Contents

MAA6616 COURSE NOTES
FALL 2019

1. σ-ALGEBRAS

Let X be a set, and let 2^X denote the set of all subsets of X. Let E^c denote the *complement* of E in X, and for $E, F \subset X$, write $E \setminus F = E \cap F^c$.

Definition 1.1. Let X be a set. A *Boolean algebra* is a nonempty collection $\mathscr{A} \subset 2^X$ that is closed under finite unions and complements. A *σ-algebra* is a Boolean algebra that is also closed under countable unions.

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Remark 1.2. If $\mathcal{E} \subset 2^X$ is any collection of sets in X, then

$$\left(\bigcup_{E \in \mathcal{E}} E^c \right)^c = \bigcap_{E \in \mathcal{E}} E. \tag{1}$$

Hence a Boolean algebra (resp. σ-algebra) is automatically closed under finite (resp. countable) intersections. It follows that a Boolean algebra (and a σ-algebra) on X always contains $\varnothing$ and X. (Proof: $X = E \cup E^c$ and $\varnothing = E \cap E^c$.) $\diamond$

Definition 1.3. A *measurable space* is a pair $(X, \mathscr{M})$ where $\mathscr{M} \subset 2^X$ is a σ-algebra. A function $f : X \to Y$ from one measurable space $(X, \mathscr{M})$ to another $(Y, \mathscr{N})$ is *measurable* if $f^{-1}(E) \in \mathscr{M}$ whenever $E \in \mathscr{N}$. $\triangleleft$

Definition 1.4. A *topological space* $X = (X, \tau)$ consists of a set X and a subset τ of 2^X such that

(i) $\emptyset, X \in \tau$;
(ii) τ is closed under finite intersections;
(iii) τ is closed under arbitrary unions.

The set τ is a *topology* on X.

(a) Elements of τ are *open sets*;
(b) A subset S of X is *closed* if $X \setminus S$ is open;
(c) S is a G_δ if $S = \cap_{j=1}^{\infty} O_j$ for open sets O_j;
(d) S is an F_σ if it is an (at most) countable union of closed sets;
(e) A subset C of X is *compact*, if for any collection $\mathcal{F} \subset \tau$ such that $C \subset \cup\{T : T \in \mathcal{F}\}$ there exist a finite subset $\mathcal{G} \subset \mathcal{F}$ such that $C \subset \cup\{T : T \in \mathcal{G}\}$; and
(f) If (X, τ) and (Y, σ) are topological spaces, a function $f : X \to Y$ is *continuous* if $S \in \sigma$ implies $f^{-1}(S) \in \tau$.

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$\triangleleft$

Example 1.5. If (X, d) is a metric space, then the collection τ of open sets (in the metric space sense) is a topology on X. There are important topologies in analysis that are not metrizable (do not come from a metric). $\triangle$

Remark 1.6. There is a superficial resemblance between measurable spaces and topological spaces and between measurable functions and continuous functions. In particular, a topology on X is a collection of subsets of X closed under arbitrary unions and finite intersections, whereas for a σ-algebra we insist only on countable unions, but require complements also. For functions, recall that a function between topological spaces is continuous if and only if pre-images of open sets are open. The definition of measurable function is plainly similar. The two categories are related by the Borel algebra construction appearing later in these notes. $\diamond$

The disjointification trick in the next Proposition is often useful.

Proposition 1.7 (Disjointification)**.** *Suppose $\emptyset \neq \mathscr{M} \subset 2^X$ is closed with respect to complements, finite intersections and countable disjoint unions. If $(G_j)_{j=1}^{\infty}$ is a sequence of sets from $\mathscr{M}$, then there exists a sequence $(F_j)_{j=1}^{\infty}$ of pairwise disjoint sets from $\mathscr{M}$ such that*

$$\bigcup_{j=1}^{n} F_j = \bigcup_{j=1}^{n} G_j$$

for n either a positive integer or ∞.

Hence, $\mathscr{M}$ is a σ-algebra if and only if $\mathscr{M}$ is closed under complement, finite intersections and countable disjoint unions. †

Proof. The proof amounts to the observation that if (G_n) is a sequence of subsets of X, then the sets

$$F_n = G_n \setminus \left(\bigcup_{k=1}^{n-1} G_k \right) = G_n \cap (\cap_{k=1}^{n-1} G_k^c) \tag{2}$$

are disjoint, in $\mathscr{M}$ and $\bigcup_{j=1}^{n} F_j = \bigcup_{j=1}^{n} G_j$ for all $n \in \mathbb{N}^+$ (and thus $\bigcup_{j=1}^{\infty} F_j = \bigcup_{j=1}^{\infty} G_j$).

To prove the second part of the Proposition, given a sequence (G_n) from $\mathscr{M}$ use the disjointification trick to obtain a sequence of disjoint sets $F_n \in \mathscr{M}$ such that $\cup G_n = \cup F_n$. $\square$

Example 1.8. Let X be a nonempty set.

(a) The power set 2^X is the largest σ-algebra on X.
(b) At the other extreme, the set $\{\varnothing, X\}$ is the smallest σ-algebra on X.
(c) Let X be an uncountable set. The collection

$$\mathscr{M} = \{E \subset X : E \text{ is at most countable or } X \setminus E \text{ is at most countable } \} \tag{3}$$

is a σ-algebra (the proof is left as an exercise).

(d) If $\mathscr{M} \subset 2^X$ a σ-algebra, and E is any nonempty subset of X, then

$$\mathscr{M}_E := \{A \cap E : A \in \mathscr{M}\} \subset 2^E$$

is a σ-algebra on E (exercise).

(e) If $\{\mathscr{M}_\alpha : \alpha \in A\}$ is a collection of σ-algebras on X, then their intersection $\cap_{\alpha \in A} \mathscr{M}_\alpha$ is also a σ-algebra (checking this statement is a simple exercise). Hence given any set $\mathscr{E} \subset 2^X$, we can define the σ-algebra

$$\mathscr{M}(\mathscr{E}) = \bigcap \{\mathscr{M} : \mathscr{M} \text{ is a } \sigma\text{-algebra and } \mathscr{E} \subset \mathscr{M}\}. \tag{4}$$

Note that the intersection is over a nonempty collection since $\mathscr{E}$ is a subset of the σ-algebra 2^X. The set $\mathscr{M}(\mathscr{E})$ is the *σ-algebra generated by $\mathscr{E}$*. It is the smallest σ-algebra on X containing $\mathscr{E}$.

(f) An important instance of the construction in item (e) is when X is a topological space and $\mathscr{E}$ is the collection of open sets of X. In this case the σ-algebra generated by $\mathscr{E}$ is the *Borel σ-algebra* and is denoted $\mathscr{B}_X$. The Borel σ-algebra over $\mathbb{R}$ is studied more closely in Subsection 1.1.

(g) If $(Y, \mathscr{N})$ is a measurable space and $f : X \to Y$, then the collection

$$f^{-1}(\mathscr{N}) = \{f^{-1}(E) : E \in \mathscr{N}\} \subset 2^X \tag{5}$$

is a σ-algebra on X (check this) called the *pull-back* σ-algebra. The pull-back σ-algebra is the smallest σ-algebra on X such that the function $f : X \to Y$ is measurable.

(h) More generally given a family of measurable spaces $(Y_\alpha, \mathscr{N}_\alpha)$, where α ranges over some index set A, and functions $f_\alpha : X \to Y_\alpha$, let

$$\mathscr{E} = \{f_\alpha^{-1}(E_\alpha) : \alpha \in A, E_\alpha \in \mathscr{N}_\alpha\} \subset 2^X$$

and let $\mathscr{M} = \mathscr{M}(\mathscr{E})$. The σ-algebra $\mathscr{M}$ is the smallest σ-algebra on X such that each of the functions f_α is measurable. Unlike the case of a single f, the collection $\mathscr{E}$ need not be σ-algebra in general. An important special case of this construction is the product σ-algebra (see Subsection 1.2).

(i) If $(X, \mathcal{M})$ is a measurable space and $f : X \to Y$, then

$$\Omega_f = \{E \subset Y : f^{-1}(E) \in \mathcal{M}\} \subset 2^Y$$

is a σ-algebra.

$\triangle$

The following proposition is trivial but useful.

Proposition 1.9. *If $\mathscr{M} \subset 2^X$ is a σ-algebra and $\mathscr{E} \subset \mathscr{M}$, then $\mathscr{M}(\mathscr{E}) \subset \mathscr{M}$.* †

The proposition is used in the following way. To prove a particular statement say P is true for every set in some σ-algebra $\mathscr{M} \subset 2^X$ (say, the Borel σ-algebra $\mathscr{B}_X$): (1) check to see if the collection of sets $\mathscr{P} \subset 2^X$ satisfying property P is itself a σ-algebra (otherwise it is time to look for a different proof strategy); and (2) find a collection of sets $\mathscr{E}$ (say, the open sets of X) such that each $E \in \mathscr{E}$ has property P and such that

$\mathscr{M}(\mathscr{E}) = \mathscr{M}$. It then follows that $\mathscr{M} = \mathscr{M}(\mathscr{E}) \subset \mathscr{P}$. (The *monotone class lemma*, which we will study later, is typically used in a similar way.)

A function $f : X \to Y$ between topological spaces is said to be *Borel measurable* if it is measurable when X and Y are equipped with their respective Borel σ-algebras.

Proposition 1.10. *If X and Y are topological spaces and if $f : X \to Y$ is continuous, then f is Borel measurable.* †

Proof. Problem 7.7. (Hint: follow the strategy described after Proposition 1.9.) □

1.1. The Borel σ-algebra over $\mathbb{R}$.

Before going further, we take a closer look at the Borel σ-algebra over $\mathbb{R}$, beginning with the following useful lemma on the structure of open subsets of $\mathbb{R}$, which may be familiar to you from advanced calculus.

Lemma 1.11. *Every nonempty open subset $U \subset \mathbb{R}$ is an (at most countable) disjoint union of open intervals.* †

Here the "degenerate" intervals $(-\infty, a), (a, +\infty), (-\infty, +\infty)$ are allowed.

Proof outline. First verify that if I and J are intervals and $I \cap J \neq \emptyset$, then $I \cup J$ is an interval. Given $x \in U$, let

$$\alpha_x = \sup\{a : [x, a) \subset U\}$$
$$\beta_x = \inf\{b : (b, x] \subset U\}$$

and let $I_x = (\alpha_x, \beta_x)$. Verify that, for $x, y \in U$ either $I_x = I_y$ or $I_x \cap I_y = \emptyset$. Indeed, $x \sim y$ if $I_x = I_y$ is an equivalence relation on U. Hence, $U = \cup_{x \in U} I_x$ expresses U as a disjoint union of nonempty intervals, say $U = \cup_{p \in P} I_p$ where P is an index set and the I_p are nonempty intervals. For each $q \in \mathbb{Q} \cap U$ there exists a unique p_q such that $q \in I_{p_q}$. On the other hand, for each $p \in P$ there is a $q \in \mathbb{Q} \cap U$ such that $q \in I_p$. Thus, the mapping from $\mathbb{Q} \cap U$ to P defined by $q \mapsto p_q$ is onto. It follows that P is at most countable. □

Proposition 1.12 (Generators of $\mathscr{B}_{\mathbb{R}}$)**.** *Each of the following collections of sets $\mathscr{E} \subset 2^{\mathbb{R}}$ generates the Borel σ-algebra $\mathscr{B}_{\mathbb{R}}$:*

(i) the open intervals $\mathscr{E}_1 = \{(a, b) : a, b \in \mathbb{R}\}$;
(ii) the closed intervals $\mathscr{E}_2 = \{[a, b] : a, b \in \mathbb{R}\}$;
(iii) the (left or right) half-open intervals $\mathscr{E}_3 = \{[a, b) : a, b \in \mathbb{R}\}$ or $\mathscr{E}_4 = \{(a, b] : a, b \in \mathbb{R}\}$;
(iv) the (left or right) open rays $\mathscr{E}_5 = \{(-\infty, a) : a \in \mathbb{R}\}$ or $\mathscr{E}_6 = \{(a, +\infty) : a \in \mathbb{R}\}$;
(v) the (left or right) closed rays $\mathscr{E}_7 = \{(-\infty, a] : a \in \mathbb{R}\}$ or $\mathscr{E}_8 = \{[a, +\infty) : a \in \mathbb{R}\}$.

†

Proof. Only the open and closed interval cases are proved, the rest are similar and left as exercises. The proof makes repeated use of Proposition 1.9. Let $\mathscr{O}$ denote the open subsets of $\mathbb{R}$. Thus, by definition, $\mathscr{B}_{\mathbb{R}} = \mathscr{M}(\mathscr{O})$. To prove $\mathscr{M}(\mathscr{E}_1) = \mathscr{B}_{\mathbb{R}}$, first note that since each interval (a,b) is open and thus in $\mathscr{O}$, $\mathscr{M}(\mathscr{E}_1) \subset \mathscr{M}(\mathscr{O})$ by Proposition 1.9. Conversely, each open set $U \subset \mathbb{R}$ is a countable union of open intervals, so $\mathscr{M}(\mathscr{E}_1)$ contains $\mathscr{O}$ and hence (after another application of Proposition 1.9) $\mathscr{M}(\mathscr{O}) \subset \mathscr{M}(\mathscr{E}_1)$.

For the closed intervals $\mathscr{E}_2$, first note that each closed set is a Borel set, since it is the complement of an open set. Thus $\mathscr{E}_2 \subset \mathscr{B}_{\mathbb{R}}$ so $\mathscr{M}(\mathscr{E}_2) \subset \mathscr{B}_{\mathbb{R}}$ by Proposition 1.9. Conversely, each open interval (a,b) is a countable union of closed intervals $[a+\frac{1}{n}, b-\frac{1}{n}]$. Indeed, for $-\infty < a < b < \infty$,

$$(a,b) = \bigcup_{n=N}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}]$$

and a similar construction handles the cases that either $a = -\infty$ or $b = \infty$. It follows that $\mathscr{E}_1 \subset \mathscr{M}(\mathscr{E}_2)$, so by Proposition 1.9 and the first part of the proof,

$$\mathscr{B}_{\mathbb{R}} = \mathscr{M}(\mathscr{E}_1) \subset \mathscr{M}(\mathscr{E}_2).$$

□

Sometimes it is convenient to use a more refined version of the above Proposition, where we consider only dyadic intervals.

Definition 1.13. A *dyadic interval* is an interval of the form

$$I = \left(\frac{k}{2^n}, \frac{k+1}{2^n} \right] \tag{6}$$

where k, n are integers. ◁

(Draw a picture of a few of these to get the idea). A key property of dyadic intervals is the *nesting property*: if I, J are dyadic intervals, then either they are disjoint, or one is contained in the other. Dyadic intervals are often used to "discretize" analysis problems.

Proposition 1.14. *Every open subset of $\mathbb{R}$ is a countable disjoint union of dyadic intervals.* †

Proof. Problem 7.5. □

It follows (using the same idea as in the proof of Proposition 1.12) that the dyadic intervals generate $\mathscr{B}_{\mathbb{R}}$. The use of half-open intervals here is only a technical convenience, to allow us to say "disjoint" in the above proposition instead of "almost disjoint."

1.2. **Product σ-algebras.** Suppose $n \in \mathbb{N}^+$ and $(X_j, \mathscr{M}_j)$ are σ-algebras for $j = 1, 2, \ldots, n$. Let $X = \prod_{j=1}^{n} X_j$, the product space. Thus $X = \{(x_1, \ldots, x_n) : x_j \in X_j,\ j = 1, \ldots, n\}$. Let $\pi_j : X \to X_j$ denote the j-th coordinate projection, $\pi(x) = x_j$. The product σ-algebra, defined below, is the smallest σ-algebra on X such that each π_j is measurable.

Definition 1.15. Given measurable spaces $(X_j, \mathscr{N}_j)$, $j = 1, \dots n$, the *product σ-algebra* $\otimes_{j=1}^n \mathscr{N}_j$ is the σ algebra on $X = \prod_{j=1}^n X_j$ generated by

$$\{\pi_j^{-1}(E_j) : E_j \in \mathscr{N}_j, j = 1, \dots n\}.$$

Given $E_j \in \mathscr{N}_j$ for $j = 1, \dots, n$, the set $\times_{j=1}^n E_j \in \otimes_{j=1}^n \mathscr{N}_j$ is a *measurable rectangle.* $\triangleleft$

Proposition 1.16. *The collection $\mathcal{R}$ of measurable rectangles in $\otimes_{j=1}^n \mathscr{N}_j$ generates the product σ-algebra.* †

Proof. Each measurable rectangle is a finite intersection of elements of

$$\mathscr{E} = \{\pi_j^{-1}(E_j) : E_j \in \mathscr{N}_j, j = 1, \dots n\}.$$

Hence $\mathcal{R} \subset \mathscr{M}(\mathscr{E})$. On the other hand $\mathscr{E} \subset \mathcal{R}$ and hence the reverse inclusion holds. □

There are now two canonical ways of constructing σ-algebras on $\mathbb{R}^n$. The Borel σ-algebra $\mathscr{B}_{\mathbb{R}^n}$ and the product σ-algebra obtained by giving each copy of $\mathbb{R}$ the Borel σ-algebra $\mathscr{B}_{\mathbb{R}}$ and forming the product σ-algebra $\otimes_1^n \mathscr{B}_{\mathbb{R}}$. It is reasonable to suspect that these two σ-algebras are the same, and indeed they are.

Proposition 1.17. $\mathscr{B}_{\mathbb{R}^n} = \otimes_{j=1}^n \mathscr{B}_{\mathbb{R}}$. †

Proof. We use Proposition 1.9 to prove inclusions in both directions. By definition, the product σ-algebra $\otimes_{k=1}^n \mathscr{B}_{\mathbb{R}}$ is generated by the collection of sets

$$\mathcal{E} = \{\pi_j^{-1}(E_j) : E_j \in \mathscr{B}_{\mathbb{R}}, \quad j = 1, \dots n\},$$

where $\pi_j(x_1, \dots x_n) = x_j$ is the projection map, $\pi : \mathbb{R}^n \to \mathbb{R}$. Summarizing, $\mathscr{M}(\mathcal{E}) = \otimes_{j=1}^n \mathscr{B}_{\mathbb{R}}$.

For each j, the projection π_j is continuous and hence, by Proposition 1.10, Borel measurable. Consequently, if $E_j \in \mathscr{B}_{\mathbb{R}}$, then

$$\pi_j^{-1}(E_j) = \mathbb{R} \times \cdots \times \mathbb{R} \times E_j \times \mathbb{R} \times \cdots \times \mathbb{R} \in \mathscr{B}_{\mathbb{R}^n}.$$

where E_j is the j^{th} factor. Hence $\mathcal{E} \subset \mathscr{B}_{\mathbb{R}^n}$ and, by Proposition 1.9, $\otimes_1^n \mathscr{B}_{\mathbb{R}} = \mathscr{M}(\mathcal{E}) \subset \mathscr{B}_{\mathbb{R}^n}$.

Let $\mathscr{O}_n$ are the open sets in $\mathbb{R}^n$. To prove the reverse inclusion, it suffices to identify a subset $\mathcal{R}_o$ of the product σ-algebra $\otimes_{j=1}^n \mathscr{B}_{\mathbb{R}}$ such that $\mathscr{M}(\mathcal{R}_o) \supset \mathscr{O}_n$, since then

$$\otimes_{j=1}^n \mathscr{B}_{\mathbb{R}} \supset \mathscr{M}(\mathcal{R}_o) \supset \mathscr{M}(\mathscr{O}_n) = \mathscr{B}_{\mathbb{R}^n}.$$

Let $\mathcal{R}_o$ denote the collection of *open rectangles*, $R = (a_1, b_1) \times \cdots \times (a_n, b_n) = \prod_{j=1}^n (a_j, b_j) \in \otimes_{j=1}^n \mathscr{B}_{\mathbb{R}}$. Each $U \in \mathscr{O}_n$ is a countable union of *open rectangles* (just take all the open boxes contained in U having rational vertices). Hence $\mathscr{O}_n \subset \mathscr{M}(\mathcal{R}_o)$. (Equality holds, of course. But we only need this inclusion.) □

2. Measures

Definition 2.1. Let X be a set and $\mathscr{M}$ a σ-algebra on X. A *measure* on $\mathscr{M}$ is a function $\mu : \mathscr{M} \to [0, +\infty]$ such that

(i) $\mu(\varnothing) = 0$; and
(ii) if $(E_j)_{j=1}^{\infty}$ is a sequence of disjoints sets in $\mathscr{M}$, then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j).$$

If $\mu(X) < \infty$, then μ is *finite*. If there exists a sequence (X_j) from $\mathscr{M}$ such that $X = \cup_{j=1}^{\infty} X_j$ and $\mu(X_j) < \infty$ for each j, then μ is *σ-finite*.

A triple $(X, \mathscr{M}, \mu)$ where X is a set, $\mathscr{M}$ is a σ-algebra and μ a measure on $\mathscr{M}$, is a *measure space*. $\triangleleft$

Almost all of the measures of importance in analysis are σ-finite.

Here are some simple measures and some procedures for producing new measures from old. Non-trivial examples of measures will have to wait for the Caratheodory and Hahn-Kolmogorov theorems in the following sections.

Example 2.2. (a) Let X be any set and, for $E \subset X$, let $|E|$ denote the cardinality of E, in the sense of a finite number or ∞. The function $\mu : 2^X \to [0, +\infty]$ defined by $\mu(E) = |E|$ is a measure on $(X, 2^X)$, called *counting measure*. It is finite if and only if X is finite, and σ-finite if and only if X is at most countable.

(b) Let X be an uncountable set and $\mathscr{M}$ the σ-algebra of (at most) countable and co-countable sets (Example 1.8(b)). The function $\mu : \mathscr{M} \to [0, \infty]$ defined by $\mu(E) = 0$ if E is countable and $\mu(E) = +\infty$ if E is co-countable is a measure.

(c) Let $(X, \mathscr{M}, \mu)$ be a measure space and $E \in \mathscr{M}$. Recall $\mathscr{M}_E$ from Example 1.8(c). The function $\mu_E(A) := \mu(A \cap E)$ is a measure on $(E, \mathscr{M}_E)$. (Why is the assumption $E \in \mathscr{M}$ necessary?)

(d) (Linear combinations) If μ is a measure on $\mathscr{M}$ and $c > 0$, then $(c\mu)(E) =: c\,\mu(E)$ is a measure, and if $\mu_1, \dots \mu_n$ are measures on the same $\mathscr{M}$, then

$$(\mu_1 + \cdots \mu_n)(E) := \mu_1(E) + \cdots \mu_n(E)$$

is a measure. Likewise a countably infinite sum of measures $\sum_{n=1}^{\infty} \mu_n$ is a measure. (The proof of this last fact requires a small amount of care. See Problem 7.9.)

$\triangle$

One can also define products and pull-backs of measures, compatible with the constructions of product and pull-back σ-algebras. These examples will be postponed until we have built up some more machinery of measurable functions.

Theorem 2.3 (Basic properties of measures). *Let $(X, \mathscr{M}, \mu)$ be a measure space.*

(a) (Monotonicity) If $E, F \in \mathscr{M}$ and $F \subset E$, then $\mu(E) = \mu(E \setminus F) + \mu(F)$. In particular, $\mu(F) \le \mu(E)$ and if $\mu(E) < \infty$, then $\mu(F \setminus E) = \mu(F) - \mu(E)$.
(b) (Subadditivity) If $(E_j)_{j=1}^\infty \subset \mathscr{M}$, then $\mu(\bigcup_{j=1}^\infty E_j) \le \sum_{j=1}^\infty \mu(E_j)$.
(c) (Monotone convergence for sets) If $(E_j)_{j=1}^\infty \subset \mathscr{M}$ and $E_j \subset E_{j+1}$ $\forall j$, then $\lim \mu(E_j)$ exists and moreover $\mu(\cup E_j) = \lim \mu(E_j)$.
(d) (Dominated convergence for sets) If $(E_j)_{j=1}^\infty$ is a decreasing ($E_j \supset E_{j+1}$ for all j) from $\mathscr{M}$ and $\mu(E_1) < \infty$, then $\lim \mu(E_j)$ exists and moreover $\mu(\cap E_j) = \lim \mu(E_j)$.

Proof. (a) Since $E = (E \setminus F) \cup F$ is a disjoint union of measurable sets, by additivity, $\mu(E) = \mu(E \setminus F) + \mu(F) \ge \mu(F)$.

(b) For $1 \le j \le g$, let

$$F_j = E_j \setminus \left(\bigcup_{k=1}^{j-1} E_k \right).$$

By proposition 1.7, the F_j are pairwise disjoint, $F_j \subset E_j$ for all j and $\cup_{j=1}^\infty F_j = \cup_{j=1}^\infty E_j$. Thus by countable additivity and (a),

$$\mu\left(\bigcup_{j=1}^\infty E_j \right) = \mu\left(\bigcup_{j=1}^\infty F_j \right) = \sum_{j=1}^\infty \mu(F_j) \le \sum_{j=1}^\infty \mu(E_j).$$

(c) With the added assumption that the sequence $(E_j)_{j=1}^\infty$ is nested increasing, $\bigcup_{k=1}^j F_k = E_j$ for each j. Thus, by countable additivity,

$$\begin{aligned}
\mu\left(\bigcup_{j=1}^\infty E_j \right) &= \mu\left(\bigcup_{j=1}^\infty F_j \right) \\
&= \sum_{k=1}^\infty \mu(F_k) \\
&= \lim_{j\to\infty} \sum_{k=1}^j \mu(F_k) \\
&= \lim_{j\to\infty} \mu\left(\bigcup_{k=1}^j F_k \right) \\
&= \lim_{j\to\infty} \mu(E_j).
\end{aligned}$$

(d) The sequence $\mu(E_j)$ is decreasing (by (a)) and bounded below, so $\lim \mu(E_j)$ exists. Let $F_j = E_1 \setminus E_j$. Then $F_j \subset F_{j+1}$ for all j, and $\bigcup_{j=1}^\infty F_j = E_1 \setminus \bigcap_{j=1}^\infty E_j$. So by

(a) and (c) applied to the F_j, and since $\mu(E_1) < \infty$,

$$\begin{aligned}\mu(E_1) - \mu(\bigcap_{j=1}^{\infty} E_j) &= \mu(E_1 \setminus \bigcap_{j=1}^{\infty} E_j) \\ &= \lim \mu(F_j) \\ &= \lim(\mu(E_1) - \mu(E_j)) \\ &= \mu(E_1) - \lim \mu(E_j).\end{aligned}$$

Again since $\mu(E_1) < \infty$, it can be subtracted from both sides. □

Example 2.4. Note that in item (d) of Theorem 2.3, the hypothesis "$\mu(E_1) < \infty$" can be replaced by "$\mu(E_j) < \infty$ for some j". However the finiteness hypothesis cannot be removed entirely. For instance, consider $(\mathbb{N}, 2^{\mathbb{N}})$ equipped with counting measure, and let $E_j = \{k : k \geq j\}$. Then $\mu(E_j) = \infty$ for all j but $\mu(\bigcap_{j=1}^{\infty} E_j) = \mu(\varnothing) = 0$. △

For any set X and subset $E \subset X$, there is a function $\mathbf{1}_E : X \to \{0, 1\}$ defined by

$$\mathbf{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases},$$

called the *characteristic function* or *indicator function* of E. It is easily verified, if $(X, \mathscr{M})$ is a measure space and $E \subset X$, then $E \in \mathscr{M}$ if and only if $\mathbf{1}_E$ is $(\mathscr{M}, \mathscr{B}_{\mathbb{R}})$ measurable. For a sequence of subsets (E_n) of X, by definition (E_n) *converges to* E *pointwise* if $\mathbf{1}_{E_n} \to \mathbf{1}_E$ pointwise[1]. This notion allows the formulation of a more refined version of the dominated convergence theorem for sets, which foreshadows (and is a special case of) the dominated convergence theorem for the Lebesgue integral. See Problems 7.12 and 7.13.

Definition 2.5. Let $(X, \mathscr{M}, \mu)$ be a measure space. A *null set* (or μ-null set) is a set $E \in \mathscr{M}$ with $\mu(E) = 0$. ◁

It follows immediately from countable subadditivity that a countable union of null sets is null. The contrapositive of this statement is a measure-theoretic version of the pigeonhole principle:

Proposition 2.6 (Pigeonhole principle for measures)**.** *If* $(E_n)_{n=1}^{\infty}$ *is a sequence of sets in* $\mathscr{M}$ *and* $\mu(\cup E_n) > 0$, *then* $\mu(E_n) > 0$ *for some* n. †

It will often be tempting to assert that if $\mu(E) = 0$ and $F \subset E$, then $\mu(F) = 0$, but one must be careful: F need not be a measurable set. This caveat is not a big deal in practice, however, because we can always enlarge the σ-algebra on which a measure is defined so as to contain all subsets of null sets, and it will usually be convenient to do so.

Definition 2.7. A measure space $(X, \mathscr{M}, \mu)$ is *complete* if it contains every subset of measure 0; i.e., if $F \subset X$ and there exists E such that

[1] What would happen if we asked for uniform convergence?

(i) $F \subset E$;
(ii) $E \in \mathscr{M}$; and
(iii) $\mu(E) = 0$,

then $F \in \mathscr{M}$. $\triangleleft$

Theorem 2.8. *Suppose $(X, \mathscr{M}, \mu)$ be a measure space and let $\mathscr{N} := \{N \in \mathscr{M} | \mu(N) = 0\}$. The collection*

$$\overline{\mathscr{M}} := \{E \cup F | E \in \mathscr{M}, F \subset N \text{ for some } N \in \mathscr{N}\}$$

is a σ-algebra, and $\overline{\mu} : \overline{\mathscr{M}} \to [0, \infty]$ given by

$$\overline{\mu}(E \cup F) := \mu(E)$$

is a well-defined function and a complete measure on $\overline{\mathscr{M}}$ such that $\overline{\mu}|_{\mathscr{M}} = \mu$.

The measure space $(X, \overline{\mathscr{M}}, \overline{\mu})$ is the *completion* of $(X, \mathscr{M}, \mu)$. It is evident that if $\mathscr{M} \subset \mathscr{N} \subset 2^X$ is a σ-algebra, ν is a measure on $\mathscr{N}$ such that $\nu|_{\mathscr{M}} = \mu$ and $(X, \mathscr{N}, \nu)$ is complete, then $\overline{\mathscr{M}} \subset \mathscr{N}$ and $\nu|_{\overline{\mathscr{M}}} = \overline{\mu}$. Thus $(X, \overline{\mathscr{M}}, \overline{\mu})$ is the smallest complete measure space *extending* $(X, \mathscr{M}, \mu)$.

Some of the proof. First note that $\mathscr{M}$ and $\mathscr{N}$ are both closed under countable unions, so $\overline{\mathscr{M}}$ is as well. To see that $\overline{\mathscr{M}}$ is closed under complements, consider $E \cup F \in \overline{\mathscr{M}}$ with $E \in \mathscr{M}, F \subset N \in \mathscr{N}$. Using, $F^c = N^c \cup (N \setminus F)$,

$$(F \cup E)^c = F^c \cap E^c = (N^c \cap E^c) \cup (N \cap F^c \cap E^c).$$

The first set on the right hand side is in $\mathscr{M}$ and the second is a subset of $\mathscr{N}$. Thus the union is in $\overline{\mathscr{M}}$ as desired. Hence $\overline{\mathscr{M}}$ is a σ-algebra.

