MAP2302
Elementary Differential Equations
Exam 3

Name: Solutions

UFID:______________________________________

Instructions:

- Read each problem carefully.
- Show all your work; you will not get credit for answers with no work even if they are correct.
- The proctor will not answer questions about the material on the exam or give hints to any of the problems; do your best to answer each question as written.
- Students should not have calculators, phones, or paper on their desk and they should not wear headphones. No student writing should be in a position where it is visible.
- All numerical answers should be left in exact form (i.e. use $\ln(2)$, not $\approx .7$).
- The proctor will have additional scratch paper if needed.

1. Three identical springs with spring constant $k = 3$ N/m and two objects of identical mass $m = 1$ kg are attached in a straight line with the ends of the outside springs fixed. If $x(t)$ denotes the position of the first object at time t and $y(t)$ denotes the position of the second object, then this system is governed by the equations

$$mx'' = -kx + k(y - x) \rightarrow x'' = -6x + 3y$$

$$my'' = -k(y - x) - ky. \qquad y'' = 3x - 6y$$

Suppose $x(0) = 6$, $y(0) = 2$, and $x'(0) = y'(0) = 0$. Solve for $x(t)$ and $y(t)$.

$$\boxed{\begin{aligned} X(T) &= 4\cos(\sqrt{3}T) + 2\cos(3T) \\ Y(T) &= 4\cos(\sqrt{3}T) - 2\cos(3T) \end{aligned}}$$

(a) $(D^2+6)X - 3Y = 0$

(b) $-3X + (D^2+6)Y = 0$

$(D^2+6)(a) + 3(b)$

$(D^2+6)(D^2+6)X - \cancel{(D^2+6)3Y} - 9X + \cancel{3(D^2+6)Y} = 0$

$(D^4 + 12D^2 + 36)X - 9X = 0$

$(D^4 + 12D^2 + 27)X = 0$

$(D^2+3)(D^2+9)X = 0$

Roots: $\pm\sqrt{3}i, \pm 3i$

$x(T) = C_1 \sin(\sqrt{3}T) + C_2\cos(\sqrt{3}T) + C_3 \sin 3T + C_4 \cos 3T$

$3Y(T) = (D^2+6)X$

$D^2 x = -3C_1\sin(\sqrt{3}T) - 3C_2\cos(\sqrt{3}T) - 9C_3\sin(3T) - 9C_4\cos(3T)$

$(D^2+6)X = 3C_1 \sin\sqrt{3}T + 3C_2\cos(\sqrt{3}T) - 3C_3\sin 3T - 3C_4\cos(3T)$

$Y(T) = C_1\sin(\sqrt{3}T) + C_2\cos(\sqrt{3}T) - C_3\sin 3T - C_4\cos 3T$

$\left.\begin{aligned} 6 &= X(0) = C_2 + C_4 \\ 2 &= Y(0) = C_2 - C_4 \end{aligned}\right\} \begin{aligned} 2C_2 &= 8 \\ C_2 &= 4 \\ C_4 &= 2 \end{aligned}$

$\left.\begin{aligned} 0 &= X'(0) = \sqrt{3}C_1 + 3C_3 \\ 0 &= Y'(0) = \sqrt{3}C_1 - 3C_3 \end{aligned}\right\} \begin{aligned} C_1 &= 0 \\ C_3 &= 0 \end{aligned}$

