Problems marked with * are to be turned in for grading.

1. Let $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$. Suppose there exist constants $0 < \alpha \leq 1$ and $C > 0$ such that for all $x, y \in E$,
   \[ |f(x) - f(y)| \leq C|x - y|^{\alpha}. \]  
   (1)

   a) Prove that $f$ is uniformly continuous on $E$.
   
   b) Prove that if $E = (a, b) \subset \mathbb{R}$ and (1) holds for some $\alpha > 1$, then $f$ is constant.

2. Suppose $f'(x) > 0$ on $(a, b)$. Prove that $f$ is strictly increasing (hence one-to-one) on $(a, b)$. Let $g$ be the inverse function to $f$. Prove that $g$ is differentiable on $(a, b)$ and $g'(f(x)) = \frac{1}{f'(x)}$.

3. Suppose $f'$ is continuous on $[a, b]$ and $\epsilon > 0$. Prove that there exists $\delta > 0$ such that
   \[ \left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon \]
   whenever $t, x \in [a, b]$ and $0 < |t - x| < \delta$.

4. Recall that a function $f : (a, b) \to \mathbb{R}$ is called convex if for every $a < x < y < b$ and every $0 < t < 1$,
   \[ f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y). \]
   (In other words, the graph of $f$ always lies below its secant line). Prove that if $f$ is differentiable on $(a, b)$, then $f$ is convex if and only if $f'$ is increasing. Prove that if $f''$ exists at every point of $(a, b)$, then $f$ is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.

5. Suppose $f : (0, \infty) \to \mathbb{R}$ is differentiable. a) Give an example of such an $f$ with $f'(x) \to 0$ and $f(x) \to +\infty$ as $x \to +\infty$. b) If $f$ is bounded and $f'(x) \to 0$ as $x \to +\infty$, does this imply that $\lim_{x \to +\infty} f(x)$ exists? Prove, or give a counterexample.

*6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and there exists a constant $C < 1$ so that $|f'(x)| \leq C$ for all $x \in \mathbb{R}$. Prove that $f$ has a fixed point, i.e. there exists at least one $x \in \mathbb{R}$ so that $f(x) = x$. (Hint: pick an arbitrary $x_1$ and define a sequence $x_n$ recursively by $x_{n+1} = f(x_n)$.)