To prove that $\overline{\mu}$ is well defined, suppose $G = E \cup F = E' \cup F'$ for $E, E' \in \mathscr{M}$ and $F, F' \in \mathscr{N}$. In particular, there exists μ-null sets $N, N' \in \mathscr{M}$ with $F \subset N$ and $F' \subset N'$. Observe that

$$\mathscr{M} \ni E \setminus E' \subset G \setminus E' \subset F' \subset N'.$$

Thus $\mu(E \setminus E') = 0$. On the other hand,

$$E = (E \cap E') \cup (E \setminus E').$$

Thus, $\mu(E) = \mu(E \cap E')$. By symmetry, $\mu(E') = \mu(E' \cap E)$.

The proof that $\overline{\mu}$ is a complete measure on $\overline{\mathscr{M}}$ that extends μ, is left as an exercise (Problem 7.14). $\square$

3. Outer measures and the Caratheodory Extension Theorem

The point of the construction of Lebesgue measure on the real line is to extend the naive notion of length for intervals to a suitably large family of subsets of $\mathbb{R}$. Indeed, this family should be a σ-algebra containing all open intervals and hence the Borel σ-algebra.

Definition 3.1. Let X be a nonempty set. A function $\mu^* : 2^X \to [0, +\infty]$ is an *outer measure* if

(i) $\mu^*(\varnothing) = 0$;
(ii) (Monotonicity) if $A \subset B$, then $\mu^*(A) \le \mu^*(B)$;
(iii) (Subadditivity) if $(A_j)_{j=1}^\infty \subset 2^X$, then

$$\mu^*\left(\bigcup_{j=1}^\infty A_j\right) \le \sum_{j=1}^\infty \mu^*(A_j).$$

$\triangleleft$

Definition 3.2. If μ^* is an outer measure on X, then a set $E \subset X$ is *outer measurable* (or μ^*-*measurable,* or *measurable with respect to* μ^*, or just *measurable*) if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \tag{7}$$

for *every* $A \subset X$. $\triangleleft$

The significance of outer measures and (outer) measurable sets stems from the following theorem.

Theorem 3.3 (Caratheodory Extension Theorem)**.** *If μ^* is an outer measure on X, then the collection $\mathscr{M}$ of outer measurable sets is a σ-algebra and the restriction of μ^* to $\mathscr{M}$ is a complete measure.*

The outer measures encountered in these notes arise from the following construction.

Proposition 3.4. *Suppose $\mathscr{E} \subset 2^X$ and $\varnothing, X \in \mathscr{E}$. If $\mu_0 : \mathscr{E} \to [0, +\infty]$ and $\mu_0(\varnothing) = 0$, then the function $\mu^* : 2^X \to [0, \infty]$ defined by*

$$\mu^*(A) = \inf\left\{\sum_{n=1}^\infty \mu_0(E_n) : E_n \in \mathscr{E} \text{ and } A \subset \bigcup_{n=1}^\infty E_n\right\} \tag{8}$$

is an outer measure. †

Remark 3.5. A few remarks are in order. Given a (nonempty) index set $\mathscr{I}$ and nonegative real numbers a_α for $\alpha \in I$, define

$$\sum_{\alpha\in\mathscr{I}} a_\alpha := \sum\{\sum_{\alpha\in F} a_\alpha : F \subset \mathscr{I},\ F \text{ finite}\}.$$

In the case $\mathscr{I}$ is countable, if $\phi : \mathbb{N}^+ \to \mathscr{I}$ is a bijection, then

$$\sum_{\alpha\in\mathscr{I}} a_\alpha = \sum_{j=1}^\infty a_{\phi(j)}.$$

In particular, the sum does not depend on the bijection ϕ. Hence in the definition of outer measure, the sums can be interpreted as the sum over countable collections of sets from $\mathscr{E}$ that cover A. For instance, in the case $\mathscr{I} = \mathbb{N}^+ \times \mathbb{N}^+$

$$\sum_{(m,n)\in\mathbb{N}^+\times\mathbb{N}^+} a_{m,n} = \sum_{s=1}^\infty \sum_{m+n=s} a_{m,n}.$$

It is also true that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{m,n} = \sum_{(m,n)\in\mathbb{N}^+\times\mathbb{N}^+} a_{m,n} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}.$$

The proofs of these assertions are left as an exercise. See Problem 7.9.

Note that we have assumed $\varnothing, X \in \mathscr{E}$, so there is at least one covering of A by sets in $\mathscr{E}$ (take $E_1 = X$ and all other E_j empty), so the definition (8) makes sense. On the other hand, Proposition 3.4 is mute on whether $E \in \mathscr{E}$ is μ^*-outer measurable or, in the case E is outer measurable, whether $\mu^*(E)$ and $\mu_0(E)$ agree. $\diamond$

Proof of Proposition 3.4. It is immediate from the definition that $\mu^*(\varnothing) = 0$ (cover the empty set by empty sets) and that $\mu^*(A) \le \mu^*(B)$ whenever $A \subset B$ (any covering of B is also a covering of A). To prove countable subadditivity, we make our first use of the "$\epsilon/2^n$" trick. Let (A_n) be a sequence in 2^X. If $\mu^*(A_j) = \infty$ for some j, then the desired subadditive inequality holds by monotonicity. Otherwise let $\epsilon > 0$ be given. For each n there exists a countable collection of sets $(E_{n,k})_{k=1}^{\infty}$ in $\mathscr{E}$ such that $A_n \subset \bigcup_{k=1}^{\infty} E_{n,k}$ and

$$\sum_{k=1}^{\infty} \mu_0(E_{n,k}) - \epsilon 2^{-n} < \mu^*(A_n).$$

But now the countable collection $(E_{n,k})_{n,k=1}^{\infty}$ covers $\cup_{n=1}^{\infty} A_n$, and, using Remark 3.5,

$$\mu^*(\cup A_n) \le \sum_{k,n=1}^{\infty} \mu_0(E_{n,k}) < \sum_{n=1}^{\infty}(\mu^*(A_n) + \epsilon 2^{-n}) = \epsilon + \sum_{n=1}^{\infty} \mu^*(A_n).$$

Since $\epsilon > 0$ was arbitrary, $\mu^*(\cup A_j) \le \sum_{n=1}^{\infty} \mu^*(A_n)$. $\square$

Example 3.6. [Lebesgue outer measure] Let $\mathscr{E} \subset 2^{\mathbb{R}}$ be the collection of all open intervals $(a,b) \subset \mathbb{R}$, with $-\infty \le a < b \le +\infty$, together with $\varnothing$. Define $m_0((a,b)) = b-a$ and $m_0(\varnothing) = 0$. The corresponding outer measure is Lebesgue outer measure and it is the mapping $m^* : 2^{\mathbb{R}} \to [0,\infty]$ defined, for $A \in 2^X$, by

$$m^*(A) = \inf\left\{\sum_{n=1}^{\infty}(b_n - a_n) : A \subset \bigcup_{n=1}^{\infty}(a_n, b_n)\right\} \tag{9}$$

where we allow the *degenerate* intervals $\mathbb{R} = (-\infty, +\infty)$ and $\emptyset$. The value $m^*(A)$ is the *Lebesgue outer measure of* A. In the next section we will construct Lebesgue measure from m^* via the Caratheodory Extension Theorem. The main issues will be to show that the outer measure of an interval is equal to its length, and that every Borel subset of $\mathbb{R}$ is outer measurable. The other desirable properties of Lebesgue measure (such as translation invariance) will follow from this construction. $\triangle$

Before proving Theorem 3.3 will make repeated use of the following observation. Namely, if μ^* is an outer measure on a set X, to prove that a subset $E \subset X$ is outer measurable, it suffices to prove that

$$\mu^*(A) \ge \mu^*(A \cap E) + \mu^*(A \setminus E)$$

for all $A \subset X$, since the opposite inequality for all A is immediate from the subadditivity of μ^*.

The following lemma will be used to show the measure constructed in the proof of Theorem 3.3 is complete. A set $E \subset X$ is called *μ^*-null* if $\mu^*(E) = 0$.

Lemma 3.7. *Every μ^*-null set is μ^*-measurable.* †

Proof. Let E be μ^*-null and $A \subset X$. By monotonicity, $A \cap E$ is also μ^*-null, so by monotonicity again,

$$\mu^*(A) \geq \mu^*(A \setminus E) = \mu^*(A \cap E) + \mu^*(A \setminus E).$$

Thus the lemma follows from the observation immediately preceding the lemma. □

Proof of Theorem 3.3. We first show that $\mathscr{M}$ is a σ-algebra. It is immediate from Definition 3.2 that $\mathscr{M}$ contains $\varnothing$ and X, and since (7) is symmetric with respect to E and E^c, $\mathscr{M}$ is also closed under complementation. Next we check that $\mathscr{M}$ is closed under finite unions (which will prove that $\mathscr{M}$ is a Boolean algebra). So, let $E, F \in \mathscr{M}$ and fix an arbitrary $A \subset X$. Since F is outer measurable,

$$\mu^*(A \cap E^c) = \mu^*((A \cap E^c) \cap F) + \mu^*((A \cap E^c) \cap F^c). \tag{10}$$

By subadditivity and the set equality $A \cap (E \cup F) = (A \cap E) \cup (A \cap (F \cap E^c))$,

$$\mu^*(A \cap (E \cup F)) \leq \mu^*(A \cap E) + \mu^*(A \cap (F \cap E^c)). \tag{11}$$

Using equations (11) and (10) and the outer measurability of E in that order,

$$\begin{aligned}
\mu^*(A \cap (E \cup F)) + &\mu^*(A \cap ((E \cup F)^c)) \\
&\leq \mu^*(A \cap E) + \mu^*(A \cap (F \cap E^c)) + \mu^*(A \cap (F^c \cap E^c)) \\
&= \mu^*(A \cap E) + \mu^*(A \cap E^c) \\
&= \mu^*(A).
\end{aligned}$$

Hence $E \cup F$ is outer measurable.

Now we show that $\mathscr{M}$ is closed under countable disjoint unions. (It then follows from Proposition 1.7 that $\mathscr{M}$ is a σ-algebra.) Let (E_n) be a sequence of disjoint outer measurable sets, and let $A \subset X$ be given. It is enough to show

$$\mu^*(A) \geq \mu^*(A \cap \bigcup_{n=1}^{\infty} E_n) + \mu^*(A \setminus \bigcup_{n=1}^{\infty} E_n).$$

For each $N \geq 1$, we have already proved that $G_N = \bigcup_{n=1}^{N} E_n$ is outer measurable, and therefore

$$\mu^*(A) \geq \mu^*(A \cap \bigcup_{n=1}^{N} E_n) + \mu^*(A \setminus \bigcup_{n=1}^{N} E_n).$$

By monotonicity, $\mu^*(A \setminus \bigcup_{n=1}^{N} E_n) \geq \mu^*(A \setminus \bigcup_{n=1}^{\infty} E_n)$. Thus it suffices to prove

$$\lim_{N \to \infty} \mu^*(A \cap \bigcup_{n=1}^{N} E_n) \geq \mu^*(A \cap \bigcup_{n=1}^{\infty} E_n). \tag{12}$$

(The limit exists as an extended real number since the sequence is increasing by monotonicity of the outer measure.) By the outer measurability of $G_N = \bigcup_{n=1}^N E_n$ and disjointness of the E_n,

$$\begin{aligned} \mu^*(A\cap \bigcup_{n=1}^{N+1} E_n) =& \mu^*(A\cap G_{N+1}) \\ =& \mu^*\left((A\cap G_{N+1})\cap G_N\right) + \mu^*\left((A\cap G_{N+1})\cap G_N^c\right) \\ =& \mu^*(A\cap \bigcup_{n=1}^{N} E_n) + \mu^*(A\cap E_{N+1}). \end{aligned} \tag{13}$$

Iterating the identity of equation (13) gives.

$$\mu^*(A\cap \bigcup_{n=1}^{N+1} E_n) = \sum_{k=0}^{N+1} \mu^*(A\cap E_k). \tag{14}$$

(In particular, choosing $A = X$, it follows that μ^* is finitely additive on $\mathscr{M}$.) Letting N tend to infinity in equation (14) gives

$$\lim_{N\to\infty} \mu^*(A\cap \bigcup_{n=1}^{N} E_n) = \sum_{N=0}^{\infty} \mu^*(A\cap E_{N+1}).$$

From countable subadditivity,

$$\lim_{N\to\infty} \mu^*(A\cap \bigcup_{n=1}^{N} E_n) = \sum_{N=0}^{\infty} \mu^*(A\cap E_{N+1}) \geq \mu^*(\bigcup_{j=1}^{\infty}(A\cap E_j)) = \mu^*(A\cap \bigcup_{j=1}^{\infty} E_j),$$

proving the inequality of equation (12) and thus that $\mathscr{M}$ is a σ-algebra. Further, from monotonicity and equation (13),

$$\mu^*(A\cap \bigcup_{n=1}^{\infty} E_n) \geq \mu^*(A\cap \bigcup_{n=1}^{N} E_n) = \sum_{n=0}^{N} \mu^*(A\cap E_n)$$

and hence

$$\mu^*(A\cap \bigcup_{n=1}^{\infty} E_n) \geq \sum_{n=1}^{\infty} \mu^*(A\cap E_n).$$

Since the reverse inequality holds by subadditivity,

$$\mu^*(A\cap \bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu^*(A\cap E_n).$$

In particular, choosing $A = X$ proves that μ^* is countably additive on the σ-algebra of μ^*-outer measurable sets and hence $\mu^*|_{\mathscr{M}}$ is a measure.

Finally, that μ^* is a *complete* measure on $\mathscr{M}$ is an immediate consequence of Lemma 3.7. $\square$

4. Construction of Lebesgue measure

In this section, by an *interval* we mean any set $I \subset \mathbb{R}$ of the from $(a, b), [a, b], (a, b], [a, b)$, including $\varnothing$, open and closed half-lines and $\mathbb{R}$ itself. Let $|I| = b - a$, the length of the interval I, interpreted as $+\infty$ in the line and half-line cases and 0 for $\varnothing$. Recall the definition of Lebesgue outer measure of a set $A \subset \mathbb{R}$ from Example 9:

$$m^*(A) = \inf \left\{ \sum_{n=1}^{\infty} |I_n| : A \subset \bigcup_{n=1}^{\infty} I_n \right\}$$

where the I_n are *open* intervals, or empty.

Theorem 4.1. *If $I \subset \mathbb{R}$ is an interval, then $m^*(I) = |I|$.*

Proof. We first consider the case where I is a finite, closed interval $[a, b]$. For any $\epsilon > 0$, the single open interval $(a - \epsilon, b + \epsilon)$ covers I, so $m^*(I) \leq (b - a) + 2\epsilon = |I| + 2\epsilon$, and thus $m^*(I) \leq |I|$. For the reverse inequality, again choose $\epsilon > 0$, and let (I_n) be a cover of I by open intervals such that $\sum_{n=1}^{\infty} |I_n| < m^*(I) + \epsilon$. Since I is compact, there is a finite subcollection $(I_{n_k})_{k=1}^{N}$ of the I_n that covers I. We claim

$$\sum_{k=1}^{N} |I_{n_k}| > b - a = |I|. \tag{15}$$

To verify this statement, observe that by passing to a further subcollection, we can assume that none of the intervals I_{n_k} is contained in another one and so that each has non-trivial intersection with $[a, b]$. Re-index the intervals as $I_1, \dots I_N$ so that the left endpoints $a_1, \dots a_N$ are listed in increasing order. Since these intervals cover I, there are no containments, and each intersects $[a, b]$ it follows that $a_1 < a$, $a_2 < b_1$, $a_3 < b_2$, ...$a_N < b_{N-1}$ and $b < b_N$. (Draw a picture.) Therefore

$$\sum_{k=1}^{N} |I_k| = \sum_{k=1}^{N} (b_k - a_k) = b_N - a_1 + \sum_{k=1}^{N-1} (b_k - a_{k+1}) \geq b_N - a_1 > b - a = |I|.$$

From the inequality (15) it follows $m^*(I) = |I|$.

Now we consider the cases of bounded, but not closed, intervals $(a, b), (a, b], [a, b)$. If I is such an interval, then $\overline{I} = [a, b]$ its closure Since m^* is an outer measure, by monotonicity $m^*(I) \leq m^*(\overline{I}) = |I|$. On the other hand, if $\epsilon > 0$, then $I_\epsilon := [a+\epsilon, b-\epsilon] \subset I$ and thus, by monotonicity again, $m^*(I) \geq m^*(I_\epsilon) = |I| - 2\epsilon$. Hence $m^*(I) \geq |I|$.

Finally, the result is immediate in the case of unbounded intervals, since any unbounded interval contains arbitrarily large bounded intervals and m^* is monotonic. □

Theorem 4.2. *Every Borel set $E \in \mathscr{B}_{\mathbb{R}}$ is m^*-measurable.*

Proof. By the Caratheodory extension theorem, the collection of m^*-measurable sets is a σ-algebra, so by Propositions 1.12 and 1.9, it suffices to show that the open rays $(a, +\infty)$ are m^*-measurable. Fix $a \in \mathbb{R}$ and an arbitrary set $A \subset \mathbb{R}$. We must prove

$$m^*(A) \geq m^*(A \cap (a, +\infty)) + m^*(A \cap (-\infty, a]).$$

To simplify the notation put $A_1 = A \cap (a, +\infty)$, $A_2 = A \cap (-\infty, a]$. Let (I_n) be a cover of A by open intervals. For each n let $I_n' = I_n \cap (a, +\infty)$ and $I_n'' = I_n \cap (-\infty, a]$. The families $(I_n'), (I_n'')$ are intervals (not necessarily open) that cover A_1, A_2 respectively. Now

$$\begin{aligned}\sum_{n=1}^{\infty} |I_n| &= \sum_{n=1}^{\infty} |I_n'| + \sum_{n=1}^{\infty} |I_n''| \\ &= \sum_{n=1}^{\infty} m^*(I_n') + \sum_{n=1}^{\infty} m^*(I_n'') \\ &\geq m^*(\bigcup_{n=1}^{\infty} I_n') + m^*(\bigcup_{n=1}^{\infty} I_n'') \\ &\geq m^*(A_1) + m^*(A_2),\end{aligned}$$

where the second equality follows from Theorem 4.1, the first inequality from subadditivity and the last inequality by monotonicity. Since this inequality holds for all coverings of A by open intervals, taking the infimum on the left hand side gives $m^*(A) \geq m^*(A_1) + m^*(A_2)$. □

Definition 4.3. A set $E \subset \mathbb{R}$ is called *Lebesgue measurable* if

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \tag{16}$$

for all $A \subset \mathbb{R}$. The restriction of m^* to the Lebesgue measurable sets is called *Lebesgue measure*, denoted m. ◁

By Theorem 3.3, m is a measure. By Theorem 4.2, every Borel set is Lebesgue measurable, and by Theorem 4.1 the Lebesgue measure of an interval is its length. It should also be evident by now that m is σ-finite. So, we have arrived at the promised extension of the length function on intervals to a measure. (A proof of uniqueness of m will have to wait until for the Hahn Uniqueness Theorem. See Corollary 5.5.)

Next we prove that m has the desired invariance properties. Given $E \subset \mathbb{R}$, $x \in \mathbb{R}$, and $t > 0$, let

$$E+x = \{y \in \mathbb{R} : y-x \in E\}, \quad -E = \{y \in \mathbb{R} : -y \in E\}, \quad \text{and } tE = \{y \in \mathbb{R} : y/t \in E\}.$$

It is evident that $m^*(E+x) = \mu^*(E)$, $m^*(-E) = m^*(E)$ and, $m^*(tE) = tm^*(E)$ since, if I is an interval, then $|I+x| = |I|$, $|-I| = |I|$ and $|tI| = t|I|$. Thus m^* has the desired invariance properties. In particular, if both E and $E+x$ are Lebesgue measurable, then $m(E+x) = m(E)$. What remains to be shown is that if E is Lebesgue measurable, then so are $E+x$, $-E$ and tE. This fact follows easily from the corresponding invariance property of m^*.

Theorem 4.4. *If $E \subset \mathbb{R}$ is Lebesgue measurable, $x \in \mathbb{R}$, and $t > 0$, then the sets $E+x$, $-E$, and tE are Lebesgue measurable. Moreover $m(E+x) = m(E)$, $m(-E) = m(E)$, and $m(tE) = tm(E)$.*

Proof. We give the proof for $E+x$. Proof of the others are similar and left as exercises. Accordingly, suppose E is measurable. To prove $E+x$ is measurable, let $A \subset \mathbb{R}$ be given and observe that $A \cap (E+x) = ((A-x)\cap E)+x$ and $A\cap(E+x)^c = ((A-x)\cap E^c)+x$. Thus,

$$\begin{aligned} m^*(A) &= m^*(A-x) \\ &= m^*((A-x)\cap E) + m^*((A-x)\cap E^c) \\ &= m^*(((A-x)\cap E)+x) + m^*(((A-x)\cap E^c)+x) \\ &= m^*(A\cap(E+x)) + m^*(A\cap(E+x)^c), \end{aligned}$$

where measurability of E is used in the second equality. Hence $E+x$ is Lebesgue measurable and $m(E+x) = m(E)$. □

4.1. **Regularity of Lebesgue measure.** The condition (16) does not make clear which subsets of $\mathbb{R}$ are Lebesgue measurable. Theorems 4.5 and 4.6 are fundamental approximation results. They say 1) up to sets of measure zero, every Lebesgue measurable set is a G_δ or an F_σ, and 2) if we are willing to ignore sets of measure ϵ, then every set of finite Lebesgue measure is a union of intervals. (Recall that a set in a topological space is called a G_δ-set if it is a countable intersection of open sets, and an F_σ-set if it is a countable union of closed sets.)

Theorem 4.5. *Let $E \subset \mathbb{R}$. The following are equivalent.*

(a) E is Lebesgue measurable.
(b) For every $\epsilon > 0$, there is an open set $U \supset E$ such that $m^*(U\setminus E) < \epsilon$.
(c) For every $\epsilon > 0$, there is a closed set $F \subset E$ such that $m^*(E\setminus F) < \epsilon$.
(d) There is a G_δ set G such that $E \subset G$ and $m^*(G\setminus E) = 0$.
(e) There is an F_σ set F such that $E \supset F$ and $m^*(E\setminus F) = 0$.

Proof. To prove (a) implies (b) let E a (Lebesgue) measurable set and $\epsilon > 0$ be given. Further, suppose for the moment that $m(E) < \infty$. Because E is measurable, $m(E) = m^*(E)$. From the definition of m^*, there is a covering of E by open intervals I_n such that $\sum_{n=1}^\infty |I_n| < m^*(E)+\epsilon$. Put $U = \bigcup_{n=1}^\infty I_n$. By subadditivity of m,

$$m(U) \le \sum_{n=1}^\infty m(I_n) = \sum_{n=1}^\infty |I_n| < m(E)+\epsilon.$$

Since $U \supset E$ and $m(E) < \infty$ (and both U and E are Lebesgue measurable), Theorem 2.3 implies $m^*(U\setminus E) = m(U\setminus E) = m(U) - m(E) < \epsilon$.

To remove the finiteness assumption on E, we apply the $\epsilon/2^n$ trick: for each $n \in \mathbb{Z}$ let $E_n = E\cap(n, n+1]$. The E_n are disjoint measurable sets whose union is E, and $m(E_n) < \infty$ for all n. For each n, by the first part of the proof we can pick an open set U_n so that $m(U_n\setminus E_n) < \epsilon/2^{|n|}$. Let U be the union of the U_n. Thus U is open and $U\setminus E \subset \bigcup_{n=1}^\infty (U_n\setminus E_n)$ since $E^c \subset E_n^c$. The subadditivity of m gives $m(U\setminus E) < \sum_{n\in\mathbb{Z}} \epsilon 2^{-|n|} = 3\epsilon$.

To prove that (b) implies (d), let $E\subset\mathbb{R}$ be given and for each $n \ge 1$ choose (using (b)) an open set $U_n \supset E$ such that $m^*(U_n\setminus E) < \frac{1}{n}$. Put $G = \bigcap_{n=1}^\infty U_n$. Thus G is a G_δ containing E, and $G\setminus E \subset U_n \setminus E$ for every n. By monotonicity of m^* we see

$m^*(G \setminus E) < \frac{1}{n}$ for every n and thus $m^*(G \setminus E) = 0$. (Note that in this portion of the proof we cannot (and do not!) assume E is measurable.)

To prove (d) implies (a), suppose G is a G_δ set such that $E \subset G$ and $\mu^*(G \setminus E) = 0$. Since G is a G_δ, it is a Borel set and hence Lebesgue measurable by Theorem 4.2. By Lemma 3.7, every m^*-null set is Lebesgue measurable, so $G \setminus E$ and hence $G^c \cup E$ is Lebesgue measurable. Finally, $E = G \cap (G^c \cap E)$ is Lebesgue measuarable.

To prove that (a) implies (c), suppose E is Lebesgue measurable and let $\epsilon > 0$ be given. Thus E^c is Lebesgue measurable and, by the already established implication (a) implies (b), there is an open set U such that $E^c \subset U$ and $m(U \setminus E^c) < \epsilon$. Since $U \setminus E^c = U \cap E = E \setminus U^c$, it follows that $\mu(E \setminus U^c) < \epsilon$. Observing that U^c is closed completes the proof.

Now suppose $E \subset \mathbb{R}$ and (c) holds. Choose a sequence of closed sets (F_n) such that $F_n \subset E$ and $m^*(E \setminus F_n) < \frac{1}{n}$. The set $F = \cup_{j=1}^\infty F_j$ is an F_σ and, by monotonicity, for each n we have $m^*(E \setminus F) \le m^*(E \setminus F_n) < \frac{1}{n}$. Hence $m^*(E \setminus F) = 0$. Thus (c) implies (e).

Finally, if (e) holds, then $E = F \cup (E \setminus F)$ for some closed set $F \subset E$ with $\mu^*(E \setminus F) = 0$. Thus, E is the union of a closed (and hence Lebesgue) set and a set of outer measure zero (which is thus Lebesgue). Since the Lebesgue sets are closed under union, E is Lebesgue and the proof is complete. □

Recall the symmetric difference of sets $A, B \subset X$ is $A \Delta B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

Theorem 4.6. *If E is Lebesgue measurable and $m(E) < \infty$, then for each $\epsilon > 0$ there exists a set A that is a finite union of open intervals such that $m(E \Delta A) < \epsilon$.*

Proof. Since E is measurable, $m(E) < \infty$ and $m(E) = m^*(E)$, there exists a sequence (I_n) of open intervals that covers E such that

$$\sum_{n=1}^\infty |I_n| < m(E) + \epsilon/2. \tag{17}$$

Since the sum is finite there exists an integer N so that

$$\sum_{n=N+1}^\infty |I_n| < \epsilon/2. \tag{18}$$

Let $U = \bigcup_{n=1}^\infty I_n$ and $A = \bigcup_{n=1}^N I_n$. Then $A \setminus E \subset U \setminus E$, so $m(A \setminus E) \le m(U) - m(E) < \epsilon/2$ by (17). Similarly $E \setminus A \subset U \setminus A \subset \bigcup_{n=N+1}^\infty I_n$, so $m(E \setminus A) < \epsilon/2$ by (18). Therefore $m(E \Delta A) < \epsilon$. □

Thus, while the "typical" measurable set can be quite complicated in the set-theoretic sense (i.e. in terms of the Borel hierarchy), for most questions in analysis this complexity is irrelevant. In fact, Theorem 4.6 is the precise expression of a useful heuristic:

Littlewood's First Principle of Analysis: *Every measurable set $E \subset \mathbb{R}$ with $m(E) < \infty$ is almost a finite union of intervals.*

Definition 4.7. Let X be a topological space. A *neighborhood* U of a point $x \in X$ is an open set such that $x \in U$.

A topological space X is *locally compact* if for each $x \in X$ there is a neighborhood U_x of x and a compact set C_x such that $x \in U_x \subset C_x$.

A topological space is *Hausdorff* if given $x, y \in X$ with $x \neq y$, there exists neighborhoods U and V of x and y respectively such that $U \cap V = \emptyset$. (Distinct points can be separated by open sets.)

A *Borel measure* is a measure on the Borel σ-algebra $\mathscr{B}_X$ of a locally compact Hausdorff space X.

A Borel measure μ is *outer regular* if, for all $E \in \mathscr{B}_X$,

$$\mu(E) = \inf\{\mu(U) : U \supset E \text{ and } U \text{ is open}\}$$

and is *inner regular* if

$$\mu(E) = \sup\{\mu(K) : K \subset E \text{ and } K \text{ is compact}\}.$$

Finally μ is *regular* if it is both inner and outer regular. ◁

Theorem 4.8. *If $E \subset \mathbb{R}$ is Lebesgue measurable, then*

$$\begin{aligned} m(E) &= \inf\{m(U) : U \supset E \textit{ and } U \textit{ is open}\} \\ &= \sup\{m(K) : K \subset E \textit{ and } K \textit{ is compact}\} \end{aligned}$$

That is, m is a regular Borel measure.

Proof. Fix E. Let $\rho(E)$ denote the infimum in the first equality. By monotonicity, $\rho(E) \geq m(E)$. If $m(E) = \infty$, then equality is evident. The case $m(E) < \infty$ follows from Theorem 4.5(b) (together with the additivity of m).

For the second equality, let $\nu(E)$ be the value of the supremum on the right-hand side. By monotonicity $m(E) \geq \nu(E)$. For the reverse inequality, first assume $m(E) < \infty$ and let $\epsilon > 0$ be given. By Theorem 4.5(c), there is a closed subset $F \subset E$ with $m(E \setminus F) < \epsilon/2$. Since $m(E) < \infty$, by additivity $m(E) < m(F) + \epsilon/2$. Thus $m(F) > m(E) - \epsilon/2$. However this F need not be compact. To fix this potential shortcoming, for each $n \geq 1$ let $K_n = F \cap [-n, n]$. Then the K_n are an increasing sequence of compact sets whose union is F. By monotone convergence for sets (Theorem 2.3(c)), there is an n so that $m(K_n) > m(F) - \epsilon/2$. It follows that $m(K_n) > m(E) - \epsilon$, and thus $\nu(E) \geq m(E)$. The case $m(E) = +\infty$ is left as an exercise. □

4.2. **Examples.**

Example 4.9. [The Cantor set] Recall the usual construction of the "middle thirds" Cantor set. Let E_0 denote the unit interval $[0, 1]$. Obtain E_1 from E_0 by deleting the middle third (open) subinterval of E_0, so $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Continue inductively as follows. At the n^{th} step delete the middle thirds of all the intervals present at that step.

So, (E_n) is nested decreasing and E_n is a union of 2^n closed intervals of length 3^{-n}. The *Cantor set* is defined as the intersection $C = \bigcap_{n=0}^\infty E_n$. It is well-known (though not obvious and not proven here) that C is uncountable. It is clear that C is a closed set (hence Borel) that contains no (non-trivial) interval, since if J is an interval of length ℓ and n is chosen so that $3^{-n} < \ell$, then $J \not\subset E_n$ and thus $J \not\subset C$. The Lebesgue measure of E_n is $(2/3)^n$, which goes to 0 as $n \to \infty$, and thus by monotonicity (or dominated convergence for sets) $m(C) = 0$. So, C is an example of an uncountable, closed set of measure 0. Another way to see that C has measure zero is to note that at the n^{th} stage $(n \geq 1)$ we have deleted a collection of 2^{n-1} disjoint open intervals, each of length 3^{-n}. Thus the Lebesgue measure of $[0,1] \setminus C$ is

$$\sum_{n=1}^\infty 2^{n-1}3^{-n} = \frac{1}{2}\frac{\frac{2}{3}}{1-\frac{2}{3}} = 1.$$

Thus $m(C) = 0$. △

Example 4.10. [Fat Cantor sets] The standard construction of the Cantor set can be modified in the following way. Fix a number $0 < c < 1$ and imitate the construction of the Cantor set, except at the n^{th} stage delete, from each interval I present at that stage, an open interval centered at the midpoint of I of length $3^{-n}c$. (In the previous construction $c = 1$.) Again at each stage we have a set E_n that is a union of 2^n closed intervals each of which has length at most $(\frac{3-c}{6})^n$ and $m([0,1] \setminus E_n) = \sum_{j=1}^n 2^{j-1}\frac{c}{3^j}$. Let $F = \bigcap_{n=0}^\infty E_n$. One can prove (in much the same way as for C) that 1) F is an uncountable, closed set; 2) F contains no intervals; and 3) $m(F) = 1 - c > 0$. Thus, F is a closed set of positive measure that contains no (non-trivial) interval. △

Example 4.11. [Construction of Vitali sets] The Vitali sets are perhaps the most elementary examples of subsets of $\mathbb{R}$ that are not Lebesgue measurable. The construction depends on the axiom of choice. Since $\mathbb{Q} \subset \mathbb{R}$ is a subgroup of the abelian group $\mathbb{R}$ under addition, declaring $x \sim y$ if and only if $x - y \in \mathbb{Q}$ defines an equivalence relation on $\mathbb{R}$. This relation partitions $\mathbb{R}$ into disjoint equivalence classes whose union is $\mathbb{R}$. In particular, for each $x \in \mathbb{R}$ its equivalence class is the set $\{x + q : q \in \mathbb{Q}\}$. Each equivalence class C contains an element of the closed interval $[0,1]$. By the axiom of choice, there is a set $E \subset [0,1]$ that contains exactly one member x_C from each class C. Each such E is a Vitali set. We claim any such E is not Lebesgue measurable.

