Problems marked with * are to be turned in for grading.

1. Let $f_n, f, g_n, g \in L^1$. Suppose $f_n \to f$ and $g_n \to g$ a.e., $|f_n| \leq g_n$, and $\int g_n \to \int g$. Prove that $\int f_n \to \int f$. (Rework the proof of the dominated convergence theorem.)

2. Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Prove that $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$. (Use Problem 1.)

3. Let $E$ be a measurable subset of $\mathbb{R}$, let $I$ be an open interval of $\mathbb{R}$, and let $f : I \times E \to \mathbb{R}$ be a function. Suppose that:
   - for each $t \in E$, the function $x \to f(x, t)$ is continuous on $I$,
   - for each $x \in I$, the function $t \to f(x, t)$ is measurable on $E$, and
   - there is a function $g \in L^1(E)$ such that for all $(x, t) \in I \times E$ we have $|f(x, t)| \leq g(t)$.

   Prove that the expression
   $$F(x) = \int_E f(x, t) \, dt$$
   defines a continuous function of $x$ on $I$.

*4. Let $E$ be a measurable subset of $\mathbb{R}$, let $I$ be an open interval of $\mathbb{R}$, and let $f : I \times E \to \mathbb{R}$ be a function. Suppose that:
   - for each $t \in E$, the function $x \to f(x, t)$ is differentiable on $I$,
   - for each $x \in I$, the function $t \to f(x, t)$ is absolutely integrable on $E$, and
   - there is a function $g \in L^1(E)$ such that for all $(x, t) \in I \times E$, we have $\left| \frac{\partial f}{\partial x}(x, t) \right| \leq g(t)$.

   Prove that the expression
   $$F(x) = \int_E f(x, t) \, dt$$
   defines a differentiable function of $x$ on $I$, with
   $$F'(x) = \int_E \frac{\partial f}{\partial x}(x, t) \, dt$$
   for all $x \in I$.

*5. (The Laplace transform.) Let $f : [0, +\infty) \to \mathbb{R}$ be a measurable function, and suppose there exist constants $a, M > 0$ so that
   $$|f(t)| \leq Me^{at} \quad \text{for all } t \geq 0.$$  
(1)
The Laplace transform of $f$ is the function $F(s)$ defined by the expression

$$F(s) = \int_0^\infty e^{-st} f(t) \, dt$$

for those values of $s \in \mathbb{R}$ for which the integral makes sense. Prove the following:

a) $F(s)$ is defined for all $s > a$.

b) $\lim_{s \to +\infty} F(s) = 0$.

c) $F(s)$ is infinitely differentiable for all $s > a$.

6. Evaluate each of the following limits. Carefully justify your answers.

a) $\lim_{n \to \infty} \int_0^\infty \left( 1 + \frac{x}{n} \right)^{-n} \sin \left( \frac{x}{n} \right) \, dx$

b) $\lim_{n \to \infty} \int_0^\infty \frac{1 + nx^2}{(1 + x^2)^n} \, dx$

c) $\lim_{y \to 0^+} \int_{-\infty}^{\infty} \frac{y}{y^2 + x^2} \, dx$

d) $\lim_{s \to 0^+} \int_0^\infty e^{-st} \sin t \, \frac{dt}{t}$