Lecture 18: Rates of Change

Physics

**ex.** Suppose the position of a particle is given by

\[ s = f(t) = 2t^3 - 15t^2 + 24t, \]

where \( t \) is measured in seconds and \( s \) in feet.

1) Find the velocity of the particle at any time \( t \).

\[ v(t) = f'(t) = 6t^2 - 30t + 24 \]

\[ v(0) = 24 \]

2) Find the velocity at \( t = 3 \) seconds.

\[ v(3) = 6(3)^2 - 30(3) + 24 \]

\[ = 54 - 90 + 24 = 54 - 66 \]

\[ = -12 \text{ ft/sec} \]

3) When is the particle at rest?

\[ 6t^2 - 30t + 24 = 0 \]

\[ 6(t^2 - 5t + 4) = 0 \]

\[ 6(t - 4)(t - 1) = 0 \]

\[ t = 4 \]

\[ t = 1 \]

4) When is the particle moving in a positive direction?
5) Draw a diagram to represent the particle’s motion.

6) Find the total distance the particle moves in the first six seconds.

Net dist: \[ |f(6) - f(0)| = |36 - 0| = 36 \]

\[ f(6) = 36 \]

Total dist: \[ |f(1) - f(0)| + |f(4) - f(1)| + |f(6) - f(4)| \]

\[ = |11 - 0| + |16 - 11| + |36 - (-16)| \]

\[ = 11 + 27 + 52 = 90 \text{ ft} \]

Acceleration

\[ a(t) = V'(t) = f''(t) \]

7) Find the acceleration of \( s(t) = 2t^3 - 15t^2 + 24t \) at any time \( t \).

\[ V(t) = 6t^2 - 30t + 24 \]

\[ a(t) = 12t - 30 \]

\[ 12t - 30 = 0 \quad t = \frac{30}{12} = \frac{15}{6} = 2.5 \]
8) When is the particle speeding up and when is it slowing down?

speeding up: \( v(t) + a(t) \) have same sign
\( (1, 2.5) \cup (4, \infty) \)
both -
both +

slowing down
\( (0, 1) \cup (2.5, 4) \)
\( v(t) + \) \( v(t) - \)
\( a(t) - \) \( a(t) + \)
Economics

The total cost of producing $x$ units of a product is called the cost function $C(x)$.

Average rate of change of cost as the number of items produced increases from $x_1$ to $x_2$:

$$
\frac{C(x_2) - C(x_1)}{x_2 - x_1}
$$

Marginal Cost

$$
\frac{C(x+1) - C(x)}{1} = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h}
$$

$$
= C'(x)
$$

ex. Suppose the cost function for a certain product is given by $C(x) = 1000 + 25x - 0.1x^2$.

1) Find the total cost of producing 100 items.

$$
C(100) = 1000 + 2500 - 1000 = 2500
$$

2) Estimate the marginal cost at the production level of 100 items.

$$
C'(x) = 25 - 0.2x
$$

$$
C'(100) = 25 - 20 = 5
$$
3) Find the actual cost of producing the 101st item given $C(101) = 1000 + 25(101) - 0.1(101)^2 = 2504.90$.

$C(101) - C(100) = 4.90$ - marginal cost

Now suppose that the unit price $p$ at which $x$ items will sell can be modeled by the demand function

$$p(x) = -0.3x + 125, \ 0 \leq x \leq 400.$$ 

4) Find the revenue from the sale of $x$ items.

$$R(x) = x \cdot p(x)$$

5) Find the profit function, $P(x)$, which gives the profit from the sale of $x$ items.

$$P(x) = R(x) - C(x)$$

$$P(x) = x \cdot p(x) - C(x)$$

6) Estimate the marginal profit when 50 items are sold.

$$C'(x) = 25 - 0.2x \quad P'(x) = p(x) + x(-0.3) - (25 - 0.2x)$$

$$P'(x) = -0.3x + 125 - 0.3x - 25 + 0.2x$$

Note: $P(51) - P(50) = 3579.80 - 3500$.

$$P'(50) = 100 - 0.4x \quad x = \frac{100}{0.4} = 250$$
Chemistry

Consider a chemical reaction \( A + B \rightarrow C \), where \( A \) and \( B \) are reactants and \( C \) is the product. The concentration of product \( C \) in moles per liter is denoted \([C](t)\).

Average rate of reaction of the product \( C \) over \([t_1, t_2]\) is

\[
\frac{\Delta [C]}{\Delta t} = \frac{[C](t_2) - [C](t_1)}{t_2 - t_1}
\]

Instantaneous rate of reaction \( = \lim_{\Delta t \to 0} \frac{\Delta [C]}{\Delta t} = \frac{d[C]}{dt} \)

ex. Assume that the initial concentration of \( A \) and \( B \) have the same value \([A] = [B] = 2 \text{ moles/L}\), and \([C](t) = \frac{16t}{8t + 1}\) moles/L.

1) Find the rate of reaction at time \( t \), \( R(t) \).

\[
R(t) = \sum C \beta(t) = \frac{(16)(8^2T+1) - (16)(8)}{(8T+1)^2}
\]

\[
= \frac{(16)(8)T + 16 - (16)(8)T}{(8T+1)^2}
\]

\[
R(T) = \frac{16}{(8T+1)^2}
\]
Biology

**ex.** A certain type of bacteria doubles in population every 3 hours. Assume that there were 10 bacteria initially.

1) Find a formula for the population of bacteria at any time \( t \).

\[
P(3) = 2 \, P(0) \\
P(6) = 4 \, P(0) \\
P(9) = 8 \, P(0)
\]

\[
\begin{align*}
2^{\frac{t}{3}} \, P(0) \\
2^2 \, P(0) \\
\text{Number of Hours} \\
\end{align*}
\]

2) Find the rate of growth of the bacteria at \( t = 6 \) hour.

\[
P'(t) = 10 \left( 2^{\frac{t}{3}} \right) (\ln 2) \left( \frac{1}{3} \right)
\]

\[
P'(6) = 10 \left( \frac{4}{3} \right) (\ln 2) \left( \frac{1}{3} \right)
\]
2) Find the rate of reaction at $t = 0$, 1, and 2 seconds. Include units.

\[ R(0) = 16 \text{ mol/s} \]
\[ R(1) = \frac{16}{81} \]
\[ R(2) = \frac{16}{(17)^2} \]

3) What happens to $[C](t)$ and $R(t)$ at $t \to \infty$. Does this make sense?

\[ \lim_{t \to \infty} [C](t) = 2 \text{ mol} \]
\[ \lim_{t \to \infty} R(t) = 0 \]

\[ [C](t) = \frac{16}{8t+1} \cdot \frac{1}{t} = \frac{16}{8t+1} \to 0 \]
\[ R(t) = \frac{16}{(8t+1)^2} \]