To prove the claim, let $y \in [0,1]$ be given. Let C denote the equivalence class of y. Thus y differs from x_C by some rational number in the interval $[-1,1]$. Hence

$$[0,1] \subset \bigcup_{q \in [-1,1]\cap\mathbb{Q}} (E+q).$$

On the other hand, since $E \subset [0,1]$ and $|q| \leq 1$,

$$\bigcup_{q \in [-1,1]\cap\mathbb{Q}} (E+q) \subset [-1,2].$$

Finally, by the construction of E the sets $E+p$ and $E+q$ are disjoint if p,q are distinct rationals. Arguing by contradiction, suppose E is measurable. In this case the sets $E+q$

are measurable by Theorem 4.4, and, by the countable additivity and monotonicity of m,

$$1 \le \sum_{q \in [-1,1]\cap\mathbb{Q}} m(E+q) \le 3. \tag{19}$$

But by translation invariance (Theorem 4.4 again), all of the $m(E+q)$ must be equal yielding the contradiction that the sum in equation (19) is either 0 or ∞. $\triangle$

Remark: The construction of Example 4.11 can be modified to show if F is any Lebesgue set with $m(F) > 0$, then F contains a nonmeasurable (i.e., a non-Lebesgue) subset. See Problem 7.29.

5. Premeasures and the Hahn-Kolmogorov Theorem

Definition 5.1. Let $\mathscr{A} \subset 2^X$ be a Boolean algebra. A *premeasure* on $\mathscr{A}$ is a function $\mu_0 : \mathscr{A} \to [0,+\infty]$ satisfying

(i) $\mu_0(\varnothing) = 0$; and
(ii) if $(A_j)_{j=1}^{\infty}$ is a sequence of disjoint sets in $\mathscr{A}$ and $\cup_1^\infty A_j \in \mathscr{A}$, then

$$\mu_0\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu_0(A_j)$$

$\triangleleft$

Finiteness and σ-finiteness are defined for premeasures in the same way as for measures. Note that a premeasure is automatically finitely additive and hence monotone.

Example 5.2. By an h-interval we mean a (finite or infinite) interval of the form $(a,b]$. (By convention $(a,+\infty)$ is an h-interval.) The collection $\mathscr{A} \subset 2^{\mathbb{R}}$ of finite unions of h-intervals is a Boolean algebra. The function $\mu_0 : \mathscr{A} \to [0,\infty]$ defined by

$$\mu_0(I) = \sum_{j=1}^{n} b_j - a_j$$

for $I \in \mathscr{A}$ written as the disjoint union $\cup_1^n (a_j, b_j]$ is a premeasure on $\mathscr{A}$. (Warning: that μ_0 is well defined and a premeasure is immediate if we have already constructed Lebesgue measure, but it is not as obvious as it seems to prove from scratch - there are many different ways to decompose a given I as a finite or countable disjoint union of h-interval. Thus verifying that μ_0 is well defined and countable additivity is somewhat delicate. See Section 6.) $\triangle$

Theorem 5.3 (Hahn-Kolmogorov Theorem)**.** *If μ_0 is a premeasure on a Boolean algebra $\mathscr{A} \subset 2^X$ and μ^* is the outer measure on X defined by (8), then every set $A \in \mathscr{A}$ is outer measurable and $\mu^*|_{\mathscr{A}} = \mu_0$. Thus, letting $\mathscr{M}$ denote the σ-algebra of μ^*-outer measurable sets and μ the (complete) measure $\mu = \mu^*|_{\mathscr{M}}$, we have μ_0 is the restriction of μ to $\mathscr{A}$.*

In particular, if $\mu_0 : \mathscr{A} \to [0, +\infty]$ is a premeasure on a Boolean algebra $\mathscr{A}$, then there exists a σ-algebra $\mathscr{B} \supset \mathscr{A}$ and a measure $\mu : \mathscr{B} \to [0, +\infty]$ such that $\mu|_{\mathscr{A}} = \mu_0$.

Proof. If $\mathscr{A} \subset 2^X$ is a Boolean algebra and $\mu_0 : \mathscr{A} \to [0, +\infty]$ is a premeasure, then μ_0 determines an outer measure μ^* by Proposition 3.4. We will prove that (1) $\mu^*|_{\mathscr{A}} = \mu_0$, and (2) every set in $\mathscr{A}$ is μ^*-outer measurable. The theorem then follows from Theorem 3.3.

To prove (1), let $E \in \mathscr{A}$. It is immediate that $\mu^*(E) \le \mu_0(E)$, since for a covering of E from $\mathscr{A}$ we can take $A_1 = E$ and $A_j = \varnothing$ for all other j. For the reverse inequality, let (A_j) be any covering of E by sets $A_j \in \mathscr{A}$ and let (A'_n) denote the disjointification of (A_n)

$$A'_n = A_n \setminus \bigcup_{j=1}^{n-1} A_j.$$

By Proposition 1.7, the $A'_n \in \mathscr{A}$, are pairwise disjoint and $\cup_1^n A_j = \cup_1^n A'_j$ for $n = 1, 2, \dots, \infty$. Let $B_n = E \cap A'_n$. Hence the B_n are disjoint sets in $\mathscr{A}$ whose union is E, and $B_n \subset A_n$ for all n. Thus by the countable additivity and monotonicity of μ_0,

$$\mu_0(E) = \sum_{n=1}^{\infty} \mu_0(B_n) \le \sum_{n=1}^{\infty} \mu_0(A_n)$$

Since the covering was arbitrary, $\mu_0(E) \le \mu^*(E)$.

For (2), let $E \in \mathscr{A}$, $A \subset X$, and $\epsilon > 0$ be given. There exists a sequence of sets $(B_j) \subset \mathscr{A}$ such that $A \subset \bigcup_{j=1}^{\infty} B_j$ and $\sum_{j=1}^{\infty} \mu_0(B_j) < \mu^*(A) + \epsilon$. By additivity and monotonicity of μ_0,

$$\begin{aligned} \mu^*(A) + \epsilon &> \sum_{j=1}^{\infty} \mu_0(B_j) \\ &= \sum_{j=1}^{\infty} \mu_0(B_j \cap E) + \sum_{j=1}^{\infty} \mu_0(B_j \cap E^c). \end{aligned}$$

Since $(B_j \cap E)$ is a sequence from $\mathscr{A}$ and $A \cap E \subset \cup_{j=1}^{\infty}(B_j \cap E)$, it follows that

$$\mu^*(A \cap E) \le \sum_{j=1}^{\infty} \mu_0(B_j \cap E)$$

and similarly for E^c. Thus $\mu^*(A) + \epsilon \ge \mu^*(A \cap E) + \mu^*(A \cap E^c)$ and since $\epsilon > 0$ as well as $A \subset \mathbb{R}$ and $E \in \mathscr{A}$ are arbitrary, the proof is complete. □

The measure μ constructed in Theorem 5.3 is the *Hahn-Kolmogorov extension* of the premeasure μ_0. The relationship between premeasures, outer measures, and measures in this construction is summarized in the following table:

	domain	additivity condition
premeasure	Boolean algebra $\mathscr{A}$	countably additive, when possible
outer measure	all of 2^X	monotone, countably subadditive
measure	σ-algebra containing $\mathscr{A}$	countably additive

The premeasure μ_0 has the right additivity properties, but is defined on too few subsets of X to be useful. The corresponding outer measure μ^* constructed in Proposition 3.4 is defined on all of 2^X, but we cannot guarantee countable additivity. By Theorem 5.3, restricting μ^* to the σ-algebra of outer measurable sets is "just right."

We have established that every premeasure μ_0 on an algebra $\mathscr{A}$ can be extended to a measure on the σ-algebra generated by $\mathscr{A}$. The next theorem addresses the uniqueness of this extension.

Theorem 5.4 (Hahn uniqueness theorem)**.** *Suppose $\mathscr{A}$ is a Boolean algebra and let $\mathscr{N}$ denote the σ-algebra it generates. If μ_0 is premeasure on $\mathscr{A}$ and μ^* is the outer measure it determines, then every extension of μ_0 to a measure on $\mathscr{N}$ agrees on sets $E \in \mathscr{N}$ of finite outer measure.*

Further, if μ_0 is σ-finite, then μ_0 has a unique extension to a measure ν on $\mathscr{N}$.

Uniqueness can fail in the non-σ-finite case. An example is outlined in Problem 7.20.

Proof. Let $\mathscr{N}$ be the σ-algebra generated by $\mathscr{A}$, let ν denote the Hahn-Kolmogorov extension of μ_0, but restricted to $\mathscr{N}$. Thus, letting μ denote the outer measure μ^* determined by μ_0 restricted to the μ^*-outer measurable sets $\mathscr{M}$, we have $\nu = \mu|_{\mathscr{N}}$. Let $\widetilde{\nu}$ be any other extension of μ_0 to $\mathscr{N}$. We first show, if $E \in \mathscr{N}$, then $\widetilde{\nu}(E) \leq \nu(E)$. Let $E \in \mathscr{N}$ and let (A_n) be a sequence in $\mathscr{A}$ such that $E \subset \bigcup_{n=1}^\infty A_n$. Then

$$\widetilde{\nu}(E) \leq \sum_{n=1}^\infty \widetilde{\nu}(A_n) = \sum_{n=1}^\infty \nu_0(A_n)$$

Taking the infimum over all such coverings of E, it follows that $\widetilde{\nu}(E) \leq \nu(E)$. (Recall the definition of μ.)

Next we show, if $E \in \mathscr{N}$ and $\nu(E) < \infty$ (the finite outer measure assumption), then $\nu(E) \leq \widetilde{\nu}(E)$. As a first observation, note that given a sequence (A_n) from $\mathscr{A}$ and letting $A = \bigcup_{n=1}^\infty A_n \in \mathscr{N}$, monotone convergence for sets (twice) implies

$$\widetilde{\nu}(A) = \lim_{N\to\infty} \widetilde{\nu}(\bigcup_{n=1}^N A_n) = \lim_{N\to\infty} \nu(\bigcup_{n=1}^N A_n) = \nu(A). \tag{20}$$

Now let $\epsilon > 0$ be given and choose a covering (A_n) of E by sets in $\mathscr{A}$ such that, letting $A = \bigcup_{n=1}^\infty A_n$, we have $\nu(A) < \nu(E) + \epsilon$. Consequently $\nu(A \setminus E) < \epsilon$. In particular,

$\widetilde{\nu}(A \setminus E) \le \nu(A \setminus E) < \epsilon$, since $A \setminus E \in \mathscr{N}$. Thus, using equation (20),

$$\begin{aligned}\nu(E) &\le \nu(A)\\ &= \widetilde{\nu}(A)\\ &= \widetilde{\nu}(E) + \widetilde{\nu}(A \setminus E)\\ &< \widetilde{\nu}(E) + \epsilon.\end{aligned}$$

Since ϵ was arbitrary, we conclude $\nu(E) \le \widetilde{\nu}(E)$. At this point the first part of the Theorem is proved.

Now suppose μ_0 is σ-finite. Thus there exists a sequence of sets (X_n) such that $X_n \in \mathscr{A}$, $\mu_0(X_n) < \infty$ and $X = \cup X_n$. By Proposition 1.7, we may assume the (X_n) are pairwise disjoint. If $E \in \mathscr{N}$, then $E \cap X_n \in \mathscr{N}$ and $\nu(E \cap X_n) \le \nu(X_n) = \mu_0(X_n) < \infty$. Therefore, from what has already been proved,

$$\nu(E) = \sum_{n=1}^{\infty} \nu(E \cap X_n) = \sum_{n=1}^{\infty} \widetilde{\nu}(E \cap X_n) = \widetilde{\nu}(E).$$

$\square$

Corollary 5.5 (Uniqueness of Lebesgue measure)**.** *If μ is a Borel measure on $\mathbb{R}$ such that $\mu(I) = |I|$ for every interval I, then $\mu(E) = m(E)$ for every Borel set $E \subset \mathbb{R}$.* †

6. Lebesgue-Stieltjes measures on $\mathbb{R}$

Let μ be a Borel measure on $\mathbb{R}$. (Thus the domain of μ contains all Borel sets, though we allow that the domain of μ may be larger.) The measure μ is *locally finite* if $\mu(I) < \infty$ for every compact interval I. (Equivalently, $\mu(I)$ is finite for every finite interval.) Given a locally finite Borel measure, define a function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \begin{cases} 0 & \text{if } x = 0,\\ \mu((0, x]) & \text{if } x > 0,\\ -\mu((x, 0]) & \text{if } x < 0.\end{cases} \tag{21}$$

It is not hard to show, using dominated and monotone convergence for sets, that F is nondecreasing and continuous from the right; that is, $F(a) = \lim_{x \to a^+} F(x)$ for all $a \in \mathbb{R}$ (see Problems 7.22 and 7.23). In this section we prove the converse: given any increasing, right-continuous function $F : \mathbb{R} \to \mathbb{R}$, there is a unique locally finite Borel measure μ such that (21) holds. The proof will use the Hahn-Kolmogorov extension theorem.

Let $\mathscr{A} \subset 2^{\mathbb{R}}$ denote the Boolean algebra generated by the half-open intervals $(a, b]$. (We insist that the interval be open on the left and closed on the right, a convention compatible with the definition of F.) More precisely, $\mathscr{A}$ consists of all finite unions of intervals of the form $(a, b]$ (with $(-\infty, b]$ and $(a, +\infty)$ allowed). Fix a nondecreasing, right-continuous function $F : \mathbb{R} \to \mathbb{R}$. Since F is monotone, the limits $F(+\infty) :=$

$\lim_{x\to+\infty} F(x)$ and $F(-\infty) := \lim_{x\to-\infty} F(x)$ exist (possibly $+\infty$ or $-\infty$ respectively). For each interval $I = (a, b]$ in $\mathscr{A}$, we define its *F-length* by

$$|I|_F := F(b) - F(a).$$

Given a set $A \in \mathscr{A}$, we can write it as a disjoint union of intervals $A = \bigcup_{n=1}^N I_n$ with $I_n = (a_n, b_n]$. Define

$$\mu_0(A) = \sum_{n=1}^N |I_n|_F = \sum_{n=1}^N F(b_n) - F(a_n). \tag{22}$$

Proposition 6.1. *The expression (22) is a well-defined premeasure on $\mathscr{A}$.* †

Proof. That μ_0 is well-defined and finitely additive on $\mathscr{A}$ is left as an exercise.

To prove that μ_0 is a premeasure, let (I_n) be a disjoint sequence of intervals in $\mathscr{A}$ and suppose $J = \bigcup_{n=1}^\infty I_n \in \mathscr{A}$. For now assume J is an h-interval. By finite additivity (and monotonicity),

$$\mu_0(J) \geq \mu_0\left(\bigcup_{n=1}^N I_n\right) = \sum_{n=1}^N \mu_0(I_n).$$

Taking limits, we conclude $\mu(\bigcup_{n=1}^\infty I_n) \geq \sum_{n=1}^\infty \mu_0(I_n)$.

For the reverse inequality, we employ a compactness argument similar to the one used in the proof of Theorem 4.1. However, the situation is more complicated since we are dealing with half-open intervals. The strategy will be to shrink J to a slightly smaller compact interval, and enlarge the I_n to open intervals, using the right-continuity of F and the $\epsilon/2^n$ trick to control their F-lengths.

We'll prove the reverse inequality assuming $J = (a, b]$ is a *finite* interval, leaving the cases of the infinite intervals as an exercise. Accordingly, fix $\epsilon > 0$. By right continuity of F, there is a $\delta > 0$ such that $F(a+\delta) - F(a) < \epsilon$. Likewise, writing $I_n = (a_n, b_n]$, there exist $\delta_n > 0$ such that $F(b_n+\delta_n) - F(b_n) < \epsilon 2^{-n}$. Let $\tilde{J} = [a+\delta, b]$ and $\tilde{I}_n = (a_n, b_n+\delta_n)$. It follows that $\tilde{J} \subset J = \cup I_n \subset \cup \tilde{I}_n$. Hence, by compactness, finitely many of the $\tilde{I}_n$ cover $\tilde{J}$, and these may be chosen so that none is contained in another, each has non-trivial intersection with $\tilde{J}$ and we may reindex so that these n intervals are relabeled as $\tilde{I}_1, \ldots \tilde{I}_N$ so their left endpoints are listed in increasing order. (This rearrangement does not change the sum

$$\sum_{j=1}^\infty [F(b_j + \delta_j) - F(a_j)]$$

though it does of course change the partial sums.)

As in the proof of Theorem 4.1, it follows that $a_1 < a + \delta$, $a_2 < b_1 + \delta_1$, $a_3 < b_2$, ...$a_N < b_{N-1} + \delta_{N-1}$ and $b < b_N + \delta_N$. Thus,

$$\begin{aligned}\mu_0(J) &\le F(b) - [F(a+\delta) - \epsilon]\\ &\le F(b_N + \delta_N) - F(a_1) + \epsilon\\ &= F(b_N+\delta_N) - F(a_N) + \sum_{j=1}^{N-1}(F(a_{j+1}) - F(a_j)) + \epsilon\\ &\le F(b_N+\delta_N) - F(a_N) + \sum_{j=1}^{N-1}(F(b_j+\delta_j) - F(a_j)) + \epsilon\\ &\le \sum_{j=1}^{\infty}(F(b_j+\delta_j) - F(a_j)) + \epsilon\\ &\le \sum_{j=1}^{\infty}\mu_0(I_j) + 2\epsilon.\end{aligned}$$

Since ϵ was arbitrary, $\mu_0(J) \le \sum_{j=1}^\infty \mu_0(I_j)$ as claimed.

Now suppose $J \in \mathscr{A}$ is not necessarily an h-interval. In any case, $J = \cup_{k=1}^m J_k$ is a finite disjoint union of h-intervals. Each J_k is the a countable disjoint union

$$J_k = \cup_{j=1}^\infty J_k \cap I_j$$

of h-intervals From what has already been proved, $\mu_0(J_k) = \sum_{j=1}^\infty \mu_0(J_k \cap I_j)$. Since μ_0 is finitely additive,

$$\sum_{k=1}^m \mu_0(J_k) = \sum_{k=1}^m\sum_{j=1}^\infty \mu_0(J_k\cap I_j) = \sum_{j=1}^\infty\sum_{k=1}^m \mu_0(J_k\cap I_j) = \sum_{j=1}^\infty \mu_0(I_j).$$

Finally, suppose $(A_n)_{n=1}^\infty$ is a disjoint sequence from $\mathscr{A}$ and $J = \cup A_n \in \mathscr{A}$. For each n, there exists disjoint h-intervals $(I_{n,\ell})_{\ell=1}^{N_n}$ such that $A_n = \cup_\ell I_{n,\ell}$. Thus $J = \cup I_{n,\ell}$ is a disjoint union of h-intervals and, by what has already been proved and finite additivity,

$$\mu_0(J) = \sum_n \sum_\ell \mu_0(I_{n,\ell}) = \sum_n \mu_0(A_n).$$

□

By the Hahn-Kolmogorov Theorem, μ_0 extends to a Borel measure μ_F, called the *Lebesgue-Stieltjes measure* associated to F. It is immediate from the definition that μ_0 is σ-finite (each interval $(n, n+1]$ has finite F-length), so the restriction of μ_F to the Borel σ-algebra is uniquely determined by F by Theorem 5.4. In particular we conclude that the case $F(x) = x$ recovers Lebesgue measure.

Example 6.2. (a) (Dirac measure) Define the *Heaviside function*

$$H(x) = \begin{cases} 1 & \text{if } x \ge 0\\ 0 & \text{if } x < 0\end{cases}$$

Then for any interval $I = (a, b]$, $\mu_H(I) = 1$ if $0 \in I$ and 0 otherwise. Since Dirac measure δ_0 is a Borel measure and also has this property, and the intervals $(a, b]$ generate the Borel σ-algebra, it follows from the Hahn Uniqueness Theorem (Theorem 5.4) that $\mu_H(E) = \delta_0(E)$ for all Borel sets $E \subset \mathbb{R}$. There is nothing special about 0 here. Given $p \in \mathbb{R}$, let δ_p denote the Borel measure defined by $\delta(E) = 1$ if $p \in E$ and 0 if $p \notin E$. For a finite set $x_1, \ldots x_n$ in $\mathbb{R}$ and positive numbers $c_1, \ldots c_n$, let $F(x) = \sum_{j=1}^n c_j H(x - x_j)$. Then $\mu_F = \sum_{j=1}^n c_j \delta_{x_j}$. Not that F is continuous except at the points x_j where $F(x_j-) = c_j$.

(b) (Infinite sums of point masses) Even more generally, if $(x_n)_{n=1}^\infty$ is an infinite sequence in $\mathbb{R}$ and (c_n) is a sequence of positive numbers with $\sum_{n=1}^\infty c_n < \infty$, define $F(x) = \sum c_n H(x - x_n) = \sum_{n:x_n \leq x} c_n$. It follows, using Theorem 5.4 and Problem 7.9, that $\mu_F(E) = \sum_{n:x_n \in E} c_n$; i.e., $\mu_F = \sum_{n=1}^\infty c_n \delta_{x_n}$. A particularly interesting case is when the x_n enumerate the rationals; the resulting function F is continuous precisely on the irrationals. We will return to this example after the Radon-Nikodym theorem.

(c) (Cantor measure) Recall the construction of the Cantor set C from Example 4.9. Each number $x \in [0, 1]$ has a base 3 expansion, of the form $x = \sum_{n=1}^\infty a_n 3^{-n}$, where $a_n \in \{0, 1, 2\}$ for all n. The expansion is unique if we insist that every terminating expansion ($a_n = 0$ for all n sufficiently large) is replaced with an expansion ending with an infinite string of 2's (that is, $a_n = 2$ for all n sufficiently large). With these conventions, it is well-known that C consists of all points $x \in [0, 1]$ such that the base 3 expansion of x contains only 0's and 2's. (Referring again to the construction of C, x belongs to E_1 if and only if a_1 is 0 or 2, x belongs to E_2 if and only if both a_1, a_2 belong to $\{0, 2\}$, etc.) Using this fact, we can define a function $F : C \to [0, 1]$ by taking the base 3 expansion $x = \sum_{n=1}^\infty a_n 3^{-n}$, setting $b_n = a_n/2$, and putting $F(x) = \sum_{n=1}^\infty b_n 2^{-n}$. (The ternary string of 0's and 2's is sent to the binary string of 0's and 1's.) If $x, y \in C$ and $x < y$, then $F(x) < F(y)$ unless x, y are the endpoints of a deleted interval, in which case $F(x) = p2^{-k}$ for some integers p and k, and $F(x)$ and $F(y)$ are the two base 2 expansions of this number. We can then extend F to have this constant value on the deleted interval (x, y). The resulting F is monotone and maps $[0, 1]$ *onto* $[0, 1]$. Since F is onto and monotone, it has no jump discontinuities, and again by monotonicity, F is continuous. This function is called the *Cantor-Lebesgue function*, or in some books the *Devil's Staircase*. Finally, if we extend F to be 0 for $x < 0$ and 1 for $x > 1$, we can form a Lebesgue-Stieltjes measure μ_F supported on C (that is, $\mu_F(E) = 0$ if $E \cap C = \varnothing$ equivalent $\mu(C^c) = 0$). This measure is called the *Cantor measure*. It is said to be *singular* because it is supported on a set of Lebesgue measure 0 (see Problem 7.30). It will be an important example of what is called a *singular continuous* measure on $\mathbb{R}$.

$\triangle$

One can prove that the Lebesgue-Stieltjes measures μ_F have similar regularity properties as Lebesgue measure; since the proofs involve no new ideas they are left as exercises.

Lemma 6.3. *Let μ_F be a Lebesgue-Stieltjes measure. If $E \subset \mathbb{R}$ is a Borel set, then*

$$\mu_F(E) = \inf\{\sum_{n=1}^{\infty} \mu_F(a_n, b_n) : E \subset \bigcup_{n=1}^{\infty} (a_n, b_n)\}$$

†

Proof. Problem 7.25. □

Theorem 6.4. *Let μ_F be a Lebesgue-Stieltjes measure. If $E \subset \mathbb{R}$ is a Borel set, then*

$$\begin{aligned} \mu_F(E) &= \inf\{\mu_F(U) : E \subset U,\ U \text{ open}\} \\ &= \sup\{\mu_F(K) : K \subset E,\ K \text{ compact}\} \end{aligned}$$

Proof. Problem 7.26. □

7. Problems

Problem 7.1. Let $X = \{0, 1, 2, 3\}$ and let

$$\mathscr{N} = \{\emptyset, X, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{2, 3\}, \{1, 3\}, \{1, 2\}\}.$$

Verify that $\mathscr{N}$ is closed under complements and countable disjoint unions, but is not a σ-algebra.

Problem 7.2. Prove the "exercise" claims in Example 1.8.

Problem 7.3. (a) Let X be a set and let $\mathscr{A} = (A_n)_{n=1}^{\infty}$ be a sequence of disjoint, nonempty subsets whose union is X. Prove that the set of all finite or countable unions of members of $\mathscr{A}$ (together with $\varnothing$) is a σ-algebra. (A σ-algebra of this type is called *atomic*.)

(b) Prove that the Borel σ-algebra $\mathscr{B}_{\mathbb{R}}$ is *not* atomic. (Hint: there exists an uncountable family of mutually disjoint Borel subsets of $\mathbb{R}$.)

Problem 7.4. Can a σ-algebra be, as a set, countably infinite?

Problem 7.5. a) Prove Proposition 1.14. (First prove that every open set U is a union of dyadic intervals. To get disjointness, show that for each point $x \in U$ there is a unique largest dyadic interval I such that $x \in I \subset U$.) b) Prove that the dyadic intervals generate the Borel σ-algebra $\mathscr{B}_{\mathbb{R}}$.

Problem 7.6. Fix an integer $n \geq 1$. Prove that the set of finite unions of dyadic subintervals of $(0, 1]$ of length at most 2^{-n} (together with $\varnothing$) is a Boolean algebra (of subsets of $(0, 1]$).

Problem 7.7. Prove that if X, Y are topological spaces and $f : X \to Y$ is continuous, then f is Borel measurable.

Problem 7.8. Let $(X, \mathscr{M})$ be a measurable space and suppose $\mu : \mathscr{M} \to [0, +\infty]$ is a finitely additive measure that satisfies item (c) of Theorem 2.3. Prove that μ is a measure.

Problem 7.9. Prove that a countably infinite sum of measures is a measure (Example 2.2(d)). You will need the following fact from elementary analysis: if $(a_{mn})_{m,n=1}^{\infty}$ is a doubly indexed sequence of nonnegative reals, then $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}$. Indeed, prove the claims about sums over countable sets appearing after Proposition 3.4.

Problem 7.10. Let $\mathscr{A}$ be an atomic σ-algebra generated by a partition $(A_n)_{n=1}^{\infty}$ of a set X (see Problem 7.3).

(a) Fix $n \geq 1$. Prove that the function $\delta_n : \mathscr{A} \to [0,1]$ defined by

$$\delta_n(A) = \begin{cases} 1 & \text{if } A_n \subset A \\ 0 & \text{if } A_n \not\subset A \end{cases}$$

is a measure on $\mathscr{A}$.

(b) Prove that if μ is any measure on $(X, \mathscr{A})$, then there exists a unique sequence (c_n) with each $c_n \in [0, +\infty]$ such that

$$\mu(A) = \sum_{n=1}^{\infty} c_n \delta_n(A)$$

for all $A \in \mathscr{A}$.

Problem 7.11. Let $E\Delta F$ denote the *symmetric difference* of subsets E and F of a set X,

$$E\Delta F := (E \setminus F) \cup (F \setminus E) = (E \cup F) \setminus (E \cap F).$$

Let $(X, \mathscr{M}, \mu)$ be a measure space. Prove the following:

(a) If $E, F \in \mathscr{M}$ and $\mu(E\Delta F) = 0$ then $\mu(E) = \mu(E \cap F) = \mu(F)$.

(b) Define $E \sim F$ if and only if $\mu(E\Delta F) = 0$. Show $\sim$ is an equivalence relation on $\mathscr{M}$.

(c) Assume now that μ is a finite measure. For $E, F \in \mathscr{M}$ define $d(E, F) = \mu(E\Delta F)$. Show d defines (determines) a metric on the set of equivalence classes $\mathscr{M}/\sim$.

Problem 7.12. Let X be a set. For a sequence of subsets (E_n) of X, define

$$\limsup E_n = \bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} E_n, \quad \liminf E_n = \bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty} E_n.$$

a) Prove that $\limsup \mathbf{1}_{E_n} = \mathbf{1}_{\limsup E_n}$ and $\liminf \mathbf{1}_{E_n} = \mathbf{1}_{\liminf E_n}$ (thus justifying the names). Conclude that $E_n \to E$ pointwise if and only if $\limsup E_n = \liminf E_n = E$. (Hint: for the first part, observe that $x \in \limsup E_n$ if and only if x lies in infinitely many of the E_n, and $x \in \liminf E_n$ if and only if x lies in all but finitely many E_n.)

b) Prove that if the E_n are measurable, then so are $\limsup E_n$ and $\liminf E_n$. Deduce that if (E_n) converges to E pointwise and all the E_n are measurable, then E is measurable.

Problem 7.13. [Fatou theorem for sets] Let $(X, \mathscr{M}, \mu)$ be a measure space, and let (E_n) be a sequence of measurable sets.

a) Prove that

$$\mu(\liminf E_n) \le \liminf \mu(E_n). \tag{23}$$

b) Assume in addition that $\mu(\bigcup_{n=1}^{\infty} E_n) < \infty$. Prove that

$$\mu(\limsup E_n) \ge \limsup \mu(E_n). \tag{24}$$

c) Prove the following stronger form of the dominated convergence theorem for sets: suppose (E_n) is a sequence of measurable sets, and there is a measurable set $F \subset X$ such that $E_n \subset F$ for all n and $\mu(F) < \infty$. Prove that if (E_n) converges to E pointwise, then $(\mu(E_n))$ converges to $\mu(E)$. Give an example to show the finiteness hypothesis on F cannot be dropped.

(For parts (a) and (b), use Theorem 2.3.)

Problem 7.14. Complete the proof of Theorem 2.8.

Problem 7.15. Complete the proof of Theorem 4.4.

Problem 7.16. Given an example of a measurable function $f : X \to Y$ between measure spaces and a subset $E \subset X$ such that $f(E)$ is not measurable.