2. Give a fundamental solution set for the homogeneous equation

$$(D^2 - 2D + 3)^3(D^4 + 2D^3 - 15D^2)[y] = 0.$$

$r^2 - 2r + 3 = 0$

$(D^2)(D+5)(D-3)$

$r = \dfrac{2 \pm \sqrt{4-12}}{2}$

$1, \quad T \qquad e^{-5T} \qquad e^{3T}$

~~$R = 2 \pm \sqrt{2}\, i$~~

$r = 1 \pm \sqrt{2}\, i$

multiplicity 3,

$Y_1(T) = e^T \cos\sqrt{2}T, \; Y_2(T) = e^T \sin(\sqrt{2}T)$

$Y_3(T) = Te^T\cos(\sqrt{2}T), \; Y_4(T) = Te^T\sin(\sqrt{2}T)$

$Y_5(T) = T^2 e^T \cos(\sqrt{2}T), \; Y_6(T) = T^2 e^T \sin(\sqrt{2}T)$

$Y_7(T) = 1$

$Y_8(T) = T$

$Y_9(T) = e^{-5T}$

$Y_{10}(T) = e^{3T}$

3. Find a differential operator A such that A annhilates $e^x + 3e^{2x}$. Then determine a particular solution to

$$y''' - y' = e^x + 3e^{2x}.$$

$(D-1)\,e^x = 0$

$(D-2)\,3e^{2x} = 0$

$$\boxed{A = (D-1)(D-2) = D^2 - 3D + 2}$$

$$(D^3 - D)[y] = e^x + 3e^{2x}$$

$(D-1)(D-2)(D)(D^2-1)\,y = 0$

$(D-1)(D-2)D(D-1)(D+1)$

$xe^x \quad e^{2x} \quad \cancel{1} \quad \cancel{e^x} \quad \cancel{e^{-x}}$

$y_p(x) = Axe^x + Be^{2x}$

$y_p'(x) = Ae^x + Axe^x + 2Be^{2x}$

$y_p''(x) = Ae^x + Ae^x + Axe^x + 4Be^{2x}$

$\quad = 2Ae^x + Axe^x + 4Be^{2x}$

$y_p'''(x) = 3Ae^x + Axe^x + 8Be^{2x}$

$(D^3 - D)[y_p] = 2Ae^x + 6Be^{2x} = e^x + 3e^{2x}$

$2A = 1 \to A = 1/2$

$6B = 3 \to B = 1/2$

$$\boxed{y_p(x) = \tfrac{1}{2}xe^x + \tfrac{1}{2}e^{2x}}$$

4. Given that $\{1, x^2, x^{-2}\}$ is a fundamental solution set for $y''' + 3x^{-1}y'' - 3x^{-2}y' = 0$ on the interval $x > 0$, determine a particular solution on this interval to

$$y''' + 3x^{-1}y'' - 3x^{-2}y' = x^{-1}.$$

$$W(1, x^2, x^{-2}) = \begin{vmatrix} 1 & x^2 & x^{-2} \\ 0 & 2x & -2x^{-3} \\ 0 & 2 & 6x^{-4} \end{vmatrix} = 12x^{-3} + 4x^{-3} = 16x^{-3}$$

$$W_3 = +W(1, x^2) = \begin{vmatrix} 1 & x^2 \\ 0 & 2x \end{vmatrix} = 2x$$

$$W_2 = -W(1, x^{-2}) = -\begin{vmatrix} 1 & x^{-2} \\ 0 & -2x^{-3} \end{vmatrix} = +2x^{-3}$$

$$W_1 = +W(x^2, x^{-2}) = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = -2x^{-1} - 2x^{-1} = -4x^{-1}$$

$$V_1(x) = \int \frac{(x^{-1})(-4x^{-1})}{16x^{-3}}\,dx = -\frac{1}{4}\int x\,dx = -\frac{1}{8}x^2$$

$$V_2(x) = \int \frac{(x^{-1})(2x^{-3})}{16x^{-3}}\,dx = \frac{1}{8}\int x^{-1}\,dx = \frac{1}{8}\ln x$$

$$V_3(x) = \int \frac{(x^{-1})(2x)}{16x^{-3}}\,dx = \frac{1}{8}\int x^3\,dx = \frac{1}{32}x^4$$

$$Y_p(x) = V_1(x)Y_1(x) + V_2(x)Y_2(x) + V_3(x)Y_3(x)$$

$$Y_p(x) = \left(-\tfrac{1}{8}x^2\right)(1) + \left(\tfrac{1}{8}\ln x\right)x^2 + \left(\tfrac{1}{32}x^4\right)x^{-2}$$

$$\boxed{Y_p(x) = -\frac{3}{32}x^2 + \frac{1}{8}x^2 \ln x}$$