) = c\delta(A)$.

(c) If $A \xrightarrow{r_i \to r_i + cr_j} B$, then $\delta(B) = \delta(A)$.

Prove that $\delta(A) = det(A)$ for all $A \in \mathcal{M}_{2\times 2}(F)$.

(9) (Due 11/13) Let V be a finite dimensional vector space over $\mathbb{C}$ and let $T\colon V \to V$ be a linear transformation. Let $\lambda_1, \lambda_2, ..., \lambda_k$ be the distinct eigenvalues of T and let m_i be the algebraic multiplicity of λ_i. Prove that $det(T) = \lambda_1^{m_1}\lambda_2^{m_2}...\lambda_k^{m_k}$.

(10) (Due 12/4) Let V be an inner product space and let $T\colon V \to V$ a linear transformation such that $T^2 = T$ and $Null(T)^\perp = Range(T)$. Prove that there exists a subspace W such that for all $v \in V$, $T(v) = Proj_W(v)$.