Problem 7.17. Prove the following dyadic version of Theorem 4.6: If $m(E) < \infty$ and $\epsilon > 0$, there exists an integer $n \ge 1$ and a set A, that is a finite union of dyadic intervals of length 2^{-n}, such that $m(E \Delta A) < \epsilon$. (This result says, loosely, that measurable sets look "pixelated" at sufficiently fine scales.)

Problem 7.18. a) Prove the following strengthening of Theorem 4.6: if $E \subset \mathbb{R}$ and $m^*(E) < \infty$, then E is Lebesgue measurable if and only if for every $\epsilon > 0$, there exists a set $A = \bigcup_{n=1}^{N} I_n$ (a finite union of open intervals) such that $m^*(E \Delta A) < \epsilon$.

b) State and prove a dyadic version of the theorem in part (a).

Problem 7.19. Prove the claims made about the Fat Cantor set in Example 4.10.

Problem 7.20. Let $\mathscr{A} \subset 2^{\mathbb{R}}$ be the Boolean algebra generated by the half-open intervals $(a, b]$. For $A \in \mathscr{A}$, let $\mu_0(A) = +\infty$ if A is nonempty and $\mu_0(\varnothing) = 0$.

(a) Prove that μ_0 is a premeasure. If μ is the Hahn-Kolmogorov extension of μ_0 and $E \subset \mathbb{R}$ is a nonempty Borel set, prove that $\mu(E) = +\infty$.
(b) Prove that if μ' is counting measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$, then μ' is an extension of μ_0 different from μ.

Here is a variant of this example. Let $\mathcal{A} \subset 2^{\mathbb{Q}}$ denote the Boolean algebra generated by the half-open intervals $(a, b]$ (intersect with $\mathbb{Q}$ of course). Note that the σ-algebra generated by $\mathcal{A}$ is $2^{\mathbb{Q}}$. For $A \in \mathcal{A}$, let $\mu_0(A) = +\infty$ if A is nonempty and $\mu_0(\varnothing) = 0$. Show μ_0 is a premeasure and its Hahn-Kolmogorov extension μ to $2^{\mathbb{Q}}$ is given by $\mu(E) = 0$ if $E = \emptyset$ and $\mu(E) = \infty$ otherwise. Show counting measure c is another extension of μ_0 to $2^{\mathbb{Q}}$. In particular, counting measure c is a σ-finite measure on $2^{\mathbb{Q}}$, but the premeasure obtained by restricting c to $\mathcal{A}$ is not σ-finite.

Problem 7.21. Suppose $(X, \mathscr{M}, \mu)$ is a measure space and $\mathscr{A} \subset 2^X$ is a Boolean algebra that generates $\mathscr{M}$ and that there is a sequence (A_n) from $\mathscr{A}$ such that $\mu(A_n) < \infty$ and $\cup A_n = X$. Prove that if $E \in \mathscr{M}$ and $\mu(E) < \infty$, then for every $\epsilon > 0$ there exists a set $A \in \mathscr{A}$ such that $\mu(E \Delta A) < \epsilon$. (Hint: let μ_0 be the premeasure obtained by restricting μ to $\mathscr{A}$. One may then assume that μ is equal the Hahn-Kolmogorov extension of μ_0. (Why?))

Problem 7.22. Prove that if μ is a locally finite Borel measure and F is defined by (21), then F is nondecreasing and right-continuous. (Note, once it has been shown that F is nondecreasing, all one sided limits of F exist. The only issue that remains is the value of these limits.)

Problem 7.23. Let μ_F be a Lebesgue-Stieltjes measure. Write $F(a^-) := \lim_{x \to a^-} F(x)$. Prove that

(a) $\mu_F(\{a\}) = F(a) - F(a^-)$,
(b) $\mu_F([a, b)) = F(b^-) - F(a^-)$,
(c) $\mu_F([a, b]) = F(b) - F(a^-)$, and
(d) $\mu_F((a, b)) = F(b^-) - F(a)$.

Problem 7.24. Complete the proof of Proposition 6.1

Problem 7.25. Prove Lemma 6.3.

Problem 7.26. Prove Theorem 6.4. (Use Lemma 6.3.)

Problem 7.27. Let $E \subset \mathbb{R}$ measurable and $m(E) > 0$.

(a) Prove that for each $0 < \alpha < 1$, there is an open interval I such that $m(E \cap I) > \alpha\, m(I)$.
(b) Show that the set $E - E = \{x - y : x, y \in E\}$ contains an open interval centered at 0. (Choose I as in part (a) with $\alpha > 3/4$; then $E - E$ contains $(-m(I)/2, m(I)/2)$.)

Problem 7.28. This problem gives another construction of a set $E \subset \mathbb{R}$ that is not Lebesgue measurable.

(a) Prove that there is a subset $E \subset \mathbb{Q}^c$ such that for each $x \in \mathbb{Q}^c$ exactly one of x or $-x$ is in E and, for all rational numbers q, $E + q = E$. Suggestion: Well order the irrationals by say $\prec$ and let E denote the set of those irrational numbers x such that
$$\min(x + \mathbb{Q}) \prec \min(-x + \mathbb{Q}).$$
(b) Prove that any set E with the properties above (for $x \in \mathbb{Q}^c$ exactly one of x or $-x$ is in E and $E + q = E$ for all $q \in \mathbb{Q}$) is not Lebesgue measurable. (Hint: suppose it is. Prove, for every interval I with rational endpoints, $m(E \cap I) = \frac{1}{2}|I|$ and apply part (a) of Problem 7.27.)

Problem 7.29. Let E be the nonmeasurable set described in Example 4.11.

(a) Show if $F \subset \mathbb{R}$ is (Lebesgue) measurable, bounded and $(F + q) \cap (F + r) = \emptyset$ for distinct rationals q, r, then $m(F) = 0$.
(b) Show that if $q \in \mathbb{Q}$, $F \subset E + q$ and F is Lebesgue measurable, then $m(F) = 0$.

(c) Prove that if $G \subset \mathbb{R}$ has positive measure, then G contains a nonmeasurable subset. (Observe $G = \cup_{q \in \mathbb{Q}} G \cap (E+q)$.)

Problem 7.30. Suppose μ is a regular Borel measure on a compact Hausdorff space and $\mu(X) = 1$. Let $\mathcal{O}$ denote the collection of μ-null open subsets of X and let $U = \cup_{O \in \mathcal{O}} O$. Prove U is also μ-null. Hence U is the largest μ-null subset of X. Prove there exists a smallest compact subset K of X such that $\mu(K) = 1$. The set K is the *support* of μ.

Problem 7.31. Given a set X and a subset $\rho \subset 2^X$, there is a smallest topology τ on X containing ρ, called the topology generated by ρ. (Proof idea: 2^X is a topology on X and the intersection of topologies is also a topology.) Let N be a positive integer and (X_j, τ_j) for $1 \le j \le N$ be topological spaces. The *product topology* π on $X = \prod X_j$ is the topology generated by the open rectangles; i.e., by the set

$$\rho = \{\times_1^N U_j = U_1 \times \cdots \times U_N : U_j \in \tau_j\} \subset 2^X.$$

Observe that each of the projection maps $\pi_j : X \to X_j$ is continuous. Prove, if every each open set W in the product topology on X is an at most countable union from ρ, then $\otimes \mathscr{B}_{X_j} = \mathscr{B}_X$; i.e., the product of the Borel σ algebras on the X_j is the same as the Borel sigma algebra on X given the product topology.

Problem 7.32. Give a proof of Theorem 4.6 based upon Theorem 4.8.

Problem 7.33. Prove, if X is a compact metric space, then every compact (closed) set in X is a G_δ and likewise every open set an F_σ. Prove, a finite Borel measure on a compact metric space is regular.

8. Measurable functions

We will state and prove a few "categorical" properties of measurable functions between general measurable spaces, however in these notes we will mostly be interested in functions from a measurable space taking values in the extended positive axis $[0, +\infty]$, the real line $\mathbb{R}$, or the complex numbers $\mathbb{C}$.

Definition 8.1. Let $(X, \mathscr{M})$ and $(Y, \mathscr{N})$ be measurable spaces. A function $f : X \to Y$ is called *measurable* (or $(\mathscr{M}, \mathscr{N})$ measurable) if $f^{-1}(E) \in \mathscr{M}$ for all $E \in \mathscr{N}$. A function $f : X \to \mathbb{R}$ is *measurable* if it is $(\mathscr{M}, \mathscr{B}_{\mathbb{R}})$ measurable unless indicated otherwise. Likewise, a function $f : X \to \mathbb{C}$ is measurable if it is $(\mathscr{M}, \mathscr{B}_{\mathbb{C}})$ measurable (where $\mathbb{C}$ is identified with $\mathbb{R}^2$ topologically). $\lhd$

It is immediate from the definition that if $(X, \mathscr{M}), (Y, \mathscr{N}), (Z, \mathscr{O})$ are measurable spaces and $f : X \to Y, g : Y \to Z$ are measurable functions, then the composition $g \circ f : X \to Z$ is measurable. The following is a routine application of Proposition 1.9. The proof is left as an exercise.

Proposition 8.2. *Suppose $(X, \mathscr{M})$ and $(Y, \mathscr{N})$ are measurable spaces and the collection of sets $\mathscr{E} \subset 2^Y$ generates $\mathscr{N}$ as a σ-algebra. Then $f : X \to Y$ is measurable if and only if $f^{-1}(E) \in \mathscr{M}$ for all $E \in \mathscr{E}$.* $\dagger$

Proof. Suppose $f^{-1}(E) \in \mathscr{M}$ for all $E \in \mathscr{E}$. Let $\Omega_f = \{E \subset Y : f^{-1}(E) \in \mathscr{M}\}$. Thus Ω_f contains $\mathscr{E}$ by assumption. Moreover, Ω_f is a σ-algebra (the pushforward σ-algebra). Since Ω_f is a σ-algebra containing $\mathscr{E}$, it follows that $\mathscr{N} = \mathscr{M}(\mathscr{E}) \subset \Omega_f$. Hence f is measurable. $\square$

Corollary 8.3. *Let X, Y be topological spaces equipped with their Borel σ-algebras $\mathscr{B}_X, \mathscr{B}_Y$ respectively. Every continuous function $f : X \to Y$ is $(\mathscr{B}_X, \mathscr{B}_Y)$-measurable (or Borel measurable for short). In particular, if $f : X \to \mathbb{F}$ is continuous and X is given its Borel σ-algebra, then f is measurable, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$,* $\dagger$

Proof. Since the open sets $U \subset Y$ generate $\mathscr{B}_Y$ and $f^{-1}(U)$ is open (hence in $\mathscr{B}_X$) by hypothesis, this corollary is an immediate consequence of Proposition 8.2. $\square$

Definition 8.4. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A function $f : \mathbb{R} \to \mathbb{F}$ is called *Lebesgue measurable* (resp. *Borel measurable*) if it is $(\mathscr{L}, \mathscr{B}_{\mathbb{F}})$ (resp. $(\mathscr{B}_{\mathbb{R}}, \mathscr{B}_{\mathbb{F}})$) measurable. Here $\mathscr{L}$ is the Lebesgue σ-algebra. $\lhd$

Remark 8.5. Note that since $\mathscr{B}_{\mathbb{R}} \subset \mathscr{L}$, being Lebesgue measurable is a *weaker* condition than being Borel measurable. If f is *Borel* measurable, then $f \circ g$ is Borel or Lebesgue measurable if g is. However if f is only Lebesgue measurable, then $f \circ g$ need not be Lebesgue measurable, even if g is continuous. (The difficulty is that we have no control over $g^{-1}(E)$ when E is a Lebesgue set.) A counterexample is described in Problem 13.7. $\diamond$

It will sometimes be convenient to consider functions that are allowed to take the values $\pm\infty$.

Definition 8.6. [The extended real line] Let $\overline{\mathbb{R}}$ denote the set of real numbers together with the symbols $\pm\infty$. The arithmetic operations $+$ and $\cdot$ can be (partially) extended to $\overline{\mathbb{R}}$ by declaring

$$\pm\infty + x = x + \pm\infty = \pm\infty$$

for all $x \in \mathbb{R}$,

$$+\infty \cdot x = x \cdot +\infty = +\infty$$

for all nonzero $x \in (0, +\infty)$ (and similar rules for the other choices of signs),

$$0 \cdot \pm\infty = \pm\infty \cdot 0 = 0,$$

The order $<$ is extended to $\overline{\mathbb{R}}$ by declaring

$$-\infty < x < +\infty$$

for all $x \in \mathbb{R}$. $\triangleleft$

The symbol $+\infty + (-\infty)$ is not defined, so some care must be taken in working out the rules of arithmetic in $\overline{\mathbb{R}}$. Typically we will be performing addition only when all values are finite, or when all values are nonnegative (that is for $x \in [0, +\infty]$). In these cases most of the familiar rules of arithmetic hold (for example the commutative, associative, and distributive laws), and the inequality $\leq$ is preserved by multiplying both sides by the same quantity. However cancellation laws are *not* in general valid when infinite quantities are permitted; in particular from $x \cdot +\infty = y \cdot +\infty$ or $x + +\infty = y + +\infty$ one *cannot* conclude that $x = y$.

The order property allows us to extend the concepts of supremum and infimum, by defining the supremum of a set that is unbounded from above, or set containing $+\infty$, to be $+\infty$; similarly for inf and $-\infty$. This also means every sum $\sum_n x_n$ with $x_n \in [0, +\infty]$ can be meaningfully assigned a value in $[0, +\infty]$, namely the supremum of the finite partial sums $\sum_{n \in F} x_n$.

The collection of sets $U \subset \overline{\mathbb{R}}$ such that either U is an open subset of $\mathbb{R}$ or U is the union of an open set in $\mathbb{R}$ with a interval of the form $(a, \infty]$ and/or of the form $[-\infty, b)$ is a topology on $\overline{\mathbb{R}}$ and, of course, we refer to these sets as open (in $\overline{\mathbb{R}}$). Similarly, the collection of open sets in $\mathbb{R}$ together with open sets in $\mathbb{R}$ union an interval of the form $(a, \infty]$ is a topology on $(-\infty, \infty]$.

Definition 8.7. [Extended Borel σ-algebra] The *extended Borel σ-algebra over $\overline{\mathbb{R}}$* is the σ-algebra over generated by open sets of $\overline{\mathbb{R}}$ and is denoted $\mathscr{B}_{\overline{\mathbb{R}}}$. Similarly $\mathscr{B}_{(-\infty,\infty]}$ is the Borel σ-algebra on $(-\infty, \infty]$. $\triangleleft$

Proposition 8.8. *The collection $\mathscr{E} = \{(a, \infty] : a \in \mathbb{R}\}$ generates $\mathscr{B}_{\overline{\mathbb{R}}}$. Similarly each of the collections $\mathscr{E}_j$ from Proposition 1.12 generates $\mathscr{B}_{(-\infty,\infty]}$.* †

Proof sketch. Since $(b, \infty]^c = [-\infty, b]$ (complement in $\overline{\mathbb{R}}$), it follows that $\mathscr{M}(\mathscr{E})$ contains the (finite) intervals of the form $(a, b]$. Hence, from Proposition 1.12, $\mathscr{M}(\mathscr{E})$ contains all open interval in $\mathbb{R}$ and hence all open sets in $\mathbb{R}$. Similarly $\mathscr{E}$ contains the intervals $[-\infty, b)$. The first part of the Proposition now follows.

For the second part of the Proposition, since $(-\infty, a)^c = [a, \infty]$ and since $(a, \infty] = \cup_{n=1}^{\infty}[a - \frac{1}{n}, \infty]$ it follows that σ-algebra of subsets of $(-\infty, \infty]$ generated by the open intervals in $\mathbb{R}$ contains the intervals of the form $(a, \infty]$. It now follows that easily that open the open intervals (in $\mathbb{R}$) generate $\mathscr{B}_{(-\infty,\infty]}$. □

Definition 8.9. [Measurable function] Let $(X, \mathscr{M})$ be a measurable space. A function $f : X \to \overline{\mathbb{R}}$ is called *measurable* if it is $(\mathscr{M}, \mathscr{B}_{\overline{\mathbb{R}}})$ measurable; that is, if $f^{-1}(U) \in \mathscr{M}$ for every open set $U \subset \overline{\mathbb{R}}$. The notion of measurable functions $f : X \to (-\infty, \infty]$ is defined similarly. ◁

In particular, the following criteria for measurability will be used repeatedly.

Corollary 8.10 (Equivalent criteria for measurability)**.** *Let $(X, \mathscr{M})$ be a measurable space.*

(a) A function $f : X \to \overline{\mathbb{R}}$ is measurable if and only if the sets

$$f^{-1}((t, +\infty]) = \{x : f(x) > t\}$$

are measurable for all $t \in \mathbb{R}$; and

(b) A function $f : X \to \mathbb{R}$ or $f : X \to (-\infty, \infty]$ is measurable if and only if $f^{-1}(E) \in \mathscr{M}$ for all $E \in \mathscr{E}$, where $\mathscr{E}$ is any of the collections of sets $\mathscr{E}_j$ appearing in Proposition 1.12.

(c) A function $f : X \to \mathbb{C}$ is measurable if and only if $f^{-1}((a, b) \times (c, d))$ is measurable for every $a, b, c, d \in \mathbb{R}$. (Here $(a, b) \times (c, d)$ is identified with the open box $\{z \in \mathbb{C} : a < Re(z) < b,\ c < Im(z) < d\}$.)

†

In item (a), $t \in \mathbb{R}$ can be replaced by $t \in \mathbb{Q}$.

Proof sketch. Combine Propositions 8.8 and 8.2. □

Example 8.11. [Examples of measurable functions]

(a) An indicator function $\mathbf{1}_E$ is measurable if and only if E is measurable. Indeed, the set $\{x : \mathbf{1}_E(x) > t\}$ is either empty, E, or all of X, in the cases $t \geq 1$, $0 \leq t < 1$, or $t < 0$, respectively.

(b) The next series of propositions will show that measurability is preserved by most of the familiar operations of analysis, including sums, products, sups, infs, and limits (provided one is careful about arithmetic of infinities).

(c) Corollary 8.21 below will show that examples (a) and (b) above in fact generate all the examples in the case of $\overline{\mathbb{R}}$ or $\mathbb{C}$ valued functions. That is, every measurable function is a pointwise limit of linear combinations of measurable indicator functions.

△

Proposition 8.12. *Let $(X, \mathscr{M})$ be a measurable space. A function $f : X \to \mathbb{C}$ is measurable if and only if Ref and Imf are measurable.* †

Proof. As a topological space, $\mathbb{C}^2$ is $\mathbb{R}^2$ and the Borel σ-algebra of $\mathbb{R}^2$ is generated by open rectangles $(a,b)\times(c,d)$. Suppose $f : X \to \mathbb{C}$ is measurable. The real part u of f is measurable since it is the composition $u = \pi_1 \circ f$, of the continuous (hence Borel measurable) projection π_1 of $\mathbb{R}^2$ onto the first coordinate with the measurable function f. Likewise the imaginary part v of f is measuarable.

Conversely, suppose u, v are measurable. Fix an open rectangle $R = (a,b)\times(c,d)$ and note that

$$f^{-1}(R) = u^{-1}((a,b)) \cap v^{-1}((c,d)),$$

which lies in $\mathscr{M}$ by hypothesis. So f is measurable by Corollary 8.10. ☐

Proposition 8.13. *Suppose $(X, \mathscr{M})$ is a measurable space and let $\mathbb{F}$ denote any of $\overline{\mathbb{R}}$, $(-\infty, \infty]$ and $\mathbb{R}$. If $f : X \to \mathbb{F}$ is measurable, then so is $-f$.* †

Proof. In any case it suffices to prove $E_t = \{-f > t\}$ is measurable for each $t \in \mathbb{R}$. We have $E_t = \{f \le -t\} = \{f > -t\}^c$ is measurable. ☐

Proposition 8.14. *Let (f_n) be a sequence of $\overline{\mathbb{R}}$-valued measurable functions.*

(a) The functions

$$\sup f_n, \quad \inf f_n, \quad \limsup_{n\to\infty} f_n, \quad \liminf_{n\to\infty} f_n$$

are measurable;

(b) The set on which (f_n) converges is a measurable set; and

(c) If (f_n) converges to f pointwise, then f is measurable.

†

Proof. Let $f(x) = \sup_n f_n(x)$. Given $t \in \mathbb{R}$, we have $f(x) > t$ if and only if $f_n(x) > t$ for some n. Thus

$$\{x : f(x) > t\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > t\}.$$

It follows that f is measurable. Likewise $\inf f_n$ is measurable, since $\inf f_n = -\sup(-f_n)$ and two applications of Proposition 8.13. Consequently, $g_N = \sup_{n\ge N} f_n$ is measurable for each positive integer N and hence $\limsup f_n = \inf g_N$ is also measurable.

If (f_n) converges pointwise to f, then $f = \limsup f_n = \liminf f_n$ is measurable. Part (b) is left as an exercise. ☐

In the Proposition 8.14 it is of course essential that the supremum is taken only over a *countable* set of measurable functions; the supremum of an uncountable collection of measurable functions need not be measurable. Problem 13.6 asks for a counterexample.

Theorem 8.15. *Let $(X, \mathscr{M})$ be a measurable space.*

(a) If $f, g : X \to \mathbb{C}$ are measurable functions, and $c \in \mathbb{C}$. Then cf, $f+g$, and fg are measurable.

(b) If $f, g : X \to [-\infty, \infty]$ are measurable and, for each x, $\{f(x), g(x)\} \ne \{\pm\infty\}$, then $f+g$ is measurable.

(c) If $f, g : X \to [-\infty, \infty]$ are measurable, then so is fg.

Proof. To prove (b), supppose $f, g : X \to [-\infty, \infty]$ are measurable and $f + g$ is defined. Using Corollary 8.10 (a), it suffices to prove, for a $t \in \mathbb{R}$, that

$$\{x \in X : f(x) + g(x) > t\} = \bigcup_{q \in \mathbb{Q}} \{x : f(x) > q\} \cap \{x : g(x) > t - q\},$$

since all the sets in the last line are measurable, the intersection is finite and the union countable. The inclusion of the set on the right into the set on the left is evident. Suppose $f(x) + g(x) > t$. In particular, $g(x) \neq -\infty$ and thus $f(x) > t - g(x)$. There is a rational $q \in \mathbb{Q}$ such that $f(x) > q > t - g(x)$ and the reverse inclusion follows.

Assuming $f, g : X \to [0, \infty]$ are measurable, a proof that fg is measurable can be modeled after the proof for $f+g$. The details are left as an exercise (Problem 13.8). From Proposition 8.14, if $f : X \to [-\infty, \infty]$ is measurable, then so are $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = -\min\{f(x), 0\}$. Of course $f = f^+ - f^-$. Now suppose $f, g : X \to \mathbb{R}$. Let $F = f^+g^+ + f^-g^-$ and $G = -f^-g^+ - f^+g^-$ and note that $fg = F + G$. Since F and G are measurable and $f^\pm, g^\pm$ take values in $[0, \infty)$ and are measurable all the products $f^\pm g^\pm$ are measurable. Hence, using (b) several times and Proposition 8.13, $F + G$, and thus fg is measurable. Finally, suppose now that $f, g : X \to \overline{\mathbb{R}}$. Let $\Omega_{\pm\infty} = (fg)^{-1}(\pm\infty)$ and $\Omega = (fg)^{-1}(\mathbb{R})$. In particular,

$$\Omega_\infty = (\{f = \infty\} \cap \{g > 0\}) \cup (\{f = -\infty\} \cup \{g < 0\}) \cup (\{g = \infty\} \cap \{f > 0\}) \cup (\{g = -\infty\} \cap \{f < 0\}) \in \mathscr{M}.$$

Likewise $\Omega_{-\infty}$ is measurable and therefore Ω is measurable too. Let $\Omega_f = f^{-1}(\mathbb{R})$ and $\Omega_g = g^{-1}(\mathbb{R})$. Both are measurable. Let $\hat{f} = f\mathbf{1}_{\Omega_f}$ and $\hat{g} = g\mathbf{1}_{\Omega_g}$. It is easily checked that both are measurable. Given $x \in \Omega$ either $f(x), g(x) \in \mathbb{R}$ or $f(x) = \pm\infty$ and $g(x) = 0$ or $g(x) = \pm\infty$ and $f(x) = 0$. In each case it is readily verified that $f(x)g(x) = \hat{f}(x)\hat{g}(x)$. Hence,

$$fg\mathbf{1}_\Omega = \hat{f}\,\hat{g}.$$

Since $\hat{f}$ and $\hat{g}$ are measurable and real-valued their product is measurable and thus $fg\mathbf{1}_\Omega$ is measurable. Finally, since

$$fg = \infty\mathbf{1}_{\Omega_\infty} - \infty\mathbf{1}_{\Omega_{-\infty}} + (fg)\mathbf{1}_\Omega,$$

an application of item (b) completes the proof.

The proof of (a) is straightforward using parts (b) and (c) and Proposition 8.12.

□

Given a measure space $(X, \mathscr{M}, \mu)$ we say that a set $E \in \mathscr{M}$ has *full measure* if $\mu(E^c) = 0$. A property $P = P(x)$ is said to hold *almost everywhere* with respect to μ, abbreviated *a.e.* μ, or just *a.e.* when the measure μ is understood from context, if there exists a set of points E of full measure such that $P(x)$ holds for all $x \in E$. In the case the measure space is complete, a property holds a.e. if and only if the set where it doesn't hold is a set of measure zero.

Proposition 8.16. *Suppose $(X, \mathscr{M}, \mu)$ is a complete measure space and $(Y, \mathscr{N})$ is a measurable space.*

(a) Suppose $f, g : X \to Y$. If f is measurable and $g = f$ a.e., then g is measurable.
(b) If $f_n : X \to \overline{\mathbb{R}}$ are measurable functions and $f_n \to f$ a.e., then f is measurable. The same conclusion holds if $\overline{\mathbb{R}}$ is replaced by $\mathbb{C}$.

†

Proposition 8.17. *Let $(X, \mathscr{M}, \mu)$ be a measure space and $(X, \overline{\mathscr{M}}, \overline{\mu})$ its completion. Let $\mathbb{F}$ denote either $\mathbb{R}$, $\overline{\mathbb{R}}$ or $\mathbb{C}$.*

(i) If $f : X \to \mathbb{F}$ is a $\overline{\mathscr{M}}$-measurable function, then there is an $\mathscr{M}$-measurable function g such that $\overline{\mu}(\{x : f(x) \neq g(x)\}) = 0$.
(ii) If (f_n) is a sequence of $\overline{\mathscr{M}}$ measurable functions, $f_n : X \to \overline{\mathbb{F}}$, that converges a.e. $\overline{\mu}$ to a function f, then there is a $\mathscr{M}$ measurable function g such that (f_n) converges a.e. $\overline{\mu}$ to g.

†

The proofs of Propositions 8.16 and 8.17 are left to the reader as Problem 13.9.

Definition 8.18. [Unsigned simple function] Recall, a function f on a set X is *unsigned* if its codomain is a subset of $[0, \infty]$. An unsigned function $s : X \to [0, +\infty]$ is called *simple* if its range is a finite set. $\triangleleft$

Many statements about general measurable functions can be reduced to the unsigned case. For instance, one simple but important application of Proposition 8.14 is that if f, g are $\overline{\mathbb{R}}$-valued measurable functions, then $f \wedge g := \min(f, g)$ and $f \vee g := max(f, g)$ are measurable; in particular $f^+ := \max(f, 0)$ and $f^- := -\min(f, 0)$ are measurable if f is. It also follows that $|f| := f^+ + f^-$ is measurable when f is. Together with Proposition 8.12, these observations show every $\mathbb{C}$ valued measurable function f is a linear combination of four unsigned measurable functions (the positive and negative parts of the real and imaginary parts of f).

A *partition* P of the set X is, for some $n \in \mathbb{N}$, a set $P = \{E_0, \dots, E_n\}$ of pairwise disjoint subsets of X whose union is X. If each E_j is measurable, then P is a *measurable partition.*

Proposition 8.19. *Suppose s is an unsigned function on X. The following are equivalent.*

(i) s is a (measurable) simple function;
(ii) there exists an n, scalars $c_1, \dots, c_n \in [0, \infty]$ and (measurable) subsets $F_j \subset X$ such that
$$s = \sum_{j=1}^{n} c_j \mathbf{1}_{F_j};$$
(iii) there exists a (measurable) partition $P = \{E_1, \dots, E_m\}$, and $c_1, \dots, c_m$ in $[0, \infty]$ such that
$$s = \sum_{k=1}^{m} c_k \mathbf{1}_{E_k}.$$

†

The proof of this proposition is an easy exercise. Letting $\{c_1, c_2, \dots, c_m\}$ denote the range of s,

$$s = \sum_{j=1}^{n} c_j E_j,$$

where $E_j = s^{-1}(\{c_j\})$. Evidently $\{E_1, \dots, E_n\}$ is a partition of X that is measurable if s is measurable. This is the *standard representation* of s.

Theorem 8.20 (The Ziggurat approximation)**.** *Let $(X, \mathscr{M})$ be a measurable space. If $f : X \to [0, +\infty]$ is an unsigned measurable function, then there exists a increasing sequence of unsigned, measurable simple functions $s_n : X \to [0, +\infty)$ such that $s_n \to f$ pointwise increasing on X. If f is bounded, the sequence can be chosen to converge uniformly.*

Proof. For positive integers n and integers $0 \le k < n2^n$, let $E_{n,k} = \{x : \frac{k}{2^n} < f(x) \le \frac{k+1}{2^n}\}$, let $E_{n,n2^n} = \{x : n < f(x)\}$ and define

$$s_n(x) = \sum_{k=0}^{n2^n} \frac{k}{2^n} \mathbf{1}_{E_{n,k}}. \tag{25}$$

Verify that (s_n) is pointwise increasing with limit f and if f is bounded, then the convergence is uniform. $\square$

By the remarks following Proposition 8.14, the following corollary is immediate (since its proof reduces to the unsigned case):

Corollary 8.21. *Every $\overline{\mathbb{R}}$- or $\mathbb{C}$-valued measurable function is a pointwise limit of measurable simple functions.* †

9. Integration of simple functions

We will build up the integration theory for measurable functions in three stages. We first define the integral for unsigned simple functions, then extend it to general unsigned functions, and finally to general ($\overline{\mathbb{R}}$ or $\mathbb{C}$-valued) functions. Throughout this section and the next, we fix a measure space $(X, \mathscr{M}, \mu)$; all functions are defined on this measure space.

Suppose $P = \{E_0, \dots, E_n\}$ is a measurable partition of X, $c_0, c_1, \dots, c_n \ge 0$ and

$$s = \sum_{j=0}^{n} c_j \mathbf{1}_{E_j}. \tag{26}$$

If $Q = \{F_0, \dots, F_m\}$ is another measurable partition, $d_0, d_1, \dots, d_n \ge 0$ and

$$s = \sum_{k=0}^{m} d_k \mathbf{1}_{F_k},$$

then it is an exercise (see Problem 13.10) to show

$$\sum_{j=0}^{n} c_n\mu(E_n) = \sum_{k=0}^{m} d_m\mu(F_m).$$

Indeed, for this exercise it is helpful to consider the *common refinement* $\{E_j \cap F_k : 1 \leq j \leq n,\ 1 \leq k \leq m\}$ of the partitions P and Q. It is now possible to make the following definition.

By convention, when writing a simple measurable function s as $s = \sum_{n=0}^{N} c_n \mathbf{1}_{E_n}$ the sets E_n are assumed measurable.

