*1. Suppose \((f_n)\) is a sequence of functions, each continuous on \([a, b]\) and differentiable on \([a, b]\). Suppose that the sequences \((f_n)\) and \((f'_n)\) are uniformly bounded on \([a, b]\). Prove that \((f_n)\) has a uniformly convergent subsequence.

*2. a) Let \(f\) be continuous on \([0, 1]\) and suppose that
\[
\int_0^1 x^n f(x) \, dx = 0
\]
for all integers \(n \geq 0\). Prove that \(f = 0\).

b) What if (1) is assumed to hold only for even integers \(n \geq 0\)? Is it still the case that \(f\) must be 0?

c) Repeat parts (a) and (b) with the interval \([0, 1]\) replaced by \([-1, 1]\).

3. Let \(T\) denote the unit circle in the complex plane: \(T = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}\). Let \(A\) denote the set of trigonometric polynomials \(f : T \to \mathbb{C}\) of the form
\[
f(e^{i\theta}) = \sum_{n=0}^{N} a_n e^{in\theta}
\]
for all integers \(N \geq 0\) and all complex numbers \(a_0, \ldots, a_N\). Prove that \(A\) is a subalgebra of \(C(T)\), and that \(A\) separates points of \(T\) and is nonvanishing on \(T\). Prove, however, that \(A\) is not dense in \(C_c(T)\). Hint: to show that \(A\) is not dense, first show that for all \(f\) in the closure of \(A\),
\[
\int_0^{2\pi} f(e^{i\theta}) e^{i\theta} \, d\theta = 0.
\]

4. Let \(X, Y\) be compact metric spaces and consider the metric space \(X \times Y\) equipped with the metric \(d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)\). Prove that for every \(f \in C(X \times Y)\) and every \(\epsilon > 0\), there exist functions \(g_1, \ldots, g_n \in C(X)\) and \(h_1, \ldots, h_n \in C(Y)\) such that
\[
|f(x, y) - \sum_{j=1}^{n} g_j(x)h_j(y)| < \epsilon \quad \text{for all } x \in X, y \in Y.
\]