MAP2302
Elementary Differential Equations
Exam 1

Name: Solutions

UFID: _____

Instructions:

- Read each problem carefully.
- Show all your work; you will not get credit for answers with no work even if they are correct.
- The proctor will not answer questions about the material on the exam or give hints to any of the problems; do your best to answer each question as written.
- Students should not have calculators, phones, or paper on their desk and they should not wear headphones. No student writing should be in a position where it is visible.
- Implicit solutions will get full credit unless the problem asks for an explicit solution.
- All numerical answers should be left in exact form (i.e. use $\ln(2)$, not $\approx .7$).
- The proctor will have additional scratch paper if needed.

1. For each of the following differential equations, identify the independent variable and the order of the equation and state whether it is linear or non-linear. (Do not solve the equation).

(a) $v\frac{d^3u}{dv^3} + \cos(v)\frac{du}{dv} + v^2u = e^v$

independent variable : v

order : 3

linear

(b) $x^2 + x\frac{dx}{dt} = t$

independent variable : t

order : 1

non-linear

2. For each of the following equations, state whether it is separable, linear, or exact, or which of these it can be turned into with a change of variables (if it is more than one of these, you only need to state one of them). Finally, give all solutions to each equation, giving explicit solutions if requested.

(a) $\frac{dy}{dx} = y + e^x$ (Give an explicit solution)

$\frac{dy}{dx} - y = e^x$ linear

$\int -1\,dx = -x$

$\mu(x) = e^{-x}$

$e^{-x}\frac{dy}{dx} - e^{-x}y = 1$

$\frac{d}{dx}(e^{-x}y) = 1$

$e^{-x}y = \int 1\,dx$

$e^{-x}y = x + C$

$y = xe^x + Ce^x$

(b) $-\frac{y^2}{x^2}dx + (\frac{2y}{x})dy = 0$

$\frac{\partial}{\partial y}\left(-\frac{y^2}{x^2}\right) = -\frac{2y}{x^2}$ exact

$\frac{\partial}{\partial x}\left(\frac{2y}{x}\right) = -\frac{2y}{x^2}$

$F(x,y) = \int -\frac{y^2}{x^2}\,dx + g(y)$

$F(x,y) = \frac{y^2}{x} + g(y)$

$\frac{2y}{x} = \frac{\partial F}{\partial y} = \frac{2y}{x} + g'(y)$

$g'(y) = 0$

$g(y) = C'$

$F(x,y) = \frac{y^2}{x} + C'$

$F(x,y) = C$

$\frac{y^2}{x} = C$

(c) $\frac{dy}{dx} = 1 + \cos^2(x-y)$

$v = x - y$

$\frac{dv}{dx} = 1 - \frac{dy}{dx}$

$1 - \frac{dv}{dx} = 1 + \cos^2(v)$

$\frac{dv}{dx} = -\cos^2(v)$

$-\sec^2(v)\,dv = dx$ Separable

$\int -\sec^2(v)\,dv = \int dx$

$-\tan(v) = x + C$

$-\tan(x-y) = x + C$

(d) $\frac{dy}{dx} + \frac{y}{x} = x^2y^2$ (Give an explicit solution)

$y^{-2}\frac{dy}{dx} + y^{-1}x^{-1} = x^2$

$v = y^{-1}$

$\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$

$-\frac{dv}{dx} + x^{-1}v = x^2$

$\frac{dv}{dx} - x^{-1}v = -x^2$ linear

$\int -x^{-1}dx = -\ln x = \ln 1/x$, $\mu(x) = e^{\ln 1/x} = 1/x$

$1/x\frac{dv}{dx} - \left(\frac{1}{x^2}\right)v = -x$

$\frac{d}{dx}\left(\frac{v}{x}\right) = -x$

$\frac{v}{x} = \int -x\,dx = -\frac{1}{2}x^2 + C$

$v = -\frac{1}{2}x^3 + C'x$

$y^{-1} = \frac{Cx - x^3}{2}$

$y = \frac{2}{Cx - x^3}$

3. Suppose a brine containing 0.3 kilograms of salt per liter runs into a tank initially filled with 400 liters of water containing 5 kilograms of salt. If the brine enters at 10 L/min, the mixture is kept uniform by stirring, and the mixture flows out at the same rate, find the mass of salt in the tank after 40 minutes.

$x(T)$ = mass of salt at time T

input: $(0.3)\ kg/L \cdot 10\ L/min = 3\ kg/min$

output: $\frac{x}{400}\ kg/L \cdot 10\ L/min = \frac{x}{40}\ kg/min$

$\frac{dx}{dt} = 3 - \frac{x}{40}$ (Separable or linear)

$\frac{dx}{dt} = \frac{120 - x}{40}$

$\int \frac{40}{120-x}\,dx = \int dT$

$-40\ln(120-x) = T + C''$

$\ln(120-x) = -\frac{T}{40} + C'$

$120 - x = C e^{-T/40}$

$x = 120 - C e^{-T/40}$

$x(0) = 5 = 120 - C$

$C = 115$

$x(T) = 120 - 115 e^{-T/40}$

$x(40) = 120 - 115 e^{-1}$

4. A cup of coffee is brewed at 200° C outdoors on a 20° C day. 30 minutes later the coffee is down to 140° C. Find an expression for the temperature of the coffee t minutes after it is brewed.

$T(t)$ = temp of coffee at time t

$\frac{dT}{dt} = K(T-20)$ (Separable or linear)

$\int \frac{1}{T-20}\,dT = \int K\,dt$

$\ln(T-20) = Kt + C'$

$T-20 = Ce^{Kt}$

$T = 20 + Ce^{Kt}$

$T(0) = 200 = 20 + C$

$C = 180$

$T(30) = 140 = 20 + 180e^{30K}$

$120 = 180e^{30K}$

$\frac{2}{3} = e^{30K}$

$K = \frac{1}{30}\ln\frac{2}{3}$

$T = 20 + 180e^{\frac{\ln(2/3)}{30}t}$

or simplify to

$T = 20 + 180\left(\frac{2}{3}\right)^{T/30}$