PROOFS
LINEAR ALGEBRA

(1) (Due 8/29) Let W be a subset of a vector space V such that $W \neq \emptyset$ and for all $a \in F$ and $x, y \in W$, $ax + y \in W$. Prove that W is a subspace of V.

(2) (Due 9/5) Let S_1 and S_2 be subsets of a vector space V. Prove that $Span(S_1 \cap S_2) \subseteq Span(S_1) \cap Span(S_2)$.

(3) (Due 9/19) Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. Prove that if S_2 is linearly independent, then S_1 is linearly independent.

(4) (Due 9/26) Let $\{u, v, w\}$ be a basis for a vector space V. Prove that $\{u - w, v - w, w\}$ is also a basis for V.

(5) (Due 10/3) Let $T \colon \mathbb{R} \to \mathbb{R}$ be a linear transformation. Prove that there exists $m \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $T(x) = mx$.

(6) (Due 10/17) Let V and W be vector spaces, and let T and U be non-zero linear transformations from V to W. If $Range(T) \cap Range(U) = \{0\}$, prove that $\{T, U\}$ is a linearly independent subset of $\mathcal{L}(V, W)$.

(7) (Due 10/24/17) Let V be a finite dimensional vector space, and let $T \colon V \to V$ be a linear transformation which is not invertible. Prove that there exists a non-zero linear transformation $U \colon V \to V$ such that UT is the zero transformation.

(8) (Due 10/31/17) Let $A \in \mathcal{M}_{m \times n}(F)$ and $B \in \mathcal{M}_{n \times m}(F)$. Suppose $n < m$. Prove that AB is not invertible. Does the same result hold if $m < n$?

(9) (Due 11/7/17) Let $\delta\colon \mathcal{M}_{2\times 2}(F) \to F$ be a function such that $\delta(I_2) = 1$ and δ satisfes the following conditions with respect to elementary row operations:
(a) If $A \xrightarrow{r_i \leftrightarrow r_j} B$, then $\delta(B) = -\delta(A)$.
(b) If $A \xrightarrow{r_i \to cr_i} B$, then $\delta(B) = c\delta(A)$.
(c) If $A \xrightarrow{r_i \to r_i + cr_j} B$, then $\delta(B) = \delta(A)$.
Prove that $\delta(A) = det(A)$ for all $A \in \mathcal{M}_{2\times 2}(F)$.

(10) (Due 11/21/17) Let V be a finite dimensional vector space over $\mathbb{C}$ and let $T\colon V \to V$ be a linear transformation. Let $\lambda_1, \lambda_2, ..., \lambda_k$ be the distinct eigenvalues of T and let m_i be the algebraic multiplicity of λ_i. Prove that $det(T) = \lambda_1^{m_1}\lambda_2^{m_2}...\lambda_k^{m_k}$.

(11) (Due 12/5/17) Let V be a vector space over either $\mathbb{R}$ or $\mathbb{C}$ and let β be a basis of V. Prove that there exists an inner product on V such that β is an orthonormal basis.