Definition 9.1. Let $(X, \mathscr{M}, \mu)$ be a measure space and $f = \sum_{n=0}^{N} c_n \mathbf{1}_{E_n}$ an unsigned measurable simple function. The *integral of f* (with respect to the measure μ) is defined to be

$$\int_X f\, d\mu := \sum_{n=0}^{N} c_n \mu(E_n).$$

$\triangleleft$

One thinks of the graph of the function $c\mathbf{1}_E$ as “rectangle” with height c and “base” E; since μ tells us how to measure the length of E the quantity $c \cdot \mu(E)$ is interpreted as the “area” of the rectangle. This intuition can be made more precise once we have proved Fubini’s theorem. Note too that the definition explains the convention $0 \cdot \infty = 0$, since the set on which s is 0 should not contribute to the integral.

Let L^+ denote the set of all unsigned measurable functions on $(X, \mathscr{M})$. We begin by collecting some basic properties of the integrals of simple functions. When X and μ are understood we drop them from the notation and simply write $\int f$ for $\int_X f\, d\mu$.

Theorem 9.2 (Basic properties of simple integrals)**.** *Let $(X, \mathscr{M}, \mu)$ be a measure space and let $f, g \in L^+$ be simple functions.*

(a) (Homogeneity) If $c \geq 0$, then $\int cf = c \int f$.
(b) (Monotonicity) If $f \leq g$, then $\int f \leq \int g$.
(c) (Finite additivity) $\int f + g = \int f + \int g$.
(d) (Almost everywhere equivalence) If $f(x) = g(x)$ for μ-almost every $x \in X$, then $\int f = \int g$.
(e) (Finiteness) $\int f < +\infty$ if and only if is finite almost everywhere and supported on a set of finite measure.
(f) (Vanishing) $\int f = 0$ if and only if $f = 0$ almost everywhere.

Proof. (a) is trivial; we prove (b) and (c) and leave the rest as (simple!) exercises.

To prove (b), write $f = \sum_{j=0}^{n} c_j \mathbf{1}_{E_j}$ and $g = \sum_{k=0}^{m} d_k \mathbf{1}_{F_k}$ for measurable partitions $P = \{E_0, \dots, E_n\}$ and $Q = \{F_0, \dots, F_m\}$ of X. It follows that $R = \{E_j \cap F_k : 0 \leq j \leq n,\ 0 \leq k \leq m\}$ is a measurable partition of X too and

$$f = \sum_{j,k} c_j \mathbf{1}_{E_j \cap F_k}$$

and similarly for g. From the assumption $f \leq g$ we deduce that $c_j \leq d_k$ whenever $E_j \cap F_k \neq \varnothing$. In particular, either $c_j \leq d_k$ or $\mu(E_j \cap F_k) = 0$. Thus,

$$\int f = \sum_{j,k} c_j \mu(E_j \cap F_k) \leq \sum_{j,k} d_k \mu(E_j \cap F_k) = \int g.$$

For item (c), since $E_j = \bigcup_{k=0}^m E_j \cap F_k$ for each j and $F_k = \bigcup_{j=0}^n F_k \cap E_j$ for each k, it follows from the finite additivity of μ that

$$\int f + \int g = \sum_{j,k} (c_j + d_k) \mu(E_j \cap F_k).$$

Since $f+g = \sum_{j,k}(c_j+d_k)\mathbf{1}_{E_j \cap F_k}$, and $\{E_j \cap F_k : 1 \leq j \leq n,\ 1 \leq k \leq m\}$ is a measurable partition, the right hand side is $\int (f+g)$. $\square$

If $f : X \to [0, +\infty]$ is a measurable simple function, then so is $\mathbf{1}_E f$ for any measurable set E. We write $\int_E f\, d\mu := \int \mathbf{1}_E f\, d\mu$.

Proposition 9.3. *Let $(X, \mathscr{M}, \mu)$ be a measure space. If f is an unsigned measurable simple function, then the function $\nu : \mathscr{M} \to [0, \infty]$ defined by*

$$\nu(E) := \int_E f\, d\mu$$

is a measure on $(X, \mathscr{M})$. †

Sketch of proof. Write f as $\sum_{j=1}^m c_j \mathbf{1}_{F_j}$ with respect to a measurable partition $\{F_1, \dots, F_m\}$ and observe, for $E \in \mathscr{M}$,

$$\nu(E) = \int \mathbf{1}_E f\, d\mu = \sum_{j=0}^m c_j \mu(F_j \cap E).$$

For a fixed measurable set F, the mapping $\nu_F : \mathscr{M} \to [0, \infty]$ defined by $\tau_F(E) = \mu(E \cap F)$ is a measure by Example 2.2 item c. Given $n \in \mathbb{N}$, numbers $c_1, \dots, c_m \geq 0$ and measurable sets $F_1, \dots, F_m$, the mapping $\tau : \mathscr{M} \to [0, \infty]$ defined by

$$\tau(E) = \sum c_j \tau_{F_j}(E) = \sum c_j \mu(E \cap F_j)$$

is a measure by Example 2.2 item d. The resutl follows. $\square$

10. Integration of unsigned functions

We now extend the definition of the integral to all (not necessarily simple) functions in L^+. First note that if $(X, \mathscr{M}, \mu)$ is a measure space and s is a measurable unsigned simple function, then, by Theorem 9.2(b),

$$\int_X s\, d\mu = \sup\{\int_X t\, d\mu : 0 \leq s \leq t,\ t \text{ is a measurable unsigned simple function}\}.$$

Hence, the following definition is consistent with that of the integral for unsigned simple functions.

Definition 10.1. Let $(X, \mathscr{M}, \mu)$ be a measure space. For an unsigned measurable function $f : X \to [0, +\infty]$, define the *integral of f with respect to μ* by

$$\int_X f\,d\mu := \sup\{\int_X s\,d\mu : 0 \leq s \leq f; s \text{ simple and measurable}\} \tag{27}$$

Often we write $\int f$ instead of $\int_X f\,d\mu$ when μ and X are understood.

For $E \in \mathscr{M}$ let

$$\int_E f\,d\mu = \int_X f\mathbf{1}_E\,d\mu.$$

◁

Theorem 10.2 (Basic properties of unsigned integrals)**.** *Let $(X, \mathscr{M}, \mu)$ be a measure space and let f, g be unsigned measurable functions on X.*

(a) (Homogeneity) If $c \geq 0$ then $\int cf = c\int f$.
(b) (Monotonicity) If $f \leq g$ then $\int f \leq \int g$.
(c) (Almost everywhere equivalence) If $f(x) = g(x)$ for μ-almost every $x \in X$, then $\int f = \int g$.
(d) (Finiteness) If $\int f < +\infty$, then $f(x) < +\infty$ for μ-a.e. x.
(e) (Vanishing) $\int f = 0$ if and only if $f = 0$ almost everywhere.
(f) (Bounded) If f is bounded measurable function and $\mu(X) < \infty$, then $\int f\,d\mu < \infty$.

The integral is also additive; however the proof is surprisingly subtle and will have to wait until we have established the Monotone Convergence Theorem.

Proof of Theorem 10.2. As in the simple case homogeneity is trivial. Monotonicity is also evident, since any simple function less than f is also less than g.

Turning to item (c) let E be a measurable set with $\mu(E^c) = 0$. If s is a simple function, then $\mathbf{1}_E s$ and s are simple functions that agree almost everywhere. Thus $\int \mathbf{1}_E s = \int s$, by Theorem 9.2(d). Further, if $0 \leq s \leq f$, then $\mathbf{1}_E s \leq \mathbf{1}_E f$. Hence, using monotonicity (item (b)) and taking suprema over simple functions,

$$\int \mathbf{1}_E f \leq \int f = \sup_{0 \leq s \leq f} \int s = \sup_{0 \leq s \leq f} \int \mathbf{1}_E s \leq \sup_{0 \leq t \leq \mathbf{1}_E f} \int t = \int \mathbf{1}_E f.$$

Now suppose $f = g$ a.e. Thus, letting $E = \{f = g\}$, the set E^c has measure zero and $\mathbf{1}_E f = \mathbf{1}_E g$. Hence, from what has already been proved (twice),

$$\int f\,d\mu = \int \mathbf{1}_E f\,d\mu = \int \mathbf{1}_E g\,d\mu = \int g\,d\mu.$$

To prove item (d) observe if $f = +\infty$ on a measurable set E and $\mu(E) > 0$, then $\int f \geq \int n\mathbf{1}_E = n\mu(E)$ for all n, so $\int f = +\infty$. (A direct proof can be obtained from Markov's inequality below.)

If $f = 0$ a.e. and $0 \leq s \leq f$ is a simple function, then $s = 0$ a.e. and hence, by Theorem 9.2 item (f) $\int s = 0$. Hence $\int f = 0$. Conversely, suppose there is a set E of positive measure such that $f(x) > 0$ for all $x \in E$. Let $E_n = \{x \in E : f(x) > \frac{1}{n}\}$.

Then $E = \bigcup_{n=1}^{\infty} E_n$, so by the pigeonhole principle $\mu(E_N) > 0$ for some N. But then $0 \le \frac{1}{N}\mathbf{1}_{E_N} \le f$, so $\int f \ge \frac{1}{N}\mu(E_N) > 0$ and item (e) is proved.

Finally, for item (f), by hypothesis there is a positive real number C so that $0 \le f(x) \le C$ for $x \in X$. With g denoting the simple function $C\mathbf{1}_X$, we have $0 \le f \le g$. Hence item (f) follows from monotonicity (item (b)). □

Theorem 10.3 (Monotone Convergence Theorem)**.** *Let $(X, \mathscr{M}, \mu)$ be a measure space and suppose (f_n) is a sequence of unsigned measurable functions $f_n : X \to [0, \infty]$. If (f_n) increases to f pointwise, then $\int f_n \to \int f$, where f is the pointwise limit of (f_n).*

Proof. Since (f_n) converges pointwise to f and each f_n is measurable, f is measurable by Proposition 8.14 item (c). By monotonicity of the integral, Theorem 10.2, the sequence $(\int f_n)$ is increasing and $\int f_n \le \int f$ for all n. Thus the sequence $(\int f_n)$ converges (perhaps to ∞) and $\lim \int f_n \le \int f$. For the reverse inequality, fix a measurable simple function with $0 \le s \le f$. Let $\epsilon > 0$ be given. Consider the sets

$$E_n = \{x : f_n(x) \ge (1-\epsilon)s(x)\}.$$

Since (f_n) is pointwise increasing, (E_n) is an increasing sequence of measurable sets whose union is X. For all n,

$$\int f_n \ge \int_{E_n} f_n \ge (1-\epsilon)\int_{E_n} s.$$

By Monotone convergence for sets (Theorem 2.3(c)) applied to the measure $\nu(E) = \int_E s$ (Proposition 9.3), we see that

$$\lim \int_{E_n} s = \int_X s.$$

Thus $\lim \int f_n \ge (1-\epsilon)\int s$. Since $1 > \epsilon > 0$ is arbitrary, $\lim \int f_n \ge \int s$. Since $0 \le s \le f$ was an arbitrary simple function, $\lim \int f_n$ is an upper bound for the set whose supremum is, by definition, $\int f$. Thus $\lim \int f_n \ge \int f$. □

Before going on we mention two frequently used applications of the Monotone Convergence Theorem:

Corollary 10.4. *(i) (Vertical truncation) If f is an unsigned measurable function, then the sequence $(\int \min(f, n))$ converges to $\int f$.*
(ii) (Horizontal truncation) If f is an unsigned measurable function and $(E_n)_{n=1}^{\infty}$ is an increasing sequence of measurable sets whose union is X, then $\int_{E_n} f \to \int f$.

†

Proof. Since $\min(f, n)$ and $\mathbf{1}_{E_n} f$ are measurable for all n and increase pointwise to f, these follow from the Monotone Convergence Theorem. □

Theorem 10.5 (Additivity of the unisgned integral)**.** *If f, g are unsigned measurable functions, then $\int f + g = \int f + \int g$.*

Proof. By Theorem 8.20, there exist sequences of unsigned, measurable simple functions f_n, g_n that increase pointwise to f, g respectively. Thus $f_n + g_n$ increases to $f + g$, so by Theorem 9.2(c) and the Monotone Convergence Theorem,

$$\int f+g = \lim \int [f_n + g_n] = \lim \left[\int f_n + \int g_n \right] = \int f + \int g.$$

□

Corollary 10.6 (Tonelli's theorem for sums and integrals)**.** *If (f_n) is a sequence of unsigned measurable functions, then $\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$.* †

Proof. Let $g_N = \sum_{n=1}^{N} f_n$. Thus (g_N) is an increasing sequence with pointwise limit $g = \sum_{n=1}^{\infty} f_n$. In particular, g is measurable and by the Monotone convergence theorem $(\int g_N)$ converges to $\int g$. By induction on Theorem 10.5,

$$\int g_N = \sum_{n=1}^{N} \int f_n$$

and the result follows by taking the limit on N. □

Pointwise convergence is not sufficient to imply convergence of the integrals (see Examples 10.8 below), however the following weaker result holds.

Theorem 10.7 (Fatou's Theorem)**.** *If f_n is a sequence of unsigned measurable functions, then*

$$\int \liminf f_n \le \liminf \int f_n.$$

Proof. For $n \in \mathbb{N}$, the function $g_n(x) := \inf_{m \ge n} f_m(x)$ is unsigned, $g_n \le f_n$ pointwise and, by Proposition 8.14, measurable. By definition of the lim inf, the sequence (g_n) increases pointwise to $\liminf f_n$. By the Monotone Convergence Theorem and monotonicity

$$\int \liminf f_n = \int \lim g_n = \lim \int g_n = \liminf \int g_n \le \liminf \int f_n.$$

□

Example 10.8. [Failure of convergence of integrals] This example highlights three modes of failure of the convergence $(\int f_n)$ to $\int f$ for sequences of unsigned measurable functions $f_n : \mathbb{R} \to [0, +\infty]$ and Lebesgue measure. In each case (f_n) converges to the zero function pointwise, but $\int f_n = 1$ for all n:

(1) (Escape to height infinity) $f_n = n\mathbf{1}_{(0,\frac{1}{n})}$
(2) (Escape to width infinity) $f_n = \frac{1}{2n}\mathbf{1}_{(-n,n)}$
(3) (Escape to support infinity) $f_n = \mathbf{1}_{(n,n+1)}$

Note that in the second example the convergence is even uniform. These examples can be though of as *moving bump* functions. In each case we have a rectangle and can vary the height, width, and position. If we think of f_n as describing a density of mass distributed over the real line, then $\int f_n$ gives the total "mass"; Fatou's theorem says mass cannot be created in the limit, but these examples show mass can be destroyed. △

Proposition 10.9 (Markov's inequality)**.** *If f is an unsigned measurable function, then for all $t > 0$*

$$\mu(\{x : f(x) > t\}) \leq \frac{1}{t} \int f$$

†

Proof. Let $E_t = \{x : f(x) > t\}$. Then by definition, $t\mathbf{1}_{E_t} \leq f$, so $t\mu(E_t) = \int t\mathbf{1}_{E_t} \leq \int f$. □

We conclude this section with a few frequently-used corollaries of the monotone convergence theorem, and a converse to it.

Theorem 10.10 (Change of variables)**.** *Let $(X, \mathscr{M}, \mu)$ be a measure space, $(Y, \mathscr{N})$ a measurable space, and $\phi : X \to Y$ a measurable function. The function $\phi_*\mu : \mathscr{N} \to [0, +\infty]$ defined by*

$$\phi_*\mu(E) = \mu(\phi^{-1}(E)). \tag{28}$$

is a measure on $(Y, \mathscr{N})$, and for every unsigned measurable function $f : Y \to [0, +\infty]$,

$$\int_Y f\, d(\phi_*\mu) = \int_X (f \circ \phi)\, d\mu. \tag{29}$$

Proof. Problem 13.16. The measure $\phi_*\mu$ is called the *push-forward* of μ under ϕ. □

Lemma 10.11 (Borel-Cantelli Lemma)**.** *Let $(X, \mathscr{M}, \mu)$ be a measure space and suppose $(E_n)_{n=1}^\infty$ is a sequence of measurable sets. If*

$$\sum_{n=1}^\infty \mu(E_n) < \infty,$$

then for almost every $x \in X$ is contained in at most finitely many of the E_n (that is, letting $N_x := \{n : x \in E_n\} \subset \mathbb{N}$, the set $\{x : |N_x| = \infty\}$ has measure 0). †

Sketch of proof. Consider the series $S = \sum_{n=1}^\infty \mathbf{1}_{E_n}$. By Tonelli (Corollary 10.6), $\int S$ is finite. Hence S is finite a.e. by Theorem 10.2(d). On the other hand, $\{x : |N_x| = \infty\} = S^{-1}(\{\infty\}$. □

There is a sense in which the monotone convergence theorem has a converse, namely that any map from unsigned measurable functions on a measurable space $(X, \mathscr{M})$ to $[0, +\infty]$, satisfying some reasonable axioms (including MCT) must come from integration against a measure. The precise statement is the following:

Theorem 10.12. *Let $(X, \mathscr{M})$ be a measurable space and let $U(X, \mathscr{M})$ denote the set of all unsigned measurable functions $f : X \to [0, +\infty]$. Suppose $L : U(X, \mathscr{M}) \to [0, +\infty]$ is a function obeying the following axioms:*

(a) (Homogeneity) For every $f \in U(X, \mathscr{M})$ and every real number $c \geq 0$, $L(cf) = cL(f)$.

(b) (Additivity) For every pair $f, g \in U(X, \mathscr{M})$, $L(f+g) = L(f) + L(g)$.
(c) (Monotone convergence) If f_n is a sequence in $U(X, \mathscr{M})$ increasing pointwise to f, then $\lim_{n\to\infty} L(f_n) = L(f)$.

Then there is a unique measure $\mu : \mathscr{M} \to [0, +\infty]$ such that $L(f) = \int_X f\, d\mu$ for all $f \in U(X, \mathscr{M})$. In fact, $\mu(E) = L(\mathbf{1}_E)$.

Proof. Problem 13.17. □

11. Integration of signed and complex functions

Again we work on a fixed measure space $(X, \mathscr{M}, \mu)$. Suppose $f : X \to \overline{\mathbb{R}}$ is measurable. Split f into its positive and negative parts $f = f^+ - f^-$. If at least one of $\int f^+, \int f^-$ is finite f is *semi-integrable* and the *integral of* f is defined as

$$\int f = \int f^+ - \int f^-.$$

If both are finite, we say f is *integrable* (or sometimes *absolutely integrable*). Note that f is integrable if and only if $\int |f| < +\infty$; this is immediate since $|f| = f^+ + f^-$ and the integral is additive on unsigned functions. We write

$$\|f\|_1 := \int_X |f|\, d\mu$$

when f is integrable. In the complex case, a measurable $f : X \to \mathbb{C}$ is *integrable* (or *absolutely integrable*) if $|f|$ is integrable. From the inequalities

$$\max(|\mathrm{Re} f|, |\mathrm{Im} f|) \le |f| \le |\mathrm{Re} f| + |\mathrm{Im} f|$$

it is clear that $f : X \to \mathbb{C}$ is (absolutely) integrable if and only if $\mathrm{Re} f$ and $\mathrm{Im} f$ are. If f is complex-valued and absolutely integrable (that is, f is measurable and $|f|$ is integrable), we define

$$\int f = \int \mathrm{Re} f + i \int \mathrm{Im} f.$$

We also write $\|f\|_1 := \int_X |f|\, d\mu$ in the complex case.

If $f : X \to \overline{\mathbb{R}}$ is absolutely integrable, then necessarily the set $\{x : |f(x)| = +\infty\}$ has measure 0 by Theorem 10.2(d). We may therefore redefine f to be 0, say, on this set, without affecting the integral of f (by Theorem 10.2(c)). Thus when working with absolutely integrable functions, we often can (and often will) always assume that f is finite-valued everywhere.

11.1. Basic properties of the absolutely convergent integral.

The next few propositions collect some basic properties of the absolutely convergent integral. Let $L^1(\mu)$ denote the set of all absolutely integrable $\mathbb{C}$-valued functions on X. If the measure space is understood, as it is in this section, we just write L^1.)

Theorem 11.1 (Basic properties of L^1 functions). *Let $f, g \in L^1$ and $c \in \mathbb{C}$. Then:*

(a) L^1 is a vector space over $\mathbb{C}$;

(b) the mapping $\Lambda : L^1 \to \mathbb{C}$ *defined by* $\Lambda(f) = \int f$ *is linear;*
(c) $\left|\int f\right| \le \int |f|$.
(d) $\|cf\|_1 = |c|\|f\|_1$.
(e) $\|f+g\|_1 \le \|f\|_1 + \|g\|_1$.

Proof. To prove L^1 is a vector space, suppose $f, g \in L^1$ and $c \in \mathbb{C}$. Since f, g are measurable, so is $f+g$, thus $|f+g|$ has an integral. Moreover, since $|f+g| \le |f| + |g|$, monotonicity and additivity of the unsigned integral (Theorems 10.2 and 10.5) gives

$$\|f+g\|_1 = \int |f+g| \le \int |f| + \int |g| \le \|f\|_1 + \|g\|_1,$$

proving item (e) and that L^1 is closed under addition. Next, $\int |cf| = |c| \int |f| = |c|\, \|f\|_1$ (using homogeneity of the unsigned integral in Theorem 10.2). Thus $cf \in L^1$ and item (d) holds. Further L^1 is a vector space.

To prove that Λ is linear, first assume f and g are real-valued and $c \in \mathbb{R}$. Checking $c\int f = \int cf$ is straightforward. For additivity, let $h = f+g$ and observe

$$h^+ - h^- = f^+ + g^+ - f^- - g^-.$$

Therefore

$$h^+ + f^- + g^- = h^- + f^+ + g^+.$$

Thus,

$$\int h^+ + f^- + g^- = \int h^- + f^+ + g^+$$

and rearranging, using additivity of the unsigned integral and finiteness of all the integrals involved, gives $\int h = \int f + \int g$. The complex case now follows essentially by definition. Hence Λ is linear proving item (b).

If f is real, then, using additivity of the unsigned integral,

$$\left|\int f\right| = \left|\int f^+ - \int f^-\right| \le \int f^+ + \int f^- = \int (f^+ + f^-) = \int |f|.$$

Hence (c) holds for real-valued functions. When f is complex, assume $\int f \neq 0$ and let $t = \overline{\operatorname{sgn} \int f}$. Then $|t| = 1$ and $|\int f| = t \int f$. It follows that, using part (c) for the real-valued function $\operatorname{Re} tf$,

$$\left|\int f\right| = t\int f = \int tf = \operatorname{Re}\int tf = \int \operatorname{Re} tf \le \int |\operatorname{Re} tf| \le \int |tf| = \int |f|.$$

$\square$

Because of cancellation, it is clear that $\int f = 0$ does not imply $f = 0$ a.e. when f is a signed or complex function. However the conclusion $f = 0$ a.e. can be recovered if we assume the vanishing of *all* the integrals $\int_E f$, over all measurable sets E.

Proposition 11.2. *Let* $f \in L^1$. *The following are equivalent:*

(a) $f = 0$ *almost everywhere,*

(b) $\int |f| = 0$,
(c) For every measurable set E, $\int_E f = 0$.

†

Proof. Since $f = 0$ a.e. if and only if $|f| = 0$ a.e., (a) and (b) are equivalent by Theorem 10.2(e). Now assuming (b), if E is measurable then by monotonicity and Theorem 11.1(b)

$$\left|\int_E f\right| \le \int_E |f| \le \int |f| = 0,$$

so item (c) holds.

Now suppose (c) holds and f is real-valued. Let $E = \{f > 0\}$ and note $f^+ = f\mathbf{1}_E$. Hence f^+ is unsigned and, by assumption $\int f^+ = \int_E f = 0$. Thus, by Theorem 10.2, $f^+ = 0$ a.e. Similarly $f^- = 0$ a.e. and thus f is the difference of two functions that are zero a.e. To complete the proof, write f in terms of its real and imaginary parts. □

Corollary 11.3. *If* $f, g \in L^1$ *and* $f = g$ μ*-a.e., then* $\int f = \int g$. †

Proof. Apply Proposition 11.2 to $f - g$. □

A consequence of Corollary 11.3 is that we can introduces an equivalence relation on $L^1(\mu)$ by declaring $f \sim g$ if and only if $f = g$ a.e. If $[f]$ denotes the equivalence class of f under this relation, we may define the integral on equivalence classes by declaring $\int [f] := \int f$. Corollary 11.3 shows that this is well-defined. It is straightforward to check that $[cf + g] = [cf] + [g]$ for all $f, g \in L^1$ and scalars c (so that $L^1/\sim$ is a vector space), and that the properties of the integral given in Theorem 11.1 all persist if we work with equivalence classes. The advantage is that now $\int [|f|] = 0$ if and only if $[|f|] = 0$. This means that the quantity $\|[f]\|_1$ is a *norm* on $L^1/\sim$. Henceforth will we agree to impose this relation whenever we talk about L^1, but for simplicity we will drop the $[\cdot]$ notation, and also write just L^1 for $L^1/\sim$. So, when we refer to an L^1 function, it is now understood that we refer to the equivalence class of functions equal to f a.e., but in practice this abuse of terminology should cause no confusion.

Just as the Monotone Convergence Theorem is associated to the unsigned integral, there is a convergence theorem for the absolutely convergent integral.

Theorem 11.4 (Dominated Convergence Theorem)**.** *Suppose* $(f_n)_{n=1}^\infty$ *is a sequence from* L^1 *that converges pointwise a.e. to a measurable function* f*. If there exists a function* $g \in L^1$ *such that for every* n*, we have* $|f_n| \le g$ *a.e., then* $f \in L^1$*, and*

$$\lim_{n\to\infty} \int f_n = \int f.$$

Proof. First observe that $|f| \le g$ and hence $f \in L^1$. By considering the real and imaginary parts separately, we may assume f and all the f_n are real valued. By hypothesis,

$g \pm f_n \geq 0$ a.e. Applying Fatou's theorem and linearity of the integral to these sequences gives

$$\int g + \int f = \int (g+f) \leq \liminf \int (g+f_n) = \int g + \liminf \int f_n$$

and

$$\int g - \int f = \int (g-f) \leq \liminf \int (g-f_n) = \int g - \limsup \int f_n.$$

It follows that $\liminf \int f \geq \int f \geq \limsup \int f$. □

The conclusion $\int f_n \to \int f$ (equivalently, $\left|\int f_n - \int f\right| \to 0$) can be strengthened somewhat:

Corollary 11.5. *If f_n, f, g satisfy the hypotheses of the Dominated Convergence theorem, then $\lim_{n\to\infty} \|f_n - f\|_1 = 0$ (that is, $\lim \int |f_n - f| = 0$).* †

Proof. Problem 13.20. □

Theorem 11.6 (Density of simple functions in L^1)**.** *If $f \in L^1$, then there is a sequence (f_n) of simple functions from L^1 such that,*

(a) $|f_n| \leq |f|$ for all n,
(b) $f_n \to f$ pointwise, and
(c) $\lim_{n\to\infty} \|f_n - f\|_1 = 0$.

Item (c) says (f_n) converges to f in L^1.

Proof. Write $f = u + iv$ with u, v real, and $u = u^+ - u^-, v = v^+ - v^-$. Each of the four functions $u^\pm, v^\pm$ is unsigned and measurable and each is in L^1 since $f \in L^1$. By the ziggurat approximation we can choose four sequences of unsigned measurable simple functions $u_n^\pm, v_n^\pm$ increasing pointwise to $u^\pm, v^\pm$ respectively. Now put $u_n = u_n^+ - u_n^-$, $v_n = v_n^+ - v_n^-$, and $f_n = u_n + iv_n$. By construction, each f_n is simple (and measurable). Moreover

$$|u_n| = u_n^+ + u_n^- \leq u^+ + u^- = |u|,$$

and similarly $|v_n| \leq |v|$, so $|f_n| \leq |u| + |v| \leq 2|f|$. Since $f \in L^1$ each f_n is in L^1, and $f_n \to f$ pointwise by construction. Thus the sequence (f_n) satisfies the hypothesis of the dominated convergence theorem (with $g = 2|f|$) and hence item (c) follows from Corollary 11.5. □

12. Modes of convergence

In this section we consider five different ways in which a sequence of functions on a measure space $(X, \mathscr{M}, \mu)$ can be said to converge. There is no simple or easily summarized description of the relation between the five modes. At the end of the section the reader is encouraged to draw a diagram showing the implications.

12.1. **The five modes of convergence.**

Definition 12.1. Let $(X, \mathscr{M}, \mu)$ be a measure space and $(f_n)_{n=1}^{\infty}$, f be measurable functions from X to $\mathbb{C}$.

(a) The sequence (f_n) converges to f *pointwise almost everywhere* if $\mu(\{\lim f_n \neq f\}) = 0$.
(b) The sequence (f_n) converges to f *essentially uniformly* or *in the L^∞ norm* if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\mu(\{|f_n - f| \geq \epsilon\}) = 0$ for all $n \geq N$.
(c) The sequence (f_n) converges to f *almost uniformly* if for every $\epsilon > 0$, there exists an (exceptional set) $E \in \mathscr{M}$ such that $\mu(E) < \epsilon$ and (f_n) converges to f uniformly on the complement of E.
(d) The sequence (f_n) converges to f *in the L^1 norm* if $(\|f_n - f\|_1 := \int_X |f_n - f| \, d\mu)$ converges to 0.
(e) The sequence (f_n) converges to f in *measure* if for every $\epsilon > 0$, the sequence $\big(\mu(\{x : |f_n - f| > \epsilon\})\big)$ converges to 0.

$\triangleleft$

The first thing to notice is that each of these modes of convergence is unaffected if f or the f_n are modified on sets of measure 0 (this is not true of ordinary pointwise or uniform convergence). Thus these are modes of convergence appropriate to measure theory. The L^1 and L^∞ modes are special cases of L^p convergence, which will be studied later in the course.

We first treat a few basic properties common to all five modes of convergence.

Proposition 12.2 (Linearity of convergence)**.** *Let $(f_n), (g_n), f, g$ be measurable functions and c a complex number.*

(a) For each of the five modes, (f_n) converges to f in the given mode if and only if $(|f_n - f|)$ converges to 0 in the given mode.
(b) If (f_n) converges to f and (g_n) converges to g, then $(cf_n + g_n)$ converges to $cf + g$ in the given mode.

†

Proof. The proof is left as an exercise. (Problem 13.24) □

Proposition 12.3. *Let (f_n) be a sequence of measurable functions and suppose f is measurable.*

(a) If (f_n) converges to f essentially uniformly, then (f_n) converges to f almost uniformly.
(b) If (f_n) converges to f almost uniformly, then (f_n) converges to f pointwise a.e. and in measure.
(c) If (f_n) converges to f in the L^1 norm, then (f_n) converges to f in measure.

†

Proof. (a) is immediate from definitions. For (b), given $k \geq 1$ we can find a measurable set E_k with $\mu(E_k) < \frac{1}{k}$ such that $f_n \to f$ uniformly (hence pointwise) on E_k^c. It follows that $f_n \to f$ pointwise on the set $\bigcup_{k=1}^{\infty} E_k^c$. The complement of this set, $\bigcap_{k=1}^{\infty} E_k$ has measure zero, since $\mu(\bigcap_{k=1}^{\infty} E_k) \leq \mu(E_m) \leq \frac{1}{m}$ for all $m \in \mathbb{N}^+$. The second part of (b) is left as an exercise.

Finally, (c) follows from Markov's inequality (Proposition 10.9). For $\epsilon > 0$ fixed,

$$\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \leq \frac{1}{\epsilon} \int_X |f_n - f| \, d\mu = \frac{1}{\epsilon} \|f_n - f\|_1$$

and the sequence $(\|f_n - f\|_1)$ converges to 0 by hypothesis. □

Of the twenty possible implications that can hold between the five modes of convergence, only the four implications ((b) is really two implications) given in the last proposition (and the ones that follow by combining these) are true without further hypotheses.

