1. If $x^2 + xy + y^3 = 1$, express y'' in terms of x and y. (Do not simplify your answer).

$$y'' = \frac{(-2 - \frac{-2x-y}{x+3y^2})(x+3y^2) - (-2x-y)(1+6y(\frac{-2x-y}{x+3y^2}))}{(x+3y^2)^2}$$

2. Find the equation of the line tangent to the curve $2\cos x \sin y = 1$ at the point $(\frac{\pi}{4}, \frac{\pi}{4})$.

$$y - \frac{\pi}{4} = x - \frac{\pi}{4}$$

3. Evaluate the following derivatives.

   (a) $\frac{d}{dx}x^x = (x^x)(\ln(x)+1)$

   (b) $\frac{d}{dx}(\cos(x))^{2x+1} = (\cos(x)^{2x+1})(2\ln(\cos(x)) - (2x+1)(\tan(x)))$

4. Suppose the cost of producing a certain commodity is given by $C(x) = 500 + 10x + \frac{1}{5}x^2$.

   (a) What is the average rate of change of cost when the production is increased from 20 to 25?

   $$\frac{C(25) - C(20)}{25 - 20} = 19$$

   (b) What is the marginal cost (or instantneous rate of change of cost) when $x = 20$?

   $$C'(20) = 18$$

5. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height after t seconds is given by $H = 10t - \frac{9}{5}t^2$. With what velocity will the rock hit the ground?
   $v(t) = H'(t) = 10 - \frac{18}{5}t$. $H(t) = 0$ when $t = \frac{50}{9}$. $v(\frac{50}{9}) = -10$ m/s.

6. The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?
   Given: $\frac{dl}{dt} = 8$ cm/sec and $\frac{dw}{dt} = 3$ cm/sec. Want: $\frac{dA}{dt}$ when $l = 20$ cm and $w = 10$ cm. $A = lw$, so $\frac{dA}{dt} = l\frac{dw}{dt} + \frac{dl}{dt}w = 3l + 8w$. When $l = 20$ and $w = 10$, $\frac{dA}{dt} = 3(20) + 8(10) = 140$ cm^2/sec.

7. An inverted cone of height 10 ft and base radius 5 ft is full of water which begins to leak out the bottom. If the height of the water is dropping at a rate of 1 ft every 4 minutes, at what rate is the water leaking out of the cone when the height of the water is 8 ft?
   When $h = 8$, $\frac{dV}{dt} = \pi$ ft^3/min.

8. Let $f(t) = t\sqrt{4 - t^2}$

   (a) Find all critical numbers of the function f.
   $t = \sqrt{2}$ and $t = -\sqrt{2}$.

   (b) Find the absolute maximum and absolute minimum of f on $[-1, 2]$.
   abs max: $f(\sqrt{2}) = 2$
   abs min: $f(-1) = -\sqrt{3}$

9. Find the local minimums and maximums of $h(x) = 2 + 3x^2 - x^3$ (use the first or second derivative test to verify your answers).

   local min at $x = 0$, local max at $x = 2$.

10. Sketch the graph of a function that is continuous on $[1, 5]$ and has an absolute minimum at 2, and absolute maximum at 3, and a local minimum at 4.

11. For the following function, find all local maximums and minimums, intervals of increase and decrease, concavity, inflection points, and vertical and horizontal asymptotes. Then use this information to sketch the graph of the function.

$$f(x) = 1 + \frac{2}{x} + \frac{3}{x^2}$$

   Local min at $x = -3$. No local max. Increasing on $(-3, 0)$, decreasing on $(-\infty, -3) \cup (0, \infty)$. Concave down on $(-\infty, -\frac{9}{2})$, concave up on $(-\frac{9}{2}, 0) \cup (0, \infty)$. Inflection point at $x = -\frac{9}{2}$. Vertical asymptote is $x = 0$, horizontal asymptote is $y = 1$.

12. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest from the point $(1, 0)$.

   Distance^2 from $(1, 0)$ to (x, y) is $(x - 1)^2 + y^2$. Constraint is $4x^2 + y^2 = 4$, so $y^2 = 4 - 4x^2$. So we want to maximize $f(x) = (x - 1)^2 + (4 - 4x^2) = 5 - 2x - 3x^2$. $f'(x) = -2 - 6x$, so the critical number is $x = -\frac{1}{3}$.$f''(x) = -6$, so f has a maximum maximum at $x = -\frac{1}{3}$. the corresponding points on the ellipse are $(-\frac{1}{3}, \frac{4\sqrt{2}}{3})$ and $(-\frac{1}{3}, -\frac{4\sqrt{2}}{3})$.

