Lecture 18: Rates of Change

Physics

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ex. Suppose the position of a particle is given by

$$s = f(t) = 2t^3 - 15t^2 + 24t,$$

where t is measured in seconds and s in feet.

1) Find the velocity of the particle at any time t.

$v(T) = f'(T) = 6T^2 - \cancel{15}\, 30T + 24$

$v(0) = 24$

2) Find the velocity at $t = 3$ seconds.

$v(3) = 6(3)^2 - 30(3) + 24$

$6 \cdot 9 - 90 + 24 = 54 - 90 + 24$

$= -12$ ft/sec

3) When is the particle at rest?

$6T^2 - 30T + 24 = 0$

$6(T^2 - 5T + 4) = 0$

$6(T-4)(T-1) = 0$

$T = 4$

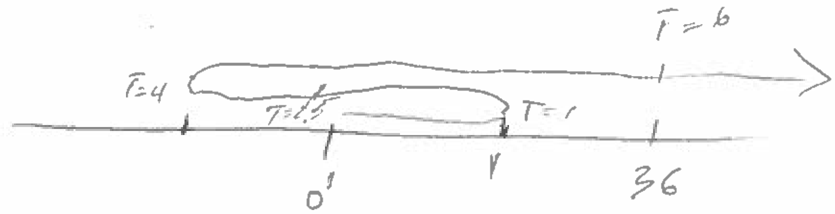
$T = 1$

4) When is the particle moving in a positive direction?

	0	1	4
$(T-4)$	$-$	$-$	$+$
$(T-1)$	$-$	$+$	$+$
$v(T)$	$+$	$-$	$+$

$\boxed{(0,1),\ (4,\infty)}$

5) Draw a diagram to represent the particle's motion.



6) Find the total distance the particle moves in the first six seconds.

net dist: $|f(6) - f(0)| = |36 - 0| = 36$

$f(6) = 36$

Total dist: $|f(1) - f(0)| + |f(4) - f(1)| + |f(6) - f(4)|$

$|11 - 0| + |-16 - 11| + |36 - (-16)|$

$= 11 + 27 + 52 = 90$ ft

Acceleration

$a(t) = v'(t) = f''(t)$

7) Find the acceleration of $s(t) = 2t^3 - 15t^2 + 24t$ at any time t.

$v(t) = 6t^2 - 30t + 24$

$a(t) = 12t - 30$

$12t - 30 = 0 \quad t = \frac{30}{12} = \frac{15}{6} = 2.5$

		2.5	
$a(t)$	$-$		$+$

8) When is the particle **speeding up** and when is it **slowing down**?

Speeding up: $v(t)$ + $a(t)$
have same sign

$(1, 2.5) \cup (4, \infty)$

both $-$ both $+$

slowing down

$(0, 1) \cup (2.5, 4)$

$v(t)$ + $v(t)$ $-$

$a(t)$ $-$ $a(t)$ +

Economics

The total cost of producing x units of a product is called the **cost function** $C(x)$.

Average rate of change of cost as the number of items produced increases from x_1 to x_2:

$$\frac{C(x_2) - C(x_1)}{x_2 - x_1}$$

Marginal Cost

$$\boxed{C(x+1) - C(x)} = \frac{C(x+1) - C(x)}{1} \approx \lim_{h \to 0} \frac{C(x+h) - C(x)}{h}$$

$$= C'(x)$$

<u>**ex.**</u> Suppose the cost function for a certain product is given by $C(x) = 1000 + 25x - 0.1x^2$.

1) Find the total cost of producing 100 items.

$$C(100) = 1000 + 2500 - 1000 = 2500$$

2) Estimate the marginal cost at the production level of 100 items.

$$C'(x) = 25 - 0.2x$$

$$C'(100) = 25 - 20 = 5$$

3) Find the actual cost of producing the 101st item given $C(101) = 1000 + 25(101) - 0.1(101)^2 = 2504.90$.