To understand the differing modes of convergence, and the failure of the remaining possible implications in Proposition 12.3, it is helpful to work out what they say in the simplest possible case, namely that of step functions. A *step function* is a function of the form $c\mathbf{1}_E$ for a positive constant c and measurable set E. Convergence of a sequence of step functions to 0 in each of the five modes, turns out to be largely determined by three objects associated to the sequence $(c_n \mathbf{1}_{E_n})_{n=1}^{\infty}$: the *heights* c_n, the *widths* $\mu(E_n)$, and the *tail supports* $T_n := \bigcup_{j \geq n} E_j$. The proof of the following theorem involves nothing more than the definitions, but is an instructive exercise.

Theorem 12.4. *Let $f_n = c_n \mathbf{1}_{E_n}$ be a sequence of step functions.*

(a) Assuming $\mu(E_n) > 0$ for each n, the sequence (f_n) converges to 0 in L^∞ if and only if (c_n) converges to 0.
(b) The sequence (f_n) converges to 0 almost uniformly if either (c_n) or $(\mu(T_n))$ converges to 0.
(c) If $(|c_n|)$ is (eventually) bounded away from 0 and (f_n) converges almost uniformly to 0, then $(\mu(T_n))$ converges to 0.
(d) The sequence (f_n) converges to 0 in measure if and only if the sequence $(\min\{c_n, \mu(E_n)\})$ converges to 0.

Proof. To prove item (c), suppose, without loss of generality, that there is a $C > 0$ such that $|c_n| \geq C$ for all n and (f_n) converges almost uniformly to 0. Given $\epsilon > 0$ there is a set F such that $\mu(F^c) < \epsilon$ and (f_n) converges uniformly on F. In particular, for each $k \in \mathbb{N}^+$ there is an N_k such that

$$F \subset \cap_{n \geq N_k} \{|f_n| < \frac{1}{k}\}.$$

Equivalently,

$$F^c \supset \cup_{n \geq N_k} \{|f_n| \geq \frac{1}{k}\}.$$

Choose k such that $\frac{1}{k} < C$. Since $|c_n| \geq C > \frac{1}{k}$, we see $\{|f_n| \geq \frac{1}{k}\} = E_n$ for each n. Hence,

$$F^c \supset T_{N_k}.$$

Thus, $\epsilon > \mu(F^c) \geq \mu(T_n)$ for all $n \geq N_k$. It follows that $(\mu(T_n))$ converges to 0.

The remaining parts of the Theorem are similar (and easier) and are left as Problem 13.29. □

The moving bump examples

(a) (Escape to height infinity) $f_n = n\mathbf{1}_{(0,\frac{1}{n})}$
(b) (Escape to width infinity) $f_n = \frac{1}{2n}\mathbf{1}_{(-n,n)}$
(c) (Escape to horizontal infinity) $f_n = \mathbf{1}_{(n,n+1)}$
(d) (Escape to horizontal infinity alternate) $f_n = \mathbf{1}_{(n,n+\frac{1}{n})}$,

together with the typewriter example below suffice to produce counterexamples to all of the implications not covered in Proposition 12.3.

Example 12.5. [The Typewriter Sequence] Consider Lebesgue measure on $(0,1]$. Let $I_{nk} \subset (0,1]$ denote the dyadic interval $(\frac{k}{2^n}, \frac{k+1}{2^n}]$ for $n \geq 1$, $0 \leq k < 2^n$. List these intervals in dictionary order (first by increasing n, then by increasing k). So the first few intervals are $I_{10} = (0,\frac{1}{2}]$, $I_{11} = (\frac{1}{2},1]$, $I_{20} = (0,\frac{1}{4}]$, $I_{21} = (\frac{1}{4},\frac{1}{2}]$, etc. (Draw a picture to see what is going on.) The sequence of indicator functions of these intervals (in this order) converges in measure to 0, since for any $0 < \epsilon < 1$ we have $m(\{x : \mathbf{1}_{I_{nk}} > \epsilon\} = m(I_{nk}) = 2^{-n}$. However since each point in $(0,1]$ lies in infinitely many I_{nk} and also lies outside infinitely many I_{nk}, the sequence $\mathbf{1}_{I_{nk}}(x)$ does not converge at *any* point of $(0,1]$. △

To go further we begin with a closer investigation of convergence in measure.

Definition 12.6. A sequence $(f_n)_{n=1}^{\infty}$ of $\mathbb{C}$-valued measurable functions is *Cauchy in measure* if for every $\epsilon, \eta > 0$, there is an N such that for $n, m \geq N$,

$$\mu(\{x : |f_n(x) - f_m(x)| > \epsilon\}) < \eta.$$

◁

It is readily seen that if (f_n) converges to f in measure, then the sequence (f_n) is Cauchy in measure. Indeed, by the triangle inequality,

$$\{x : |f_n(x) - f_m(x)| > \epsilon\} \subset \{x : |f_n(x) - f(x)| > \epsilon/2\} \cup \{x : |f_m(x) - f(x)| > \epsilon/2\}, \quad (30)$$

and thus the result follows by subadditivity of μ.

Proposition 12.7. *A sequence $(f_n)_{n=1}^{\infty}$ of measurable functions $f_n : X \to \mathbb{C}$ is Cauchy in measure if and only if for every $\epsilon > 0$ there exists an integer $N \geq 1$ such that*

$$\mu(\{x : |f_n(x) - f_m(x)| > \epsilon\}) < \epsilon$$

for all $n, m \geq N$. Similarly (f_n) converges to f in measure if and only if for every $\epsilon > 0$ there exists N such that

$$\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) < \epsilon$$

for all $n \geq N$. †

Proof. Problem 13.27. □

We have already seen that convergence in measure does not imply pointwise a.e. convergence (the typewriter sequence). Note, however, that in that example there is at least a subsequence converging pointwise a.e. to 0 (give an example).

Proposition 12.8. *If $(f_n)_{n=1}^\infty$ is a sequence of measurable functions that is Cauchy in measure, then*

- *a) there is a measurable function f and a subsequence $(f_{n_k})_{k=1}^\infty$ such that $(f_{n_k})_k$ converges to f almost uniformly; and*
- *b) with f as in part (a), (f_n) converges to f in measure, and if also (f_n) converges to g in measure, then $f = g$ a.e.*

†

In other words, if the sequence (f_n) is Cauchy in measure, then it converges in measure to a unique (a.e.) f, and a subsequence of (f_n) converges to f a.e.

Proof. With $\epsilon = 2^{-1}$, there is an n_1 such that if $m \ge n_1$, then $\mu\{|f_m - f_{n_1}| > 2^{-1}\} < 2^{-1}$. Now with $\epsilon = 2^{-2}$, there is an n_2 such that $n_2 > n_1$ and if $m \ge m_2$, then $\mu\{|f_m - f_{n_2}| > 2^{-2}\} < 2^{-2}$. In particular, $\mu\{|f_{n_2} - f_{n_1}| > 2^{-1}\} < 2^{-1}$. Taking $\epsilon = 2^{-k} = \eta$ in the definition of convergence in measure,choose inductively a sequence of integers $n_1 < n_2 < \dots n_k < \dots$ such that

$$\mu(\{x : |f_{n_k}(x) - f_{n_{k+1}}(x)| > 2^{-k}\}) < 2^{-k}. \tag{31}$$

Put $g_k = f_{n_k}$. Let

$$E_k = \{x : |g_k(x) - g_{k+1}(x)| > 2^{-k}\}.$$

By (31) $\mu(E_k) < 2^{-k}$. Let $F_N = \bigcup_{k=N}^\infty E_k$ and $F = \bigcap_{N=1}^\infty F_N = \limsup E_k$ and observe $\mu(F_N) \le \sum_{k=N}^\infty 2^{-k} = 2^{-N-1}$. For $x \notin F_N$ and $m \ge n \ge N$, the estimate

$$|g_n(x) - g_m(x)| \le \sum_{k=n}^{m-1} |g_{k+1}(x) - g_k(x)| \le \sum_{k=n}^{m-1} 2^{-k} \le 2^{-n-1} \le 2^{N-1} \tag{32}$$

shows that (g_n) is uniformly Cauchy on F_N^c. Hence (g_n) is pointwise Cauchy on F^c and thus converges pointwise a.e. to a measurable function f by Proposition 8.17. Finally (g_n) converges almost uniformly to f.

Part (b) is a version of the fact that if a Cauchy sequence has a convergent subsequence, then the sequence converges; and, if a sequence has a limit, then the limit is unique. Thus, to prove part (b), let (g_k) and f be as in the proof of part (a). Since (g_k) converges to f almost uniformly, it converges to f in measure by Proposition 12.3. Using the triangle inequality,

$$\{x : |f_n(x) - f(x)| > \epsilon\} \subset \{x : |f_n(x) - g_k(x)| > \epsilon/2\} \cup \{x : |g_k(x) - f(x)| > \epsilon/2\},$$

and, using the Cauchy in measure assumption on (f_n) and that (g_k) converges to f in measure, the measures of the sets on the right can be made less than ϵ by choosing n, k sufficiently large. The details are left to the gentle reader.

Finally, suppose also (f_n) converges to g in measure. By one more application of the triangle inequality trick, for a fixed $\epsilon > 0$,

$$\{x : |f(x) - g(x)| > \epsilon\} \subset \{x : |f(x) - f_n(x)| > \epsilon/2\} \cup \{x : |f_n(x) - g(x)| > \epsilon/2\},$$

and the measures of the sets on the right-hand side go to 0 by hypothesis. So $\mu(\{x : |f(x) - g(x)| > \frac{1}{n}\}) = 0$ for all $n \in \mathbb{N}^+$. By the pigeon hole principle, Proposition 2.6, $\mu(\{|f - g| \neq 0\}) = 0$. □

Corollary 12.9. *If (f_n) converges to f in L^1, then (f_n) has a subsequence converging to f a.e.* †

Proof. Combine Proposition 12.3(c) and Proposition 12.8. □

Proposition 12.10. *Let $(X, \mathscr{M}, \mu)$ be a measure space. The normed vector space $L^1 = L^1(\mu)$ is complete. In particular, if (f_n) is an L^1 Cauchy sequence, then there is an $f \in L^1$ and a subsequence (g_k) of (f_n) such that (g_k) converges pointwise a.e. to f and (f_n) converges in L^1 to f.* †

Proof. Suppose (f_n) is L^1-Cauchy. In this case (f_n) is Cauchy in measure and hence has a subsequence (h_m) that converges pointwise a.e. to some measurable function f by Corollary 12.9. Choose a subsequence $(g_k = h_{m_k})$ such that

$$\|g_{k+1} - g_k\| < \frac{1}{2^k}.$$

(Such a subsequence is *super Cauchy*.) Let

$$G_m = \sum_{k=1}^{m} |g_{k+1} - g_k|.$$

The sequence G_m is an increasing sequence of non-negative functions and hence converges to some G. By Tonelli (Corollary 10.6),

$$1 = \sum_{k=1}^{\infty} \frac{1}{2^k} \geq \int_X G \, d\mu.$$

In particular G is in L^1. Further,

$$g_{m+1} = \sum_{k=1}^{m} [g_{k+1} - g_k] + g_1$$

is dominated by $|g_1| + G$ and converges pointwise a.e. to f. Hence by Corollary 11.5, $f \in L^1$ and (g_m) converges to f in L^1. Finally, since (f_n) is L^1 Cauchy and a subsequence converges (in L^1) to f, the full sequence converges in L^1 to f. □

12.2. **Finite measure spaces.** Observe that two of the "moving bump" examples (escape to width infinity and escape to horizontal infinity) exploit the fact that Lebesgue measure on $\mathbb{R}$ is infinite. Moreover, in some cases these are the only counterexamples (of the four) to particular implications–for example, escape to width infinity is the only example of convergence in L^∞ but not convergence in L^1, and escape to horizontal infinity is the only one of pointwise a.e. convergence but not convergence in measure. It is then plausible that if we work on a finite measure space ($\mu(X) < \infty$), where these examples are "turned off," we might recover further convergence results. This is indeed the case.

Proposition 12.11. *Suppose $(X, \mathscr{M}, \mu)$ is a finite measure space, (f_n) is an L^1 sequence and $f : X \to \mathbb{C}$ is measurable. If (f_n) converges to f essentially uniformly, then $f \in L^1$ and (f_n) converges to f in L^1.* †

Proof. Problem 13.28. □

Theorem 12.12 (Egorov's Theorem)**.** *Let $(X, \mathscr{M}, \mu)$ be a measure space with $\mu(X) < \infty$. If $f_n : X \to \mathbb{C}$ is a sequence of measurable functions, $f : X \to \mathbb{C}$ is measurable and (f_n) converges to f a.e., then (f_n) converges to f almost uniformly.*

Proof. There is no loss of generality in assuming (f_n) converges to f everywhere. For $N, k \geq 1$, let

$$E_{N,k} = \bigcup_{n=N}^{\infty} \{x : |f_n(x) - f(x)| \geq \frac{1}{k}\}.$$

Fix k. For each x, there is an N such that $|f_n(x) - f(x)| < \frac{1}{k}$ for all $n \geq N$. Hence $\bigcap_{N\geq 1} E_{N,k} = \varnothing$. Since the $E_{N,k}$ are decreasing with N and $\mu(X) < \infty$, by dominated convergence for sets for each k the sequence $(\mu(E_{N,k}))_N$ converges to 0.

Now let $\epsilon > 0$ be given. Choose, for each k, an N_k such that $\mu(E_{N_k,k}) < \epsilon 2^{-k}$. Let $E = \bigcup_{k=1}^{\infty} E_{N_k,k}$ and observe $\mu(E) < \epsilon$. To prove (f_n) converges to f uniformly on E^c, given $\eta > 0$ choose k such that $\frac{1}{k} < \eta$. Suppose now that $x \in E^c$ and $n \geq N_k$. Since $E^c \subset E^c_{N_k,k}$, the inequality $|f_n(x) - f(x)| < \frac{1}{k} < \eta$ holds and we conclude that (f_n) converges uniformly to f on E^c. □

Remark 12.13. Note that in Egorov's theorem, almost uniform convergence cannot be improved to essential uniform convergence, as the moving bump example $\mathbf{1}_{[0,\frac{1}{n}]}$ shows. ⋄

12.3. **Uniform integrability.** In the last section we saw that some convergence implications could be recovered by making an assumption ($\mu(X) < \infty$) that "turns off" some of the failure modes. In this section we do something similar. In particular note that the moving bump examples show that of the five modes, L^1 convergence is the *only* one that implies $\int f_n \to \int f$ (assuming the f_n and f are integrable). The main result of this section, a version of the Vitali convergence theorem, says that if we make certain assumptions on f_n (namely "uniform integrability") that turn off the moving bump examples, then we can conclude that L^1 convergence is equivalent to convergence in measure. This result is similar in spirit to the classical Vitali Convergence Theorem (which we will cover later in the context of L^p spaces), though the approach used here (borrowed from T. Tao, *An Introduction to Measure Theory*, Section 1.5) is slightly different.

Definition 12.14. [Uniform integrability] A subset $\mathscr{F}$ of L^1 is *uniformly integrable* provided

(i) [Uniform bound on L^1 norm] The set $\{\|f\|_1 : f \in \mathscr{F}\}$ is bounded;
(ii) [No escape to vertical infinity] $\sup(\{\int_{\{|f|\geq M\}} |f|\,d\mu : f \in \mathscr{F}\}) \to 0$ as $M \to +\infty$
(iii) [No escape to width infinity] $\sup(\{\int_{\{|f|\leq \delta\}} |f|\,d\mu : f \in \mathscr{F}\}) \to 0$ as $\delta \to 0$.

A sequence $f_n : X \to \mathbb{C}$ of L^1 functions is *uniformly integrable* if the set $\{f_n : n\}$ is uniformly integrable. ◁

(To visualize the conditions in items (ii) and (iii), work out what they say for sequences of step functions.) We warm up by observing that, for a single L^1 function f, by the Dominated Convergence theorem,

$$\lim_{M\to+\infty} \int_{|f|>M} |f|\,d\mu = 0$$

and

$$\lim_{\delta\to 0} \int_{|f|\leq\delta} |f|\,d\mu = 0.$$

Thus $\mathscr{F} = \{f\}$ is uniformly integrable.

Uniform integrability for a sequence (f_n) says the quantities $\int_{|f_n|>M} |f_n|\,d\mu$ and $\int_{|f_n|\leq\delta} |f_n|\,d\mu$ can be made arbitrarily small by choice of large M and small δ, *independently of* n. Proposition 12.17 below says if (f_n) is an L^1 Cauchy sequence, then (f_n) is uniformly integrable. Note too a finite union of uniformly integrable sets is uniformly integrable. In particular, if (f_n) converges to $f \in L^1$, then $\mathscr{F} = \{f_n : n\} \cup \{f\}$ is uniformly integrable.

Before going further we give an equivalent formulation of item (ii) (assuming item (i)):

Lemma 12.15. *If (f_n) is a sequence of L^1 functions such that $\sup_n \|f_n\|_1 < \infty$, then the condition of item (ii) in Definition 12.14 is equivalent to,*

(iib) for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $\mu(E) < \delta$, then $\int_E |f_n| < \epsilon$ for all n.

†

Proof. Suppose (ii) holds and let $\epsilon > 0$ be give. Choose M such that $\sup_n \int_{|f_n|\geq M} |f_n| < \frac{\epsilon}{2}$, and let $\delta < \frac{\epsilon}{2M}$. If E is a measurable set with $\mu(E) < \delta$, then for all n,

$$\begin{aligned}\int_E |f_n|\,d\mu &= \int_{E\cap\{|f_n|\geq M\}} |f_n|\,d\mu + \int_{E\cap\{|f_n|<M\}} |f_n|\,d\mu \\ &\leq \int_{\{|f_n|\geq M\}} |f_n|\,d\mu + \int_E M\,d\mu \\ &< \frac{\epsilon}{2} + M\mu(E) \\ &< \epsilon.\end{aligned}$$

Conversely, suppose item (iib) holds. Fix $\epsilon > 0$, and for $\delta > 0$ satisfying (iib) choose M large enough that $\frac{1}{M}\sup_n \|f_n\|_1 < \delta$. Then by Markov's inequality (Proposition 10.9), for all n we have $\mu(\{|f_n| \geq M\} \leq \frac{\|f_n\|_1}{M} < \delta$. Thus by (iib)

$$\int_{|f_n|\geq M} |f_n| < \epsilon$$

for all n. Hence (i) and (ii) implies (iib).

Remark 12.16. On a finite measure space, escape to width infinity is impossible, and a sequence is uniformly integrable if and only if $\sup_n \|f\|_1 < \infty$ and (ii) (equivalently, (iib)) is satisfied. $\diamond$

$\square$

Proposition 12.17. *If (f_n) is a sequence of L^1 functions and (f_n) converges to f in L^1, then (f_n) is uniformly integrable.* $\dagger$

Proof. Problem 13.35. Since convergent sequence are bounded, there $K = \sup\{\|f_n\| : n\}$ is finite.

We next turn to verifying item (iib). Accordingly, let $\epsilon > 0$ be given. By Markov's inequality, $\mu(\{|f_n| > M\}) \leq \frac{K}{M}$ for $M > 0$.

Since (f_n) converges to f, there is an N such that $\|f - f_n\| < \epsilon$ for all $n \geq N$. Since $\{f\}$ is absolutely integral, by item (iib), there is a $\eta > 0$ such that if $\mu(E) < \eta$, then $\int_E |f| < \epsilon$. Choose $M > \frac{K}{\eta}$. If $n \geq N$, then

$$\int_{|f_n|>M} |f_n| \leq \int_{|f_n|>M} |f_n - f| + \int_{|f_n|>M} |f| < 2\epsilon.$$

To prove item (iii), let $\epsilon > 0$ be given. Using the fact that $\{f\}$ is uniformly integrable, choose $\delta > 0$ such that $\int_{|f|<2\delta} |f| < \epsilon$. There is an N such that if $n \geq N$, then $\|f - f_n\| < \min\{\delta\eta, \epsilon\}$. It follows from Markov's inequality that $\mu(\{|f - f_n| > \delta\}) < \eta$. Finally, observe $\{|f_n| < \delta\} \subset \{|f| < 2\delta\} \cup \{|f - f_n| > \delta\}$ for all $n \geq N$. Hence

$$\int_{|f_n|<\delta} |f_n| \leq \int |f_n - f| + \int_{|f|<2\delta} |f| + \int_{|f-f_n|>\delta} |f| < 3\epsilon.$$

□

Theorem 12.18. *Suppose $f_n : X \to \mathbb{C}$ is a sequence of L^1 functions and $f : X \to \mathbb{C}$ is measurable. The sequence (f_n) converges to f in L^1 if and only if (f_n) is uniformly integrable and converges to f in measure.*

Proof. The forward implication follows from Propositions 12.10, 12.3 and 12.17.

For the converse, suppose (f_n) is uniformly integrable and converges to f in measure. To show that $f \in L^1$, first note, by uniform integrability, there is a constant C such that $\int_X |f_n| \le C$ for all n, and by Proposition 12.8 there is a subsequence (f_{n_k}) of (f_n) converging to f a.e. Applying Fatou's theorem to this subsequence, we conclude

$$\int_X |f| \le \liminf \int_X |f_{n_k}| \le C, \tag{33}$$

so $f \in L^1$.

Since $f \in L^1$ and (f_n) is uniformly integrable, the set $\{f_n : n\} \cup \{f\}$ is also uniformly integrable. Thus, by condition (iii) in the definition of uniformly integrable, given $\epsilon > 0$, there is a $\delta > 0$ such that for all n

$$\int_{|f_n| \le \delta} |f_n| \, d\mu \le \epsilon \tag{34}$$

and at the same time

$$\int_{|f| \le \delta} |f| \, d\mu \le \epsilon. \tag{35}$$

From conditions (i) and (ii) and Lemma 12.15, there exists a $\gamma > 0$ such that $\mu(E) \le \gamma$ implies

$$\int_E |f_n| \, d\mu < \epsilon$$
$$\int_E |f| \, d\mu < \epsilon$$

for all n. Now choose $0 < \eta = \min\{\frac{\delta}{2}, \frac{\epsilon\delta}{2C}, \gamma\}$.

From (34) and (35)

$$\int_{|f_n - f| < \eta, \ |f| \le \delta/2} |f_n| \, d\mu \le \epsilon$$

and

$$\int_{|f_n - f| < \eta, \ |f| \le \delta/2} |f| \, d\mu \le \epsilon.$$

So by the triangle inequality

$$\int_{|f_n - f| < \eta, \ |f| \le \delta/2} |f_n - f| \, d\mu \le 2\epsilon. \tag{36}$$

We now estimate the integral of $|f_n - f|$ over the region $|f_n - f| < \eta, |f| > \delta/2$ via Markov's inequality. Indeed,

$$\mu(\{x : |f(x)| > \delta/2\}) \leq \frac{C}{\delta/2}.$$

Thus

$$\int_{|f_n - f| < \eta,\ |f| > \delta/2} |f_n - f| \, d\mu \leq \frac{C}{\delta/2}\eta \leq \epsilon. \tag{37}$$

Combining Equations (37) and (36) gives

$$\int_{|f_n - f| < \eta} |f_n - f| \, d\mu \leq 3\epsilon. \tag{38}$$

Finally the convergence in measure hypothesis is used to estimate the integral of $|f_n - f|$ over the set $|f_n - f| \geq \eta$. With $\epsilon = \eta$ in the definition of convergence in measure, there is an N such that for all $n \geq N$,

$$\mu(\{x : |f_n(x) - f(x)| \geq \eta\} \leq \eta.$$

Hence, by the choice of γ,

$$\int_{|f_n - f| \geq \eta} |f_n| \, d\mu \leq \epsilon$$

and

$$\int_{|f_n - f| \geq \eta} |f| \, d\mu \leq \epsilon.$$

Thus again by the triangle inequality

$$\int_{|f_n - f| \geq \eta} |f_n - f| \, d\mu \leq 2\epsilon$$

for all $n \geq N$. Putting this last inequality together with (38),

$$\int_X |f_n - f| \, d\mu < 5\epsilon$$

for all $n \geq N$ and the proof is complete. □

Remark 12.19. Say that a sequence of measurable functions $f_n : X \to \mathbb{C}$ is *dominated* if there is an L^1 function g such that $|f_n| \leq |g|$ for all n. It is not hard to show (Problem 13.34) that if a sequence (f_n) is dominated, then it is uniformly integrable. On the other hand, a sequence (f_n) can converge in L^1 yet not be dominated. The main utility of Theorem 12.18 is a criteria for proving L^1 convergence for sequences that are not dominated. ◇

13. Problems

13.1. **Measurable functions.**

Problem 13.1. Suppose $f : X \to \mathbb{C}$ is a measurable function. Prove that the functions $|f|$ and $\operatorname{sgn} f$ are measurable. (Recall that for a complex number z, $\operatorname{sgn}(z) = z/|z|$ if $z \neq 0$, and $\operatorname{sgn}(0) = 0$.)

Problem 13.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For each of the following, prove or give a counterexample.

a) Suppose f^2 is Lebesgue measurable. Does it follow that f is Lebesgue measurable?
b) Suppose f^3 is Lebesgue measurable. Does it follow that f is Lebesgue measurable?

Problem 13.3. Recall the definition of an *atomic σ-algebra* (Problem 7.3). Prove that if $(X, \mathscr{M})$ is a measurable space and $\mathscr{M}$ is an atomic σ-algebra, then a function $f : X \to \overline{\mathbb{R}}$ is measurable if and only if it is constant on each atom A_n.

Problem 13.4. Prove, if $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

Problem 13.5. Let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of measurable functions. Prove that the set $\{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is measurable.

Problem 13.6. Give an example of an uncountable collection $\mathcal{F}$ of Lebesgue measurable functions $f : \mathbb{R} \to [0, +\infty]$ such that the function $f(x) = \sup_{f\in\mathcal{F}} f(x)$ is not Lebesgue measurable.

Problem 13.7. Let $f : [0, 1] \to [0, 1]$ denote the Cantor-Lebesgue function of Example 6.2(c) and define $g(x) = f(x) + x$.

(i) Prove that g is a homeomorphism of $[0, 1]$ onto $[0, 2]$. (Hint: it suffices to show g is a continuous bijection.)
(ii) Let $C \subset [0, 1]$ denote the Cantor set. Prove that $m(g(C)) = 1$. (Here m is Lebesgue measure.)
(iii) By Problem 7.29, $g(C)$ contains a nonmeasurable set S. Show that $g^{-1}(S)$ is Lebesgue measurable, but not Borel.
(iv) Prove that there exists functions F, G on $\mathbb{R}$ such that F is Lebesgue measurable, G is continuous, but $F \circ G$ is not Lebesgue measurable.

Problem 13.8. Prove that if $f, g : X \to [0, \infty]$ are measurable functions, then fg is measurable.

Problem 13.9. Prove Propositions 8.16 and 8.17. For 8.17, you may wish to use the observation that $f : X \to \overline{\mathbb{R}}$ is measurable if and only if $\{x : f(x) > q\}$ is measurable for every $q \in \mathbb{Q}$. (The following example shows that $\overline{\mu}$ can not be replaced by μ in the conclusion. Let $X = \overline{\mathbb{R}}$ and $\mathscr{M} = \{\emptyset, X\}$. Let μ denote the zero measure on $\mathscr{M}$. In this case the completion of $\mathscr{M}$ is 2^X and $\overline{\mu}$ is the zero measure on 2^X. Let $f : X \to X$ denote the identity function. It is 2^X measurable. A function g is $\mathscr{M}$ measurable if and only if it is constant, say with value c. Hence $\{f \neq g\} = X \setminus \{c\}$ and this set is not in $\mathscr{M}$.)

13.2. **The unsigned integral.**

Problem 13.10. Prove the claim made immediately before Definition 9.1.

Problem 13.11. Complete the proof of Theorem 9.2.

Problem 13.12. Prove that if f is an unsigned measurable function and $\int f < +\infty$, then the set $E := \{x : f(x) > 0\}$ is σ-finite. (That is, E can be written as a disjoint union of measurable sets $E = \bigcup_{n=1}^{\infty} E_n$ with each $\mu(E_n) < +\infty$.)

Problem 13.13. Suppose that f is an unsigned measurable function and $\int f < +\infty$.

a) Prove that for every $\epsilon > 0$ there is a measurable set E with $\mu(E) < +\infty$, such that $\int f - \int_E f < \epsilon$.
b) Prove that for every $\epsilon > 0$ there is a positive integer n such that $\int f - \int \min(f, n) < \epsilon$.

Problem 13.14. Prove Fatou's Theorem (Theorem 10.7) without using the Monotone Convergence Theorem. Then use Fatou's theorem to prove the Monotone Convergence Theorem.

Problem 13.15. Let X be any set and let μ be counting measure on X. Prove that for every unsigned function $f : X \to [0, +\infty]$, we have $\int_X f\, d\mu = \sum_{x \in X} f(x)$.

Problem 13.16. Prove Theorem 10.10. (Hint: to verify the integral formula, use the Monotone Convergence Theorem.)

Problem 13.17. Prove Theorem 10.12. (Hint: show first that $\mu(E) := L(\mathbf{1}_E)$ is a measure, then that $L(f) = \int f\, d\mu$. Problem 7.8 may be helpful.)

Problem 13.18. Let f be an unsigned measurable function on the measure space $(X, \mathscr{M}, \mu)$. Prove that the function $\nu : \mathscr{M} \to [0, \infty]$ defined by $\nu(E) := \int_E f\, d\mu$ is a measure and, if g is an unsigned measurable function on X, then $\int g\, d\nu = \int gf\, d\mu$.

Problem 13.19. Prove (using monotone convergence and *without* using the dominated convergence theorem) that if f_n is a sequence of unsigned measurable functions that decreases pointwise to f, and $\int f_N < \infty$ for some N, then $\int f = \lim \int f_n$. Give an example to show that the finiteness hypothesis is necessary.

13.3. **The signed integral.**

Problem 13.20. Prove Corollary 11.5.

Problem 13.21. Prove the following generalization of the dominated convergence theorem: suppose (f_n) converges to f a.e. If g_n is a sequence of L^1 functions converging a.e. to an L^1 function g, if $|f_n| \le g_n$ for all n, and $\int g_n \to \int g$, then $\int f_n \to \int f$.

Problem 13.22. Suppose $f_n, f \in L^1$ and (f_n) converges to f a.e. Prove that $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$. (Use the previous problem.)

Problem 13.23. Evaluate each of the following limits, and carefully justify your claims.

a) $\displaystyle\lim_{n\to\infty}\int_0^\infty \frac{\sin(x/n)}{(1+(x/n))^n}\,dx$
b) $\displaystyle\lim_{n\to\infty}\int_0^\infty \frac{1+nx^2}{(1+x^2)^n}\,dx$
c) $\displaystyle\lim_{n\to\infty}\int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)}\,dx$
d) $\displaystyle\lim_{n\to\infty}\int_0^\infty \frac{n}{1+n^2x^2}\,dx$

13.4. **Modes of convergence.**

Problem 13.24. Prove Proposition 12.2

Problem 13.25. Prove that if (f_n) converges to f almost uniformly, then (f_n) converges to f in measure.

Problem 13.26. Show that the implications between modes of convergence not given in Proposition 12.3 are false.

Problem 13.27. Prove Proposition 12.7.

Problem 13.28. . Prove Proposition 12.11.

Problem 13.29. Prove Theorem 12.4.

Problem 13.30. Let $f_n = \mathbf{1}_{(n,n+\frac{1}{n})}$. Show that (f_n) converges pointwise and in measure, but not almost uniformly, to 0.

Let $f_{2n} = \mathbf{1}_{(n,n+\frac{1}{n^2})}$ and $f_{2n+1} = \frac{1}{2n}\mathbf{1}_{(-1,1)}$. Show (f_n) converges almost uniformly to 0, but, writing $f_n = c_n\mathbf{1}_{E_n}$, neither (c_n) nor $(\mu(E_n))$ converges to 0.