13. Suppose cylindrical can is made to hold 1000 cm^3 of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

   radius $= \sqrt[3]{\frac{500}{\pi}}$, height $= 2\sqrt[3]{\frac{500}{\pi}}$

14. Let A denote the area under the curve $\cos x$ on the interval $[0, \frac{\pi}{2}]$. Draw and label a picture of the area under this curve being approximated by the *right approximating sum* with 4 rectangles. Use the picture to decide if R_4 is an over approximation or an under approximation of A.

   under approximation

15. State the definition of $\int_a^b f(x)dx$.

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

   Where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

16. Fundamental Theorem of Calculus

    (a) State (both parts of) the Fundamental Theorem of Calculus.
    Suppose f is continuous on $[a, b]$. Then for $a \leq x \leq b$,

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

    and if $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

(b) Explain why the following expression cannot be evaluated using the Fundamental Theorem of Calculus.

$$\int_{\pi/3}^{2\pi/3} \sec^2\theta d\theta$$

$\sec^2(\theta)$ is not continuous on $[\frac{\pi}{3}, \frac{2\pi}{3}]$.

(c) Simplify the following expressions:

i. $\frac{d}{dx}\int_0^{\cos x}(t^4+6)dt = ((\cos(x))^4+6)(-\sin(x))$

ii. $\frac{d}{dx}\int_x^{\pi}\sin t dt = -\sin(x)$

17. Evaluate the following definite integrals.

(a) $\int_{-\pi/7}^{\pi/7}(2-5\sin\theta)d\theta = 2\theta + 5\cos\theta|_{-\pi/7}^{\pi/7} = \frac{4\pi}{7}$

(b) $\int_0^{\pi}\cos x dx = \sin x|_0^{\pi} = 0$

(c) $\int_0^{\pi/4}\tan\theta\sec^2\theta d\theta = \frac{1}{2}(\tan\theta)^2|_0^{\pi/4} = \frac{1}{2}$

(d) $\int_1^4 \frac{x-\sqrt{x}}{x^3}dx = (-x^{-1}+\frac{2}{3}x^{-\frac{3}{2}})|_1^4 = \frac{1}{6}$

(e) $\int_0^1(2x+1)^{-3}dx = \frac{1}{2}\int_1^3 u^{-3}du = -\frac{1}{4}u^{-2}|_1^3 = \frac{23}{104}$

18. Suppose the velocity of a particle at time t is given by $v(t) = 10\sin 2t$, $0 \leq t \leq \pi$.

(a) Find the total displacement of the particle.

$$\int_0^{\pi} 10\sin(2t)dt = -5\cos(2t)|_0^{\pi} = 0$$

(b) Find the total distance travelled by the particle.

$$\int_0^{\frac{\pi}{2}} 10\sin(2t)dt - \int_{\frac{\pi}{2}}^{\pi} 10\sin(2t)dt = -5\cos(2t)|_0^{\frac{\pi}{2}} + 5\cos(2t)|_{\frac{\pi}{2}}^{\pi} = 5+5+5-5 = 10$$

19. The population of an endangered species changes at a rate given by $P'(t) = 30-20t$ (individuals / year). Assume the initial population of the species is 300 individuals.

(a) What is the population after 5 years?

net change is $\int_0^5 30-20tdt = 30t-10t^2|_0^5 = -100$, so population is $300-100=200$.

(b) When will the species become extinct? (population decreases to 0)

$t = \frac{3+\sqrt{129}}{2} \sim 7.36$ years

20. Evaluate the following indefinite integrals.

(a) $\int 2x(x^2-1)^{99}dx = \frac{1}{100}(x^2-1)^{100}+C$

(b) $\int \frac{\cos x}{1-\cos^2 x}dx = -\csc(x)+C$

(c) $\int(\sin 5y + \sec 2y\tan 2y)dy = -\frac{1}{5}\cos(y) + \frac{1}{2}\sec(2y)+C$

(d) $\int x^3\sqrt{x^2+1}dx = \frac{1}{5}(x^2+1)^{\frac{5}{2}} - \frac{1}{3}(x^2+1)^{\frac{3}{2}}+C$

(e) $\int(6x^2-4)dx = 2x^3-4x+C$