MAP2302
Elementary Differential Equations
Exam 2

Name: Solutions

UFID:

Instructions:

- Read each problem carefully.
- Show all your work; you will not get credit for answers with no work even if they are correct.
- The proctor will not answer questions about the material on the exam or give hints to any of the problems; do your best to answer each question as written.
- Students should not have calculators, phones, or paper on their desk and they should not wear headphones. No student writing should be in a position where it is visible.
- Implicit solutions will get full credit unless the problem asks for an explicit solution.
- All numerical answers should be left in exact form (i.e. use ln(2), not $\approx .7$).
- The proctor will have additional scratch paper if needed.

1. Suppose a cannonball of mass 5 kg is fired from ground level straight up in the air with an initial velocity of 200 m/s. Suppose that acceleration due to gravity is 9.8 m/s^2 and the force due to air resistance is proportional to the square of the velocity of the cannonball with proportionality constant .3.

   (a) Write an initial value problem which models the motion of the cannonball from the time it is fired until it reaches its maximum height.

$h(t)$ = height of cannonball at time t

$v(t) = \frac{dh}{dt}$

Force due to gravity: -49

force due to Air resistance: $-0.3v^2$

$m = 5$

$m \cdot a = F$

$g = 9.8$

$mg = 49$

$$\boxed{5\frac{dv}{dt} = -49 - 0.3v^2 \qquad h(0)=0, \quad v(0)=200}$$

   (b) After 2 seconds the cannonball reaches its maximum height of 56 meters above the ground and begins to fall. Write an initial value problem which models the motion of the cannonball as it falls to the ground.

gravity Same as above,
air resistance flips Sign.

$$\boxed{5\frac{dv}{dt} = 0.3v^2 - 49 \qquad h(2) = 56, \quad v(2) = 0}$$

2. Give a general solution to the following differential equations.

(a) $y'' - 3y' - 10y = 0$.

$r^2 - 3r - 10 = 0$

$(r-5)(r+2) = 0$

$r = 5, \ r = -2$

$Y_1(T) = e^{5T} \quad Y_2(T) = e^{-2T}$

$$\boxed{Y(T) = c_1 e^{5T} + c_2 e^{-2T}}$$

(b) $y'' - 2y' + 5y = 0$.

$r^2 - 2r + 5$

$r = \frac{2 \pm \sqrt{4-20}}{2}$

$r = 1 \pm 2i$

$Y_1(T) = e^T \sin 2T$

$Y_2(T) = e^T \cos 2T$

$$\boxed{Y(T) = c_1 e^T \sin 2T + c_2 e^T \cos 2T}$$

$A_2 = 1$

$A_1 + 4A_2 = 0 \rightarrow A_1 = -4$

(c) $y'' + 2y' + y = t^2 - 6.$

$A_0 + 2A_1 + 2A_2 = 6 \rightarrow A_0 + 6 = 6$

$A_0 = 0$

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$Y_p = T^2 - 4T$

$Y_1 = e^{-T}, Y_2 = Te^{-T}$

$Y_p(T) = A_2T^2 + A_1T + A_0$

$\boxed{Y(T) = T^2 - 4T + C_1e^{-T} + C_2Te^{-T}}$

$Y_p'(T) = 2A_2T + A_1$

$Y_p''(T) = 2A_2$

$2A_2 + 4A_2T + 2A_1 + A_2T^2 + A_1T + A_0 = T^2 - 6$

$A_2T^2 + (A_1 + 4A_2)T + (A_0 + 2A_1 + 2A_2) = T^2 - 6$

3. Give a particular solution to the following differential equations.

(a) $y'' + 2y' + y = e^{-t}.$

$Y_p = CT^2e^{-T}$

$Y_p' = 2CTe^{-T} - CT^2e^{-T}$

$Y_p'' = 2Ce^{-T} - 2CTe^{-T} - 2CTe^{-T} + CT^2e^{-T}$

$= CT^2e^{-T} - 4CTe^{-T} + 2Ce^{-T}$

$\cancel{CT^2e^{-T}} - \cancel{4CTe^{-T}} + 2Ce^{-T} + \cancel{4CTe^{-T}} - \cancel{2CT^2e^{-T}} + \cancel{CT^2e^{-T}} = e^{-T}$

$2Ce^{-T} = e^{-T}$

$2C = 1$

$C = 1/2$

$\boxed{Y_p(T) = \tfrac{1}{2}T^2e^{-T}}$

(b) $y'' + 2y' + y = t^2 - 6 - 2e^{-t}$.

By Superposition,

$$Y_p(T) = (T^2 - 4T) - 2(\tfrac{1}{2}T^2 e^{-T})$$

$$\boxed{Y_p(T) = T^2 - 4T - T^2 e^{-T}}$$

(c) $y'' - 2y' + 5y = \sin t$.

$$Y_p = A \sin T + B \cos T$$

$$Y_p' = A \cos T - B \sin T$$

$$Y_p'' = -A \sin T - B \cos T$$

$$-A \sin T - B \cos T - 2A \cos T + 2B \sin T + 5A \sin T + 5B \cos T = \sin T$$

$$(4A + 2B) \sin T + (-2A + 4B) \cos T = \sin T$$

$$4A + 2B = 1$$

$$-2A + 4B = 0$$

$$\boxed{Y_p(T) = \frac{1}{5} \sin T + \frac{1}{10} \cos T}$$

$$10B = 1$$

$$B = 1/10$$

$$4A = 1 - 2/10$$

$$A = (\tfrac{1}{4})(\tfrac{8}{10}) = 1/5$$

4. Given that $y_1(t) = 1$ and $y_2(t) = \ln(t)$ are solutions to the homogeneous equation $y'' + t^{-1}y' = 0$, solve the initial value problem

$$y'' + t^{-1}y' = t^{-2}$$
$$y(1) = 2, \; y'(1) = -1.$$

$Y_p(T) = V_1(T)Y_1(T) + V_2(T)Y_2(T)$

$W(Y_1, Y_2) = Y_1Y_2' - Y_1'Y_2 = \left(\frac{1}{T}\right) - 0(\ln T) = 1/T$

$V_1(T) = \int -(T)(T^{-2})\ln T \, dT = -\int \frac{\ln T}{T} dT$

$= -\int u \, du = -\frac{1}{2}u^2$ $\quad u = \ln T$, $du = \frac{1}{T} dT$

$= -\frac{1}{2}(\ln T)^2$

$V_2(T) = \int (T)(T^{-2})(1) dT = \int \frac{1}{T} dT = \ln T$

$Y_p(T) = -\frac{1}{2}(\ln T)^2 + (\ln T)^2 = \frac{1}{2}(\ln T)^2$

general solution: $Y(T) = \frac{1}{2}(\ln T)^2 + C_1 + C_2 \ln T$

$2 = Y(1) = 0 + C_1 + 0 \Rightarrow 2 = C_1$

$Y'(T) = (\ln T)\left(\frac{1}{T}\right) + (C_2)\left(\frac{1}{T}\right)$

$-1 = Y'(1) = 0 + C_2 \Rightarrow -1 = C_2$

$$\boxed{Y(T) = \frac{1}{2}(\ln T)^2 - \ln T + 2}$$