Problem 13.31. Let $(X, \mathscr{M}, \mu)$ be a *finite* measure space. For measurable functions $f, g : X \to \mathbb{C}$, define
$$d(f,g) = \int_X \frac{|f-g|}{1+|f-g|}\,d\mu.$$
Prove that d is a metric on the set of measurable functions (where we identify f and g when $f = g$ a.e.).

Problem 13.32. Let $(X, \mathscr{M}, \mu)$ be a finite measure space. Prove that (f_n) converges to f in measure if and only if $d(f_n, f) \to 0$, where d is the metric in Problem 13.31.

Problem 13.33. [Fast L^1 convergence] Suppose (f_n) converges to f in L^1 in such a way that $\sum_{n=1}^\infty \|f_n - f\|_1 < \infty$. Prove that (f_n) converges to f almost uniformly. (Note that this hypothesis "turns off" the typewriter sequence.) (Hint: first show that, given $\epsilon > 0$ and $m \geq 1$, there exists an integer N such that
$$\mu\left(\bigcup_{k=N}^{\infty}\{x : |f_k(x) - f(x)| \geq 2^{-m}\}\right) < \epsilon.$$
To construct your exceptional set E, use the $\epsilon/2^n$ trick.)

Problem 13.34. Prove that if f_n is a dominated sequence, then it is uniformly integrable. Give an example of a sequence (f_n) that converges in L^1 (and is thus uniformly integrable), but is not dominated.

Problem 13.35. Prove that if (f_n) converges to f in L^1, then (f_n) is uniformly integrable (Proposition 12.17).

Problem 13.36. Suppose (f_n) converges to f in measure and f_n is dominated. Give a direct proof that (f_n) converges to f in L^1 (without using Theorem 12.18).

Problem 13.37. Prove that if (f_n) is a dominated sequence, and (f_n) converges to f a.e., then (f_n) converges to f almost uniformly. (Hint: imitate the proof of Egorov's theorem.) (Thus for dominated sequences, a.e. and a.u. convergence are equivalent.)

Problem 13.38. [Defect version of Fatou's theorem] Let (f_n) be a sequence of *unsigned* L^1 functions with $\sup_n \|f_n\|_1 < \infty$. Suppose (f_n) converges to f a.e. Prove that (f_n) converges to f in L^1 if and only if $\int f_n \to \int f$. [Suggestion: Consider the sequence $|f - f_n| + (f - f_n)$.]

14. The Riesz-Markov Representation Theorem

Let X be a compact Hausdorff space. Recall the space $C(X)$ is the vector space $\{f : X \to \mathbb{C} : f \text{ is continuous}\}$ with the norm

$$\|f\|_\infty = \|f\| = \max\{|f(x)| : x \in X\} \tag{39}$$

and that, as a metric space (the distance from f to g is $\|f-g\|$), $C(X)$ is a complete. Generally, a complete normed vector space is called a *Banach space*.

For a locally compact Hausdorff space X, a function $f : X \to \mathbb{C}$ has *compact support* if there exists a compact set K such that $f(x) = 0$ for $x \notin K$; i.e., the closure of $\{x \in X : f(x) \neq 0\}$ is compact. Assuming X is a locally compact Hausdorff space, let $C_c(X)$ denote those $f \in C(X)$ with compact support.; The space $C_c(X)$ is also given the supremum norm as in Equation (39).

Given a vector space V, a linear mapping $\lambda : V \to \mathbb{C}$ is called a *linear functional.* A linear functional $\lambda : C_c(X) \to \mathbb{C}$ is *positive*, if $\lambda(f) \geq 0$ whenever $f \geq 0$ (meaning $f(x) \geq 0$ is *pointwise positive*).

Example 14.1. Suppose μ is a regular Borel measure on the locally compact set X and $\mu(K) < \infty$ for compact subsets of X. (This last condition is automatic if X is compact and $\mu(X) < \infty$). Thus, by Theorem 10.2(f), μ determines a positive linear functional, λ, on $C_c(X)$ by

$$\lambda(f) = \int_X f\, d\mu.$$

As a second example, let $X = [0,1]$ and note that the mapping $I : C([0,1]) \to \mathbb{C}$ defined by

$$I(f) = \int_0^1 f\, dx,$$

where the integral is in the Riemann sense, is a positive linear functional on $C([0,1])$. $\triangle$

Theorem 14.2 (Riesz-Markov Representation Theorem)**.** *Let $X = (X, \tau)$ be a locally compact Hausdorff space. If $\lambda : C_c(X) \to \mathbb{C}$ is a positive linear functional, then there exists a unique Borel measure μ on the Borel σ-algebra $\mathscr{B}_X$, such that*

$$\lambda(f) = \int f\, d\mu$$

for $f \in C_c(X)$ and such that μ is regular in the sense that

(i) if K is compact, then $\mu(K) < \infty$;
(ii) if $E \in \mathscr{B}_X$, then $\mu(E) = \inf\{\mu(U) : E \subset U,\ U \text{ open}\}$; and
(iii) if $E \in \mathscr{B}_X$ and $\mu(E) < \infty$, then $\mu(E) = \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}$.

We will prove the result for the case X is compact. In this case, item (i) implies μ is a a finite measure, and hence item (iii) applies to all $E \in \mathscr{B}_X$. In the next subsection, topological preliminaries are gathered. The proof itself is in Subsection 14.2

14.1. **Urysohn's Lemma and partitions of unity.** A topological space X is *normal* if for each pair C_1, C_2 of disjoint closed subsets of X, there exists disjoint open sets U_1, U_2 such that $C_j \subset U_j$.

Theorem 14.3. A compact Hausdorff space X is normal.

Theorem 14.4 (Urysohn's lemma)**.** If X is a compact Hausdorff space and A, B are disjoint closed subsets of X, then there exists a function $f : X \to [0,1]$ such that f is zero on A and f is 1 on B. In particular, if K is compact, V is open and $K \subset V$, then there is a continuous $f : X \to \mathbb{R}$ such that $\mathbf{1}_K \le f \le \mathbf{1}_V$ and $\operatorname{supp}(f) \subset V$.

Remark 14.5. Urysohn's Lemma implies that X is normal by choosing $U = f^{-1}((-1, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, 2))$.

Note that the lemma does not say $A = f^{-1}(\{0\})$ or $B = f^{-1}(\{1\})$, though this can be arranged in the case that X is a metric space. $\diamond$

Theorem 14.6 (Partition of Unity)**.** Suppose $V_1, \dots, V_n$ are open subsets of a compact Hausdorff space X. If C is closed and $C \subset \cup V_j$, then there exists continuous functions $h_j : X \to [0,1]$ such that

(i) $h_j \le \mathbf{1}_{V_j}$;
(ii) $\operatorname{supp}(h_j) \subset V_j$; and
(iii) for $x \in C$,

$$\sum_{j=1}^{n} h_j(x) = 1.$$

14.2. **Proof of Theorem 14.2.** Suppose X is a compact metric space and $\lambda : C(X) \to \mathbb{C}$ is a positive linear functional. To get an idea how to define $\mu(V)$ for an open set $V \in \tau$, note that $K = X \setminus V$ is compact and the function $g : X \to \mathbb{R}$

$$g(x) = d(x, K) = \min\{d(x,k) : k \in K\}$$

is continuous. The sequence

$$f_n = (\frac{g}{1+g})^{\frac{1}{n}}$$

is pointwise increasing to the *characteristic function* (or *indicator function*) of V, denoted $\mathbf{1}_V$. Thus $\mathbf{1}_V : X \to \mathbb{R}$ is defined by $\mathbf{1}_V(x) = 0$ for $x \notin V$ and $\mathbf{1}_V(x) = 1$ for $x \in V$. If μ exists, then, by the MCT,

$$\mu(V) = \int \mathbf{1}_V \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

We are led to make the following definitions. For V open, define

$$\mu_0(V) = \sup\{\lambda(f) : f \in C(X),\, 0 \le f \le \mathbf{1}_V,\, \operatorname{supp}(f) \subset V\}.$$

Thus, letting τ denote the topology on X, $\mu_0 : \tau \to [0, \infty)$.

Define $\mu^* : 2^X \to \mathbb{R}$ by

$$\mu^*(E) = \inf\{\mu_0(V) : V \text{ is open and } E \subset V\}.$$

(Note that this definition is forced upon us to achieve outer regularity.)

The proof is now broken down into a series of Lemmas. The functions, unless otherwise noted, are continuous.

Lemma 14.7. The mapping μ_0 is monotone and countably subadditive (on τ). †

Proof. That μ_0 is monotone is evident. To prove that it is countably subadditive, suppose (V_j) is a sequence of open sets and let $V = \cup V_j$. Suppose f is continuous, nonnegative valued, $f \le \mathbf{1}_V$ and $K = \operatorname{supp}(f) \subset V$. Since K is compact and $K \subset \cup V_j$, there exists an N such that $K \subset \cup_{j=1}^N V_j$. By Theorem 14.6, there exists functions $h_j \in C(X)$ such that $0 \le h_j \le \mathbf{1}_{V_j}$ the support of h_j lies in V_j and $\sum_{j=1}^N h_j = 1$ on K. It follows that $f = \sum f h_j$ and $f h_j \le \mathbf{1}_{V_j}$ as well as $\operatorname{supp}(f h_j) \subset V_j$. Hence,

$$\lambda(f) = \sum \lambda(f h_j) \le \sum_{j=1}^N \mu_0(V_j) \le \sum_{j=1}^\infty \mu_0(V_j)$$

and inequality that completes the proof. □

Lemma 14.8. If V_1 and V_2 are disjoint open sets, then $\mu_0(V_1 \cup V_2) = \mu_0(V_1) + \mu_0(V_2)$. †

Proof. Let $W = V_1 \cup V_2$. By Lemma 14.7, it suffices to show that $\mu_0(W) \ge \mu_0(V_1) + \mu_0(V_2)$. To this end, let $\epsilon > 0$ be given and suppose $f_j \le \mathbf{1}_{V_j}$ are such that $\operatorname{supp}(f_j) \subset V_j$ and $\mu_0(V_j) \le \lambda(f_j) + \epsilon$. By disjointness, $f_1 + f_2 \le \mathbf{1}_W$ and $\operatorname{supp}(f_1 + f_2) \subset W$ too. Hence,

$$2\epsilon + \mu_0(W) \ge 2\epsilon + \lambda(f) = 2\epsilon + \sum \lambda(f_j) \ge \sum \mu_0(V_j).$$

Since $\epsilon > 0$ is arbitrary, the conclusion follows. □

Lemma 14.9. *The mapping μ^* is an outer measure. Further, if W is open, then $\mu^*(W) = \mu_0(W)$.* †

Proof. Since $\mu_0(\emptyset) = 0$, to prove μ^* is an outer measure, it suffices to prove that, for $E \subset X$,

$$\mu^*(E) = \inf\{\sum_{j=1}^\infty \mu_0(V_j) : (V_j) \text{ is a sequence of open sets and } E \subset \cup V_j\}.$$

Hence, it is enough to show that if (V_j) is a sequence of open sets such that $E \subset \cup V_j$, then $\mu_0(V) \le \sum_{j=1}^\infty \mu_0(V_j)$ for some open set $E \subset V$. Choose $V = \cup V_j$ and apply Lemma 14.7. □

Lemma 14.10. If K is compact and $\mathbf{1}_K \le f$, then $\mu^*(K) \le \lambda(f)$. †

Proof. Given $0 < \delta < 1$, let $V_\delta = \{f > \delta\}$. Note that V_δ contains K and is open. Moreover, if $g \le \mathbf{1}_{V_\delta}$ and $\operatorname{supp}(g) \subset V_\delta$, then $\delta g \le f$ and hence $\lambda(g) \le \frac{1}{\delta}\lambda(f)$. It follows that,

$$\mu_0(V_\delta) \le \frac{1}{\delta}\lambda(f).$$

Thus, by monotonicity of outer measure, $\mu^*(K) \le \mu_0(V_\delta) \le \frac{1}{\delta}\lambda(f)$. Letting $\delta < 1$ tend to 1 gives the result. $\square$

Lemma 14.11. If W is open, K is compact and $K \subset W$, then

$$\mu_0(W) = \mu_0(W \setminus K) + \mu^*(K).$$

†

Proof. One inequality follows from subadditivity of outer measure. To prove the other inequality note that $W \setminus K$ is open and let $\epsilon > 0$ be given. Choose $0 \le g \le 1_{W\setminus K}$ with $\operatorname{supp}(g) \subset W \setminus K$ and $\lambda(g) + \epsilon \ge \mu_0(W \setminus K)$. Let $C = \operatorname{supp}(g)$. Choose, by Theorem 14.4, $\mathbf{1}_K \le f \le 1_{C^c \cap W}$ such that $\operatorname{supp}(f) \subset C^c \cap W$. In particular, $0 \le f + g \le 1_W$ and the support of $f + g$ lies in W and, by Lemma 14.10, $\lambda(f) \ge \mu^*(K)$. Thus,

$$\epsilon + \mu_0(W) \ge \epsilon + \lambda(f + g) = \lambda(f) + (\epsilon + \lambda(g)) \ge \mu^*(K) + \mu_0(W \setminus K).$$

$\square$

Lemma 14.12. If W is open, K is compact, $K \subset W$ and $\epsilon > 0$, then there exists $\mathbf{1}_K \le f \le \mathbf{1}_W$ such that $\operatorname{supp}(f) \subset W$ and $\lambda(f) \le \mu^*(K) + \epsilon$. †

Proof. Choose V an open set such that $K \subset V$ and $\mu_0(V) \le \mu^*(K) + \epsilon$. Replacing W by $V \cap W$ if needed, it may be assumed that $V \subset W$. By Theorem 14.4, there exists $\mathbf{1}_K \le f \le \mathbf{1}_V$ and $\operatorname{supp}(f) \subset V$. It follows that

$$\lambda(f) \le \mu_0(V) \le \mu^*(K) + \epsilon.$$

$\square$

Lemma 14.13. *If W is open and $\epsilon > 0$, then there is a compact set K such that $K \subset W$ and $\mu_0(W) \le \mu^*(K) + \epsilon$.* †

Proof. There is a $0 \le g \le \mathbf{1}_W$ such that $\operatorname{supp}(g) \subset W$ and $\lambda(g) + \epsilon > \mu_0(W)$. Let K denote the support of g. Hence, $K \subset W$ and K is compact. By Lemma 14.12, there exists an f such that $\mathbf{1}_K \le f \le \mathbf{1}_W$, the support of f lies in W and $\lambda(f) \le \mu^*(K) + \epsilon$. In particular $g \le f$ and hence $\lambda(g) \le \lambda(f)$. It follows that

$$\mu_0(W) \le \lambda(g) + \epsilon \le \lambda(f) + \epsilon \le \mu^*(K) + 2\epsilon.$$

$\square$

Lemma 14.14. *If W is open, then W is outer measurable.* †

Proof. Let $A \subset X$ be given. Given $\epsilon > 0$, choose an open set $A \subset V$ such that $\mu_0(V) \le \mu^*(A) + \epsilon$. Choose, by Lemma 14.13, a compact set $K \subset W$ such that $\mu_0(W) \le \mu^*(K) + \epsilon$. Now, by monotonicity and Lemma 14.11,

$$\begin{aligned}\mu^*(A \cap W) &\le \mu_0(V \cap W)\\ &\le \mu^*(V \cap K) + \mu_0(V \cap (W \setminus K))\\ &\le \mu^*(V \cap K) + \mu_0(W \setminus K)\\ &\le \mu^*(V \cap K) + \epsilon.\end{aligned}$$

Further, by monotonicity,

$$\mu^*(A \cap W^c) \le \mu^*(V \cap W^c).$$

Now K and W^c are disjoint compact sets. Hence, by Theorem 14.3, there exist disjoint open sets S, T such that $K \subset S$ and $W^c \subset T$. Consequently, using Lemma 14.8 and monotonicity,

$$\begin{aligned}\mu^*(A \cap W) + \mu^*(A \cap W^c) &\le \mu^*(V \cap K) + \epsilon + \mu^*(V \cap W^c)\\ &\le \mu_0(V \cap S) + \mu_0(V \cap T) + \epsilon\\ &= \mu_0(V \cap (S \cup T)) + \epsilon\\ &\le \mu_0(V) \le \mu^*(A) + 2\epsilon.\end{aligned}$$

It follows that $\mu^*(A) \ge \mu^*(A \cap W) + \mu^*(A \cap W^c)$ and thus W is outer measurable. □

Let $\mathscr{M}$ denote the collection of outer measurable sets. Thus, μ, the restriction of μ^* to $\mathscr{M}$ is a complete measure. Further, $\mathscr{M}$ contains all open sets by Lemma 14.14 and by Lemma 14.9 if W is open then $\mu(W) = \mu_0(W)$. In particular, $\mathscr{B}_X \subset \mathscr{M}$.

Lemma 14.15. The measure μ satisfies the regularity conditions of the theorem. †

Proof. Outer regularity follows immediately from the definition of μ^*. As for inner regularity, suppose E is measurable. Thus E^c is measurable. By outer regularity, there is an open set V such that $E^c \subset V$ and $\mu(V \setminus E^c) < \epsilon$. Thus $K = V^c \subset E$ is compact and $\mu(E \setminus K) < \epsilon$. □

Lemma 14.16. *If $f \in C(X)$, then*

$$\lambda(f) = \int_X f\, d\mu.$$

†

Proof. Suppose $f \in C(X)$ is real-valued and that $[a, b]$ contains the range of f. Given $\epsilon > 0$, choose $t_0 < a < t_1 < \dots t_n = b$ such that $t_j - t_{j-1} < \epsilon$. Let $E_j = f^{-1}((t_{j-1}, t_j])$ for $j = 1, n$. The E_j are Borel sets, hence, by outer regularity, there exists open sets $V_j \supset E_j$ such that $\mu(V_j) \le \mu(E_j) + \frac{\epsilon}{n}$. By Theorem 14.6, there exists $h_j \in C(X)$ such that $0 \le h_j \le \mathbf{1}_{V_j}$, the support of h_j lies in V_j and $\sum h_j = 1$. Now,

$$\begin{aligned}\lambda(f) &= \sum \lambda(f h_j)\\ &\le \sum \lambda(t_j h_j)\\ &\le \sum t_j \mu(V_j)\\ &\le (\sum t_{j-1} + \epsilon)(\mu(E_j) + \frac{\epsilon}{n})\\ &\le \int_X f\, d\mu + \epsilon(\mu(X) + \epsilon).\end{aligned}$$

Consequently,

$$\lambda(f) \le \int_X f \, d\mu. \tag{40}$$

The reverse inequality follows by replacing f by $-f$ in Equation (40).

Finally, the case of general continuous $f : X \to \mathbb{C}$ is reduced to the case f is real by considering the real and imaginary parts of f separately. □

Lemma 14.17. *If μ_1, μ_2 are regular Borel measures such that*

$$\lambda(f) = \int_X f \, d\mu_j$$

for $j = 1, 2$ and $f \in C(X)$, then $\mu_1 = \mu_2$. †

Proof. Let K be a given compact set. By outer regularity, given $\epsilon > 0$ there exists an open set V such that $K \subset V$ and $\mu_j(V) \le \mu_j(K) + \epsilon$. By Theorem 14.4, there is an $f \in C(X)$ such that $\mathbf{1}_K \le f \le \mathbf{1}_V$. Hence,

$$\mu_j(V) - \epsilon \le \mu_j(K) = \int_X \mathbf{1}_K \, d\mu_j \le \int_X f \, d\mu_j = \lambda(f) \le \int_X \mathbf{1}_V \, d\mu_j = \mu_j(V).$$

Hence $|\mu_1(K) - \mu_2(K)| \le 2\epsilon$ and therefore $\mu_1(K) = \mu_2(K)$. By inner regularity, it now follows that $\mu_1 = \mu_2$. □

15. Product measures

We now revisit measures and σ-algebras. Recall, given measure spaces $(X, \mathscr{M})$ and $(Y, \mathscr{N})$, the product σ-algebra $\mathscr{M} \otimes \mathscr{N} \subset 2^{X \times Y}$ is the σ-algebra generated by the measurable rectangles $\{E \times F : E \in \mathscr{M},\ F \in \mathscr{N}\}$.

Example 15.1. (a) If X, Y are finite sets and X, Y are given the discrete σ-algebras $2^X, 2^Y$, then $2^X \otimes 2^Y = 2^{X \times Y}$.
(b) If we take two copies of $\mathbb{R}$ with the Borel σ-algebra $\mathscr{B}_{\mathbb{R}}$, then $\mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}} = \mathscr{B}_{\mathbb{R}^2}$. (See Proposition 1.17.)

△

Given a pair of measure spaces $(X, \mathscr{M}, \mu)$, $(Y, \mathscr{N}, \nu)$, we would like to construct a "product" measure $\mu \times \nu$ on the Cartesian product measurable space $(X \times Y, \mathscr{M} \times \mathscr{N})$. It is natural to insist, if $E \in \mathscr{M}$ and $F \in \mathscr{N}$ have finite measure, then $\mu \times \nu(E \times F) = \mu(E)\,\nu(F)$; i.e., the measure of a measurable rectangle is the product of the measures. We would also like conditions guaranteeing uniqueness. We will state and prove theorems for only for two factors, but there is no difficulty in extending to finitely many factors $(X_j, \mathscr{M}_j, \mu_j)$, $j = 1, \ldots n$. It turns out that the product is associative too. There is also a construction valid for infinitely many factors when each factor is a probability space (that is, $\mu(X) = 1$) but this requires more care. In these notes we consider only the finite case.

To express, in what follows, integrals of unsigned functions $f : X \times Y \to [0, +\infty]$ against product measures as iterated integrals, we introduce the *slice functions* $f_x : Y \to [0, +\infty]$ and $f^y : X \to [0, +\infty]$, defined for each $x \in X$ (respectively, each $y \in Y$) by

$$f_x(y) := f(x, y), \quad f^y(x) = f(x, y).$$

In other words, starting with $f(x, y)$ we get functions defined on Y by holding x fixed, and functions defined on X by holding y fixed.

In addition to these, given a set $E \subset X \times Y$, we can define for all $x \in X$, $y \in Y$ the *slice sets* $E_x \subset Y$, $E^y \subset X$ by

$$E_x := \{y \in Y : (x, y) \in E\}, \quad E^y := \{x \in X : (x, y) \in E\}$$

We first show that these constructions preserve measurability.

Lemma 15.2. *Let $(X, \mathscr{M}), (Y, \mathscr{N})$ be measurable spaces.*

(i) If E belongs to the product σ-algebra $\mathscr{M} \otimes \mathscr{N}$, then for all $x \in X$ and $y \in Y$ the slice sets E_x and E^y belong to $\mathscr{N}$ and $\mathscr{M}$ respectively.
(ii) If $(Z, \mathscr{O})$ is another measurable space and $f : X \times Y \to Z$ is a measurable function, then for all $x \in X$ and $y \in Y$, the functions f_x and f^y are measurable on Y and X respectively.

†

Proof. Let $\mathscr{S}$ denote the set of all $E \in 2^{X \times Y}$ with the property that $E^y \in \mathscr{M}$ and $E_x \in \mathscr{N}$ for all $x \in X$, $y \in Y$. It suffices to prove that $\mathscr{S}$ is a σ-algebra containing all measurable rectangles. First observe that $\mathscr{S}$ contains all rectangles in $\mathscr{M} \otimes \mathscr{N}$, since if $E = F \times G$ then E_x is equal to either G or $\varnothing$, if $x \in F$ or $x \notin F$ respectively. In either case $E_x \in \mathscr{N}$. The same proof works for E^y. Next, suppose (E_n) is a sequence of disjoint sets in $\mathscr{S}$ and $E = \bigcup_{n=1}^{\infty} E_n$. Then $E_x = \bigcup_{n=1}^{\infty} (E_n)_x \in \mathscr{N}$; similarly $E^y = \bigcup_{n=1}^{\infty} E_n^y \in \mathscr{M}$. Likewise $(E \cap F)_x = E_x \cap F_x$. Thus if $E, F \in \mathscr{S}$, then so is $E \cap F$. Finally, $(E^c)_x = (E_x)^c$ for all $x \in X$; similarly for E^y. Thus, by Proposition 1.7, $\mathscr{S}$ is a σ-algebra.

Item (ii) follows from item (i) by observing that for any $O \subset \mathscr{O}$ and $x \in X$,

$$(f_x)^{-1}(O) = (f^{-1}(O))_x$$

and similarly for y. □

Remark 15.3. Even if both $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ are complete measure spaces and if τ is a measure on $\mathscr{M} \otimes \mathscr{N}$ such that $\tau(E \times F) = \mu(E)\,\nu(F)$ for all measurable rectangles $E \times F$ with $\mu(E), \nu(F) < \infty$, it need not be the case that the product measure $(X \times Y, \mathscr{M} \otimes \mathscr{N}, \tau)$ is complete. Indeed, if there is an set $E \subset X$ such that $E \notin \mathscr{M}$ that is contained in a set G of finite measure and a nonempty $F \in \mathcal{N}$ of measure zero, then $E \times F \subset G \times F$ and $\tau(G \times F) = 0$, but, for any $p \in F$, the slice set $(E \times F)^p = \{x \in X : (x, p) \in F\} = E$ is not in $\mathscr{M}$, and hence $E \times F$ is not in $\mathscr{M} \otimes \mathscr{N}$ by Lemma 15.2. ⋄

The following Lemma is of independent interest.

Definition 15.4. Let X be a set. A *monotone class* is a collection $\mathscr{C} \subset 2^X$ of subsets of X such that

(i) if $E_1 \subset E_2 \subset \cdots$ belong to $\mathscr{C}$, so does $\bigcup_{n=1}^{\infty} E_n$; and
(ii) if $E_1 \supset E_2 \supset \cdots$ belong to $\mathscr{C}$, so does $\bigcap_{n=1}^{\infty} E_n$.

$\triangleleft$

Remark 15.5. It is immediate that intersections of monotone classes are monotone classes. Hence, given a collection $\mathcal{A} \subset 2^X$, there is a smallest monotone class containing $\mathcal{A}$. If $\mathscr{C}$ is a monotone class, then so is $\mathscr{C}' = \{E^c : E \in \mathscr{C}\}$. Trivially, every σ-algebra is a monotone class. The next lemma is a partial converse to this statement. $\diamond$

Lemma 15.6 (Monotone class lemma). *If $\mathscr{A} \subset 2^X$ is a Boolean algebra, then the smallest monotone class containing $\mathscr{A}$ is equal to the σ-algebra generated by $\mathscr{A}$.* $\dagger$

Proof. Let $\mathscr{M}$ denote the σ-algebra generated by $\mathscr{A}$ and $\mathscr{C}$ the smallest monotone class containing $\mathscr{A}$. Since $\mathscr{M}$ is a monotone class containing $\mathscr{A}$, we have $\mathscr{C} \subset \mathscr{M}$ and hence it suffices to prove that $\mathscr{M} \subset \mathscr{C}$.

Now $\mathscr{C}'$ is a monotone class and, since $E \in \mathscr{A}$ implies $E^c \in \mathscr{A}$, it contains $\mathscr{A}$. Hence $\mathscr{C} \subset \mathscr{C}'$. Thus, if $E \in \mathscr{C}$, then there is an $F \in \mathscr{C}$ such that $E = F^c$ and $E^c = F \in \mathscr{C}$. Thus $\mathscr{C}$ is closed under complements; that is $\mathscr{C}' = \mathscr{C}$.

Given $E \subset X$, let $\mathscr{C}_E$ denote the set of all $F \in \mathscr{C}$ such that the sets

$$F \setminus E, \quad E \setminus F, \quad F \cap E, \quad X \setminus (E \cup F)$$

belong to $\mathscr{C}$. A quick check of the definitions shows that $\mathscr{C}_E$ is a monotone class. Moreover, if $E \in \mathscr{A}$ it is immediate that $\mathscr{C}_E$ contains $\mathscr{A}$ and hence $\mathscr{C}_E = \mathscr{E}$. Let

$$\mathscr{D} = \{E \in \mathscr{C} : \mathscr{C}_E = \mathscr{C}\}.$$

In particular, $\mathscr{A} \subset \mathscr{D}$. Another definition check shows $\mathscr{D}$ is a monotone class. Thus, $\mathscr{C} \subset \mathscr{D}$ and hence $\mathscr{C} = \mathscr{D}$.

Now suppose that $E, F \in \mathscr{C}$. Since $E \in \mathscr{D}$ and $F \in \mathscr{C}$, it follows that $E \cap F \in \mathscr{C}$. Hence $\mathscr{C}$ is closed under finite intersections. Since it is also closed under complements, it is closed under finite unions. Since $\varnothing \neq \mathscr{C}$ is closed under finite unions, complements and countable increasing unions, it is a σ-algebra. $\square$

The proof strategy employing the monotone class lemma should be clear. To prove that a statement P holds for a σ-algebra $\mathscr{M}$ generated by a Boolean algebra $\mathscr{A}$, it suffices to prove 1) P is true for all $E \in \mathscr{A}$, and 2) the collection of all $E \in \mathscr{M}$ for which P is true is a monotone class.

We can now construct the product measure.

Theorem 15.7 (Existence and uniqueness of product measure). *Let $(X, \mathscr{M}, \mu)$, $(Y, \mathscr{N}, \nu)$ be σ-finite measure spaces. If $P \in \mathscr{M} \otimes \mathscr{N}$, then*

(i) $f : X \to [0, \infty]$ defined by $f(x) = \nu(P_x) = \int_Y \mathbf{1}_{P_x} \, d\nu$ is measurable;
(ii) $g : Y \to [0, \infty]$ defined by $g(y) = \mu(P^y) = \int_X \mathbf{1}_{P^y} \, d\mu$ is measurable; and

(iii)

$$\int_X f\,d\mu = \int_Y g\,d\nu.$$

The function $\mu \times \nu : \mathscr{M} \otimes \mathscr{N} \to [0,\infty]$ *defined*

$$\mu \times \nu(P) = \int_X f\,d\mu = \int_Y g\,d\nu$$

is a σ*-finite measure on the product* σ*-algebra and is uniquely determined by* $\mu \times \nu(E \times F) = \mu(E)\,\nu(F)$ *for* $E \in \mathscr{M}$ *and* $F \in \mathscr{N}$*.*

The measure $\mu \times \nu$ is the *product measure.*

Proof. We will give the proof assuming at one point both measures are finite, and then sketch out how this assumption can be relaxed to σ-finiteness. Given sets $E \in \mathscr{M}$ and $F \in \mathscr{N}$, the set $E \times F$ is (measurable) rectangle. The collection of finite disjoint unions of measurable rectangles, denoted $\mathscr{E}$ is a Boolean algebra. Let $\mathscr{P}$ denote the collection of sets $P \in \mathscr{M} \otimes \mathscr{N}$ satisfying $(i), (ii)$ and (iii). Define $\mu \times \nu : \mathscr{P} \to [0,\infty]$ as in the statement of the theorem.

That each measurable rectangle belongs to $\mathscr{P}$ is evident. In fact, if $P = \cup E_j \times F_j$ is a finite disjoint union of rectangles, then

$$\nu(P_x) = \sum \mathbf{1}_{E_j}(x)\,\nu(F_j).$$

Hence, $\nu(P_x)$ is measurable and similarly for P^y. Moreover,

$$\int_Y \nu(P_x)\,d\nu = \sum \mu(E_j)\nu(F_j) = \int_X \mu(P^y)\,d\mu.$$

Now suppose $P_1 \subset P_2 \subset \dots$ is an increasing sequence from $\mathscr{P}$ and let $P = \cup P_j$. Let

$$f_j(x) = \nu((P_j)_x)$$

and define g_j similarly. Since $P_x = (\cup P_j)_x = \cup (P_j)_x$, it follows from monotone convergence for sets (Theorem 2.3 (iii)) that (f_j) monotone increases to

$$f(x) = \nu(P_x).$$

Hence f and likewise g are measurable and moreover, by MCT twice,

$$\int_X f\,d\mu = \lim \int_X f_j\,d\mu = \lim \int_Y g_j\,d\nu = \int_Y g\,d\nu.$$

Hence $P \in \mathscr{P}$ and $\mu \times \nu(P) = \lim \mu \times \nu(P_j)$.

