Problems marked with * are to be turned in for grading.

1. Let \((s_n)\) be a sequence of complex numbers. For each \(n \geq 1\) define

\[
\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}
\]

a) Prove that if \(s \in \mathbb{C}\) and \(s_n \to s\), then \(\sigma_n \to s\).

b) Give an example of a divergent sequence \((s_n)\) for which \((\sigma_n)\) converges.

c) Give an example of an unbounded sequence \((s_n)\) for which \((\sigma_n)\) converges.

*2. Let \(X\) be a complete metric space. Prove that if \(E_n\) is a sequence of nonempty, closed subsets of \(X\) such that \(E_{n+1} \subseteq E_n\) for all \(n\) and

\[
\lim_{n \to \infty} \text{diam } E_n = 0,
\]

then \(\cap_{n=1}^{\infty} E_n\) consists of exactly one point. Does the conclusion still hold if \(X\) is not assumed complete? Prove, or give a counterexample.

3. Let \(X\) be a complete metric space. Prove that if \(E\) is a perfect subset of \(X\), then \(E\) is uncountable. Give an example to show that the completeness hypothesis cannot be dropped. (Hint: Imitate the proof we gave for \(X = \mathbb{R}\) and modify it so that the sets \(V_n\) satisfy \(\text{diam}(V_n) \to 0\).)

4. Prove that for any bounded sequences of real numbers \((s_n), (t_n)\)

\[
\limsup_{n \to \infty} (s_n + t_n) \leq \limsup_{n \to \infty} s_n + \limsup_{n \to \infty} t_n.
\]

Give an example where the inequality is strict. Formulate and prove an analogous statement for \(\liminf\).

5. Let \((s_n)\) be a sequence of real numbers and \(s \in \mathbb{R}\). Prove that \(s_n \to s\) if and only if

\[
\limsup_{n \to \infty} s_n = \liminf_{n \to \infty} s_n = s
\]

6. Let \(s_1 = 1\) and define for \(n > 1\)

\[
s_{n+1} = \sqrt{1 + s_n}.
\]

Prove that \((s_n)\) converges and find its limit.

7. Find \(\lim_{n \to \infty} (\sqrt{n^2 + n} - n)\).