$C(101) - C(100) = \$4.90$ – marginal cost estimate: $5

Now suppose that the unit price p at which x items will sell can be modeled by the **demand function**

$$p(x) = -0.3x + 125,\ 0 \le x \le 400.$$

4) Find the **revenue** from the sale of x items.

$R(x) = x\,p(x)$

5) Find the profit function, $P(x)$, which gives the profit from the sale of x items.

$P(x) = R(x) - C(x)$

$P(x) = x\,p(x) - C(x)$

6) Estimate the marginal profit when 50 items are sold.

$C'(x) = 25 - 0.2x$

$p'(x) = -0.3$

$P'(x) = p(x) + x(-0.3) - (25 - 0.2x)$

$P'(x) = -0.3x + 125 - 0.3x - 25 + 0.2x$

Note: $P(51) - P(50) = 3579.80 - 3500$.

$P'(x) = 125 - 0.4x - 25 = 100 - 0.4x$

$P'(50) = 100 - 20 = 80$

$x = \frac{100}{0.4} = 250$

Chemistry

Consider a chemical reaction $A + B \to C$, where A and B are **reactants** and C is the **product**. The **concentration** of product C in moles per liter is denoted $[C](t)$.

Average rate of reaction of the product C over $[t_1, t_2]$ is

$$\frac{\Delta[C]}{\Delta t} = \frac{[C](t_2) - [C](t_1)}{t_2 - t_1}$$

$$\text{Instantaneous rate of reaction} = \lim_{\Delta t \to 0} \frac{\Delta[C]}{\Delta t} = \frac{d[C]}{dt}$$

ex. Assume that the initial concentration of A and B have the same value $[A] = [B] = 2$ moles/L, and $[C](t) = \dfrac{16t}{8t+1}$ moles/L.

1) Find the rate of reaction at time t, $R(t)$.

$$R(t) = [C]'(t) = \frac{(16)(8T+1) - (16T)(8)}{(8T+1)^2}$$

$$= \frac{(16)(8)T + 16 - (16)(8)T}{(8T+1)^2}$$

$$R(T) = \frac{16}{(8T+1)^2}$$

Biology

ex. A certain type of bacteria doubles in population every 3 hours. Assume that there were 10 bacteria initially.

1) Find a formula for the population of bacteria at any time t.

$P(3) = 2P(0)$

$P(6) = 4P(0)$ (or) $2^{(T/3)}P(0)$

$P(9) = 8P(0)$

~~$P(t) = 10e^{2t}$~~

number of hours

$$\boxed{P(t) = (10)2^{T/3}}$$

initial

2) Find the rate of growth of the bacteria at $t = 6$ hour.

$P'(t) = 10(2^{T/3})(\ln 2)(1/3)$

$P'(6) = (10)(4)(\ln 2)(1/3)$

2) Find the rate of reaction at $t = 0$, 1, and 2 seconds. Include units.

$$R(0) = 16 \text{ moles/L/sec}$$

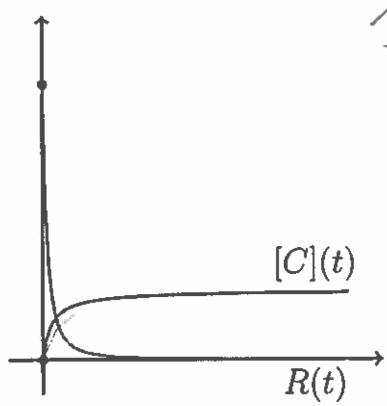
$$R(1) = \frac{16}{81}$$

$$R(2) = \frac{16}{(17)^2}$$

3) What happens to $[C](t)$ and $R(t)$ at $t \to \infty$. Does this make sense?

$$\lim_{T \to \infty} [C](T) = 2 \text{ moles/L}$$

$$\lim_{T \to \infty} R(t) = 0$$



$$[C](T) = \frac{16T}{8T+1} \cdot \frac{\frac{1}{T}}{\frac{1}{T}} = \frac{16}{8 + \frac{1}{T}} \qquad \frac{1}{T} \to 0$$

$$R(T) = \frac{16}{(8T+1)^2}$$