At this point we add the assumption that μ and ν are both finite. Suppose $P_1 \supset P_2 \supset \dots$ is a decreasing sequence from $\mathscr{P}$ and let $P = \cup P_j$. Proceeding as above, but using the DCT instead of the MCT by invoking the finiteness assumptions on μ and ν it follow that $P \in \mathscr{P}$ and the proof that $\mathcal{P} = \mathscr{M} \otimes \mathscr{N}$ is complete under the assumption that the measures μ and ν are finite. For disjoint sets $P_1, \dots, P_n \in \mathscr{M} \otimes \mathscr{N}$, the identity $\cup_{j=1}^n (P_j)_x = (\cup_{j=1}^n P_j)_x$ implies $\mu \times \nu$ is finitely additive. The argument above shows

$\mu \times \nu$ satisfies monotone convergence for sets and hence, by Problem 7.8, $\mu \times \nu$ is indeed a measure on $\mathscr{M} \otimes \mathscr{N}$ under the added finiteness assumption.

In the case the measures are σ-finite, express $X = \cup X_n$ and $Y = \cup Y_n$ as increasing unions of measurable sets of finite measure. Let $Z_n = X_n \times Y_n \in \mathscr{E}$ and note that each $Z_n \in \mathscr{P}$, each $\mu \times \nu(Z_n)$ is finite and $X \times Y = \cup Z_n$. In particular, once it is shown that $\mu \times \nu$ is a measure on the product σ-algebra, the σ-finite conclusion will automatically follow. For positive integers n, let $\mathscr{Q}_n$ denote those sets P such that

$$P_n = P \cap [X_n \times Y_n] \in \mathscr{P}.$$

In particular, $P_n \subset Z_n$ and the measures $\mu_n : \mathscr{M} \to [0, \infty]$ and $\nu : \mathscr{N} \to [0, \infty]$ defined by $\mu_n(E) = \mu(E \cap X_n)$ and $\nu_n(F) = \nu(F \cap Y_n)$ respectively are finite and $P \in \mathscr{Q}$ if and only if $P \in \mathscr{M} \otimes \mathscr{N}$ and

$$\int_X \nu_n(P_x)\, d\mu_n = \int_X \nu((P_n)_x)\, d\mu = \int_Y \mu((P_n)^y)\, d\nu = \int_Y \mu_n(P_x)\, d\nu_n.$$

Hence, by what has already been proved, $\mathscr{Q}_n = \mathscr{M} \otimes \mathscr{N}$ for each n. Given $P \in \mathscr{P}$ let $f_n(x) = \mu((P_n)_x)$ and $f(x) = \mu(P_x)$ and likewise for g_n. The monotone convergence argument above shows f is measurable and $(\int f_n\, d\mu)$ converges to $\int f\, d\mu$ and likewise for g. On the other hand $\int f_n\, d\mu = \int g_n\, d\nu$ since $P \in \mathscr{Q}_n$. Thus $\mathscr{P} = \mathscr{M} \otimes \mathscr{N}$. That $\mu \times \nu$ is measure on $\mathscr{M} \otimes \mathscr{N}$ is left as an exercise.

To prove uniqueness suppose ρ is any other measure on $\mathscr{M} \otimes \mathscr{N}$ such that $\rho(E \times F) = \mu(E)\, \nu(F)$ for measurable rectangles. Thus ρ agrees with $\mu \times \nu$ on the Boolean algebra $\mathscr{E}$ and $\mu \times \nu$ is σ-finite on $\mathscr{E}$. Hence ρ agrees with $\mu \times \nu$ on all of $\mathscr{M} \otimes \mathscr{N}$ by the Hahn Uniqueness Theorem (Theorem 5.4). □

Corollary 15.8. *Let $(X, \mathscr{M}, \mu)$, $(Y, \mathscr{N}, \nu)$ be σ-finite measure spaces. If E is a null set for $\mu \times \nu$, then $\nu(E_x) = 0$ for μ-a.e. $x \in X$, and $\mu(E^y) = 0$ for ν-a.e. $y \in Y$.* †

Proof. Problem 19.5. □

Example 15.9. a) If X, Y are at most countable, and μ_X, μ_Y denote counting measure on $X \times Y$ respectively, then $2^X \otimes 2^Y = 2^{X \times Y}$ and $\mu_X \times \mu_Y$ is counting measure on $X \times Y$.

b) For two copies of $\mathbb{R}$ with the Borel σ-algebra and Lebesgue measure m (restricted to $\mathscr{B}_{\mathbb{R}}$), the product measure is a σ-finite measure on $\mathscr{B}_{\mathbb{R}^2}$ that has the value $m(E)m(F)$ on measurable rectangles. The *completion* of this measure is Lebesgue measure on $\mathbb{R}^2$. (By iterating this construction we of course obtain Lebesgue measure on $\mathbb{R}^n$.)

Let $\mathscr{L}$ denote the Lebesgue σ-algebra on $\mathbb{R}$ and $\mathscr{L}_{\mathbb{R}^2}$ denote Lebesgue measure on $\mathbb{R}^2$. If $E \in \mathscr{L}$, then $E = B \cup W$, where B is Borel and W has Lebesgue measure zero (and hence is a subset of a Borel set of measure zero) by the regularity properties of Lebesgue measure. It follows that if $E, F \in \mathscr{L}$, then $E \times F$ is the union of a set in $\mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}}$ with a set contained within a set of measure zero in $\mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}}$ and hence $E \times F \in \mathscr{L}_{\mathbb{R}^2}$. Thus, $\mathscr{L} \otimes \mathscr{L} \subset \mathscr{L}_{\mathbb{R}^2}$. On the other hand, equality does not hold by Remark 15.3.

△

Theorem 15.10 (Tonelli's theorem, first version)**.** *Suppose $(X,\mathscr{M},\mu)$ and $(Y,\mathscr{N},\nu)$ are σ-finite measure spaces. If $f : X\times Y \to [0,+\infty]$ is a $\mathscr{M}\otimes\mathscr{N}$-measurable function, then*

(a) the slice integrals $g(x) := \int_Y f_x(y)\,d\nu(y)$ and $h(y) := \int_X f^y(x)\,d\mu(x)$ are measurable on X and Y respectively;

(b) $\displaystyle\int_{X\times Y} f\,d(\mu\times\nu) = \int_X \left(\int_Y f_x(y)\,d\nu(y)\right) d\mu(x) = \int_Y \left(\int_X f^y(x)\,d\mu(x)\right) d\nu(y)$; and

(c) if $f\in L^1(\mu\times\nu)$, then f_x and f^y are in $L^1(\nu)$ and $L^1(\mu)$ for a.e. x and a.e y.

Proof. First suppose $P\in\mathscr{M}\otimes\mathscr{N}$ and let $f = \mathbf{1}_P$. In this case, the result is the conclusion of Theorem 15.7.

To move to general unsigned f, first note that by linearity we conclude immediately that items (a) and (b) also hold for simple functions. For a general unsigned measurable $f : X\times Y\to[0,+\infty]$, use the Ziggurat approximation to choose an increasing a sequence (f_n) of measurable simple functions converging to f pointwise. Let

$$g_n(x) := \int_Y (f_n)_x(y)\,d\nu(y) \quad\text{and}\quad h_n(y) := \int_X (f_n)^y(x)\,d\mu(x).$$

The monotone convergence theorem implies that the sequences (g_n) and (h_n) increase and converge pointwise to g and h respectively. Thus g and h are measurable. Two more applications of monotone convergence gives

$$\int_X g\,d\mu = \lim\int_X g_n\,d\mu = \lim\int_{X\times Y} f_n\,d(\mu\times\nu) = \int_{X\times Y} f\,d(\mu\times\nu)$$

and similarly for h. Thus, finally, items (a) and (b) hold for all unsigned measurable functions on $X\times Y$.

Item (c) follows immediately from item (b) and Theorem 10.2 item (d). □

As noted above, the product of complete measures is almost never complete. Typically we pass to the completion $\overline{\mu\times\nu}$ of a product measure. To prove the complete version of Tonelli's theorem recall a couple of facts about measurability on complete measure spaces encountered earlier (see Propositions 8.16 and 8.17).

Proposition 15.11. *Let $(X,\mathscr{M},\mu)$ be a measure space and $(X,\overline{\mathscr{M}},\overline{\mu})$ its completion.*

a) If $f : X\to\mathbb{C}$ is $\overline{\mathscr{M}}$-measurable, then there exists a $\mathscr{M}$-measurable function $\widetilde{f}$ such that $f = \widetilde{f}$ $\overline{\mu}$-a.e.

b) If $f : X\to\mathbb{C}$ is $\overline{\mathscr{M}}$-measurable and $g : X\to\mathbb{C}$ is a function with $g(x) = f(x)$ for $\overline{\mu}$-a.e. x, then g is $\overline{\mathscr{M}}$-measurable.

†

Proof. Problem 19.4. □

Theorem 15.12 (Tonelli's theorem, complete version)**.** *Let* $(X, \mathscr{M}, \mu)$, $(Y, \mathscr{N}, \nu)$ *be complete* σ*-finite measure spaces. If* $f : X \times Y \to [0, +\infty]$ *is an* $\overline{\mathscr{M} \otimes \mathscr{N}}$*-measurable function, then*

(i) for μ*-a.e.* x *and* ν*-a.e.* y*, the functions* f_x *and* f^y *are* $\mathscr{N}$ *and* $\mathscr{M}$ *measurable respectively;*

(ii) there exists $\mathscr{M}$ *and* $\mathscr{N}$ *measurable functions* g *and* h *such that*
$$g(x) = \int_Y f_x(y)\, d\nu(y), \quad h(y) = \int_X f^y(x)\, d\mu(x)$$
μ*-a.e. and* ν*-a.e. respectively;*

(iii) $\displaystyle\int_{X\times Y} f(x,y)\, d\overline{\mu \times \nu} = \int_X \left(\int_Y f_x(y)\, d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f^y(x)\, d\mu(x) \right) d\nu(y)$*; and*

(iv) If f *is* L^1*, then* f_x *and* f^y *are in* L^1 *for* μ*-a.e.* x *and* ν*-a.e.* y *respectively.*

In item (ii), the integrals are defined only almost everywhere since the integrands are defined only almost everywhere.

Proof. From Proposition 15.11 that there exists an $\mathscr{M} \otimes \mathscr{N}$-measurable function $\hat{f}$ such that $\hat{f}(x,y) = f(x,y)$ for $\overline{\mu \times \nu}$-a.e. (x,y). Let E be the exceptional set on which $f \neq \hat{f}$. Since $\overline{\mu \times \nu}(E) = 0$, there is an $\mathscr{M} \otimes \mathscr{N}$-measurable set $\hat{E}$ containing E such that $(\mu \times \nu)(\hat{E}) = 0$ by Theorem 2.8. By Corollary 15.8, $\nu(\hat{E}_x) = 0$ for μ-a.e. x, thus since $E_x \subset \hat{E}_x$ (and since ν is complete!) E_x is in $\mathscr{N}$ and $\nu(E_x) = 0$ as well for almost every x. Since $E_x = \{y : \hat{f}_x \neq f_x\}$, it follows that $f_x = \hat{f}_x$ ν-a.e. y for μ-a.e. x. Thus, by Lemma 15.2 and completeness of ν, the function f_x is $\mathscr{N}$-measurable (Proposition 15.11 again) μ-a.e. x. Of course, the analogous proof holds for f^y.

By Theorem 15.10 (Tonelli),
$$\hat{g}(x) = \int_Y \hat{f}_x d\nu$$
is measurable. Hence, as $\hat{f}_x = f_x$ ν-a.e. y for μ-a.e. x,
$$g(x) = \int_Y \hat{f}_x\, d\nu = \int_Y f_x\, d\nu \quad \mu\text{-a.e. } x.$$

Finally (iii) and (iv) follow from (i) and (ii) and Theorem 15.10 applied to $\hat{f}$. $\square$

Theorem 15.13 (Fubini's theorem)**.** *Let* $(X, \mathscr{M}, \mu)$, $(Y, \mathscr{N}, \nu)$ *be complete* σ*-finite measure spaces. If* $f : X \times Y \to \mathbb{C}$ *belongs to* $L^1(\overline{\mu \times \nu})$*, then*

a) for μ*-a.e.* x *and* ν*-a.e.* y*, the functions* f_x *and* f^y *belong to* $L^1(\nu)$ *and* $L^1(\mu)$ *respectively, and the functions*
$$g(x) = \int_Y f_x(y)\, d\nu(y), \quad h(y) = \int_X f^y(x)\, d\mu(x)$$
belong to $L^1(\mu)$ *and* $L^1(\nu)$ *respectively; and*

$$b)\ \iint_{X\times Y} f(x,y)\, d\overline{\mu\times\nu} = \int_X \left(\int_Y f_x(y)\, d\nu(y)\right) d\mu(x) = \int_Y \left(\int_X f^y(x)\, d\mu(x)\right) d\nu(y)$$

Proof. By taking real and imaginary parts, and then positive and negative parts, it suffices to consider the case that f is unsigned, but then the theorem follows from Theorem 15.12. Indeed, when f is unsigned and belongs to $L^1(\overline{\mu\times\nu})$, by Tonelli

$$\int_X \left(\int_Y f_x(y)\, d\nu(y)\right) d\mu(x) = \iint_{X\times Y} f(x,y)\, d(\overline{\mu\times\nu}) < \infty,$$

but then $\int_Y f_x(y)\, d\nu(y) < \infty$ for μ-a.e. x; similarly for f^y. □

Corollary 15.14 (Integral as the area under a graph)**.** *Let $(X, \mathscr{M}, \mu)$ be a σ-finite measure space, and give $\mathbb{R}$ the Borel σ-algebra $\mathscr{B}_\mathbb{R}$ and Lebesgue measure m (restricted to $\mathscr{B}_\mathbb{R}$). An unsigned function $f : X \to [0, +\infty)$ is measurable if and only if the set*

$$G_f := \{(x,t) \in X \times \mathbb{R} : 0 \le t \le f(x)\}$$

is measurable. In this case,

$$(\mu \times m)(G_f) = \int_X f\, d\mu.$$

†

Corollary 15.15 (Distribution formula)**.** *Let $(X, \mathscr{M}, \mu)$ be a σ-finite measure space. If $f : X \to [0, +\infty]$ an unsigned measurable function, then*

$$\int_X f(x)\, d\mu(x) = \int_{[0,+\infty]} \mu(\{f \ge t\})\, dt.$$

†

Proof. Let G_f be the region under the graph of f as in Corollary 15.14. Then for fixed $t \ge 0$,

$$\int_X \mathbf{1}_{G_f}(x,t)\, d\mu(x) = \mu(\{f \ge t\})$$

so by Tonelli's theorem and Corollary 15.14,

$$\int_X f(x)\, d\mu(x) = (\mu\times m)(G_f) = \int_{[0,+\infty]} \left(\int_X \mathbf{1}_{G_f}(x,t)\, d\mu(x)\right) dt = \int_{[0,+\infty]} \mu(\{f \ge t\})\, dt.$$

□

Corollary 15.16 (Compatibility of the Riemann and Lebesgue integrals)**.** *If $f : [a,b] \to \mathbb{R}$ is continuous, then if we extend f to be 0 off $[a,b]$, the extended f is Lebesgue integrable on $\mathbb{R}$ and $\int_\mathbb{R} f\, dm = \int_a^b f(x)\, dx$.* †

Proof (sketch). We assume $f \geq 0$. For a partition P, $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$, define $C_j = \sup\{f(x) : x_j \leq x \leq x_{j+1}\}$ and $c_j = \sup\{f(x) : x_j \leq x \leq x_{j+1}\}$, and consider the sums

$$U(P,f) := \sum_{j=1}^{n} C_j(x_j - x_{j-1}) \quad \text{and} \quad L(P,f) := \sum_{j=1}^{n} c_j(x_j - x_{j-1}).$$

Let G_f denote the region enclosed by the graph of f, $G_f = \{(x,y) : 0 \leq y \leq f(x)\}$. Thus G_f is a closed set in $\mathbb{R}^2$ (hence Borel measurable), and by 15.14 $m^2(G_f) = \int_{\mathbb{R}} f\, dm$. Let R_+ and R_- denote the finite unions of closed rectangles corresponding to the over- and under-estimates for the Riemann integral given by $U(P,f)$ and $L(P,f)$. Then $R_- \subset G_f \subset R_+$, and $m^2(R_+) = U(P,f), m^2(R_-) = L(P,f)$. It then follows that $\sup_P L(P,f) \leq m^2(G_f) \leq \inf_P U(P,f)$. But by the definition of the Riemann integral, the inf and sup are equal to each other, and their common value is $\int_a^b f(x)\, dx$. □

Remark 15.17. The above proof can be modified to drop the continuity hypothesis (where was it used?), and conclude that every Riemann integrable function on $[a,b]$ is Lebesgue integrable, and the values of the two integrals agree. With more work it can be shown that a function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if the set of points where f is discontinuous has Lebesgue measure 0. We will not prove this fact in these notes.

We also note that it is not difficult to extend these facts about the Riemann integral to "improper" Riemann integrals, defined over $[0,+\infty)$ or $\mathbb{R}$. In particular, note that the distribution function $\mu(\{|f| \geq t\})$ is a decreasing function of t on $[0,+\infty)$, hence Riemann integrable. Thus Corollary 15.15 says that, in principle, the calculation of any Lebesgue integral can be reduced to the computation of a Riemann integral. ◇

16. Integration in $\mathbb{R}^n$

In this section we briefly discuss Lebesgue measure and Lebesgue integration on $\mathbb{R}^n$.

We begin with the observation that we can construct Lebesgue measure m^n on $\mathbb{R}^n$ in the same way as on $\mathbb{R}$, namely by introducing *boxes* $B = I_1 \times I_2 \times \cdots \times I_n$, where each I_j is an interval in $\mathbb{R}$, and declaring $|B| = \prod_{j=1}^n |I_j|$. One can then define *Lebesgue outer measure* m^{n*} by defining, for all $E \subset \mathbb{R}^n$,

$$m^{n*}(E) = \inf\{\sum_{j=1}^{\infty} |B_j| : E \subset \bigcup_{j=1}^{\infty} B_j\};$$

the infimum taken over all coverings of E by boxes. By imitating the constructions of Section 4, we are led to a σ-finite Borel measure on $\mathbb{R}^n$ such that the measure of a box B is its volume $|B|$. Since the construction proceeds through outer measure, the σ-algebra $\mathscr{L}_{\mathbb{R}^n}$ of measurable sets is complete and is of course called the *Lebesgue* σ-algebra. In particular, the following analog of Theorem 4.5 holds.

Theorem 16.1. *Let $E \subset \mathbb{R}^n$. The following are equivalent:*

a) E is Lebesgue measurable.

b) For every $\epsilon > 0$, there is an open set $U \supset E$ such that $m^{n*}(U \setminus E) < \epsilon$.
c) For every $\epsilon > 0$, there is a closed set $F \subset E$ such that $m^{n*}(E \setminus F) < \epsilon$.
d) There is a G_δ set G such that $E \subset G$ and $m^{n*}(G \setminus E) = 0$.
e) There is an F_σ set F such that $E \supset F$ and $m^{n*}(E \setminus F) = 0$.

We drop the superscript and just write m for Lebesgue measure on $\mathbb{R}^n$ when the dimension is understood. It follows from Theorem 16.1, if $E \subset \mathbb{R}^n$ is Lebesgue measurable and $\mu(E) = 0$, then there is a Borel set $G \supset E$ such that $m(G) = 0$. Thus m is the completion of m restricted to $\mathscr{B}_{\mathbb{R}^n}$ as described in Theorem 2.8. Now, let m' denote the n-fold product of Lebesgue measure restricted to $\mathscr{B}_{\mathbb{R}}$ defined on $\mathscr{B}_{\mathbb{R}} \otimes \cdots \otimes B_{\mathbb{R}} = \mathscr{B}_{\mathbb{R}^n}$. The measures m and m' agree on the Boolean algebra of disjoint union of boxes and thus, by the Hahn Uniqueness theorem, agree on $\mathscr{B}_{\mathbb{R}^n}$. Finally, the completion of m' agrees with m and the completion of $\mathscr{B}_{\mathbb{R}^n}$ (with respect to m') is $\mathscr{L}_{\mathbb{R}^n}$, the Lebesgue σ-algebra.

Definition 16.2. *Lebesgue measure m^n* on $\mathbb{R}^n$ is the completion of the n-fold product of $(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, m)$ and the completion of $\mathscr{B}_{\mathbb{R}^n}$ is the Lebesgue σ-algebra denoted $\mathscr{L}_{\mathscr{R}^n}$. ◁

$\mathbb{R}^n$ possesses a larger group of symmetries than $\mathbb{R}$ does. In particular we would like to analyze the behavior of Lebesgue measure under invertible linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$. We have the following analog of Theorem 4.4:

Theorem 16.3. *If $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation, then T is $\mathscr{L}_{\mathbb{R}^n} - \mathscr{L}_{\mathbb{R}^n}$ measurable; i.e., if $E \subset \mathbb{R}^n$ is a Lebesgue set, then $T^{-1}(E) \subset \mathbb{R}^n$ is a Lebesgue set too. Moreover,*

(a) if $f \in L^1(\mathbb{R}^n)$, $\displaystyle\int_{\mathbb{R}^n} (f \circ T)(x)\,dx = \frac{1}{|\det T|} \int_{\mathbb{R}^n} f(x)\,dx$; and
(b) if $E \in \mathbb{R}^n$ is Lebesgue measurable, then $m(T(E)) = |\det T| m(E)$.

Problem 13.7 gives an example of a Lebesgue measurable F and continuous G such that $F \circ G$ is not measurable. The difficulty is that it is possible $E = F^{-1}(B)$ is not Borel for a Borel measurable B and in this case there is no guarantee the inverse image of E under G will be a Lebesgue set. In the course of the proof it will be shown that if $E \subset \mathbb{R}^n$ is a Lebesgue set and $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear and invertible, then $T^{-1}(E)$ is Lebesgue.

Proof. Let $\mathcal{F}$ denote those $f \in L^1(\mathbb{R}^n)$ such that the composition $f \circ S$ is measurable for all invertible linear transformations $S : \mathbb{R}^n \to \mathbb{R}^n$.

Note that, if T_1 and T_2 are both invertible linear transformations and the result of (a) holds for any $f \in \mathcal{F}$ and both T_1 and T_2, then the result of (a) holds for all $f \in \mathcal{F}$ and $T = T_1 T_2$ (and $T_2 T_1$). From linear algebra, every invertible linear transformation of $\mathbb{R}^n$ is a finite product of transformations of one of the following types (we write vectors in $\mathbb{R}^n$ as $x = (x_1, \ldots x_n)$, in the standard basis).

i) (Scaling a row) $T(x_1, \ldots x_j, \ldots x_n) = (x_1, \ldots cx_j, \ldots x_n)$, for some $j = 1, \ldots n$ and some $c \in \mathbb{R}$

ii) (adding a row to another) $T(x_1, \dots x_j, \dots x_k, \dots x_n) = (x_1, \dots x_j, \dots x_j + x_k, \dots x_n)$, some $j, k = 1, \dots n$

iii) (interchanging rows) $T(x_1, \dots x_j, \dots x_k, \dots x_n) = (x_1, \dots x_k, \dots x_j, \dots x_n)$, some $j, k = 1, \dots n$.

In the first case, $\det T = c$, in the second, $\det T = 1$, and in the third, $\det T = -1$. By the multiplicativity of the determinant, it suffices to prove the theorem for T of each of these types. We may also assume $f \geq 0$ (why?) But (a) now follows easily from Tonelli's theorem and the invariance properties of one-dimensional Lebesgue measure. For example, for T of type (i) we integrate with respect to x_j first and use the one-dimensional fact

$$\int_{\mathbb{R}} g(ct)\, dt = \frac{1}{|c|} \int g(t)\, dt$$

for all $c \neq 0$. In case (ii) we integrate with respect to x_k first and use translation invariance of one-dimensional Lebesgue measure: for fixed x_j,

$$\int_{\mathbb{R}} g(x_j + x_k)\, dx_k = \int_{\mathbb{R}} g(x_k)\, dx_k,$$

while for case (iii) we simply interchange the order of integration with respect to x_j and x_k. Thus (a) holds in all three cases. By composition (a) holds for any invertible T and $f \in \mathcal{F}$.

If $f \in L^1(\mathbb{R}^n)$ is Borel measurable, then $f \in \mathcal{F}$ and hence (a) holds. In particular, if G is a Borel set, then (a) applied to $\mathbf{1}_{T(G)}$ (using T^{-1} is linear and continuous shows $T(G)$ is also a Borel set) shows (b) holds for G. In particular, if $m(G) = 0$, then $m(T(G)) = 0$ too. Now suppose E is a Lebesgue set. In this case there exists a Borel set G with $m(G) = 0$, a subset $N \subset G$ and a Borel set F such that $E = F \cup N$. Hence, as T is one-one, $T(E) = T(F) \cup T(N)$ and $T(N)$ is a subset of the Borel set $T(G)$ of measure zero. It follows that $T(E)$ is a Lebesgue measurable set and $\det(T)\, m(T(E)) = \det(T)\, (m(T(F)) = m(F) = m(E)$. We conclude, if T is an invertible linear transformation, then T maps Lebesgue sets to Lebesgue sets (as does T^{-1}.

Finally, since T^{-1} maps Lebesgue sets to Lebesgue sets, if f is measurable, then so is $f \circ T$ and hence $\mathcal{F} = L^1(\mathbb{R}^n)$ completing the proof. □

Corollary 16.4. *Lebesgue measure on $\mathbb{R}^n$ is rotation invariant.* †

Proof. A rotation of $\mathbb{R}^n$ is just a linear transformation satisfying $T^t = T^{-1}$, which implies that $|\det T| = 1$, so the claim follows from Theorem 16.3. □

One result we will use frequently in the rest of the course is the following fundamental approximation theorem. We already know that absolutely integrable functions can be approximated in L^1 by simple functions, we now show that in $\mathbb{R}^n$ we can approximate in L^1 with *continuous* functions.

Definition 16.5. We say a function $f : X \to \mathbb{C}$ is *supported* on a set $E \subset X$ if $f = 0$ on the complement of E. When X is a topological space, the *closed support* of f is equal to the smallest closed set E such that f is supported on E. Say f is *compactly supported* if it is supported on a compact set E. $\triangleleft$

Note that since every bounded set in $\mathbb{R}^n$ has compact closure, a function $f : \mathbb{R}^n \to \mathbb{C}$ is compactly supported if and only if it is supported in a bounded set. Since bounded sets have finite Lebesgue measure, it follows that if $f : \mathbb{R}^n \to \mathbb{C}$ is continuous and compactly supported, then it belongs to $L^1(\mathbb{R}^n)$.

Theorem 16.6. *If $f \in L^1(\mathbb{R}^n)$ then there is a sequence of (f_n) of continuous, compactly supported functions such that (f_n) converges to f in L^1.*

Proof. We work in $\mathbb{R}$ first, and reduce to the case where f is simple. Let $\epsilon > 0$; since simple functions are dense in L^1, there is an L^1 simple function ψ such $\|\psi - f\|_1 < \frac{\epsilon}{2}$. Since ψ is simple and in L^1, it is supported on a set of finite measure. If we can find a continuous $g \in L^1$ such that $\|\psi - g\|_1 < \epsilon/2$ we are done. For this, it suffices (by linearity) to assume $\psi = \mathbf{1}_E$ for a set E with $m(E) < \infty$ and show, given $\delta > 0$ there is a continuous g of compact support such that $\|\mathbf{1}_E - g\| < \delta$. By Littlewood's first principle Theorem 4.6, we can find a set A, a finite union of disjoint open intervals $A = \bigcup_{j=1}^n (a_j, b_j)$, such that $m(A\Delta E) < \frac{\delta}{2}$. It follows that $\|\mathbf{1}_A - \mathbf{1}_E\|_1 = \|\mathbf{1}_{A\Delta E}\|_1 < \frac{\delta}{2}$. Let $\eta = \frac{\delta}{2n}$ and let $J_j = (a_j - \frac{\eta}{2}, b_j + \frac{\eta}{2})$ and choose a continuous function $g_j : \mathbb{R} \to [0, 1]$ such that $g_j = 1$ on I_j and $g_j = 0$ on J_j^c. Thus, $\|g_j - \mathbf{1}_{I_j}\| \leq \eta$. Thus, with $g = \sum g_j$, it follows that $\|g - \mathbf{1}_A\| \leq \sum_{j=1}^n \|g_j - \mathbf{1}_{I_j}\| < \frac{\delta}{2}$.

In higher dimensions, the same approximation scheme works; it suffices (using linearity, Littlewood's first principle, and the $\epsilon/2^n$ trick as before) to approximate the indicator function of a box $B = I_1 \times \cdots \times I_n$ (where each I_j has finite measure) again a piecewise linear function that is 1 on the box and 0 outside a suitably small neighborhood of the box suffices. The details are left as an exercise. $\square$

Remark 16.7. There is a more sophisticated way to do continuous approximation in L^1, using *convolutions*. This will be covered in depth later in the course. Also note that since every L^1 convergent sequence has a pointwise a.e. convergent subsequence, every L^1 function f can be approximated by a sequence of continuous functions f_n that converge to f both in the L^1 norm and pointwise a.e. $\diamond$

As an application of the above approximation theorem, we prove a very useful fact about integration in $\mathbb{R}^n$, namely that translation is continuous in $L^1(\mathbb{R}^n)$. The proof strategy is to first prove the result from scratch for continuous, compactly supported f, then use the density of these functions in L^1 to get the general result. This *density argument* is frequently used; we will see it again in the next section when we prove the Lebesgue Differentiation Theorem.

Proposition 16.8. *For $h \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{C}$ a function, define $f_h(x) := f(x - h)$ (the translation of f by h). If $f \in L^1(\mathbb{R}^n)$, then $f_h \in L^1$ and $f_h \to f$ in the L^1 norm as $h \to 0$.* $\dagger$

Proof. First suppose f is continuous and compactly supported. In this case f is uniformly continuous, each f_h is continuous, and $f_h \to f$ uniformly on K as $h \to 0$. It follows from Proposition 12.11 that $f_h \to f$ in L^1.

Now let $f \in L^1(\mathbb{R}^n)$ and $\epsilon > 0$ be given. We can choose a continuous, compactly supported g such that $\|g - f\|_1 < \epsilon/3$. Note, by the translation invariance of Lebesgue measure, that $\|g_h - f_h\|_1 = \|g - f\|_1 < \epsilon/3$ as well. (Here we have used the readily verified fact that $(|f - g|)_h = |f_h - g_h|$). Now, since the result holds for g, there is a $\delta > 0$ such that for all $|h| < \delta$, $\|g_h - g\|_1 < \epsilon/3$. Thus

$$\|f_h - f\|_1 \leq \|f_h - g_h\|_1 + \|g_h - g\|_1 + \|g - f\|_1 < \epsilon,$$

proving the theorem. □

The section concludes with some remarks on integration in polar coordinates. Write $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$ for the Euclidean length of a vector x. Let $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^n$. Each nonzero vector x can be expressed uniquely in the form $x = \|x\| \frac{x}{\|x\|}$ (positive scalar times a unit vector), so we may identify $\mathbb{R}^n \setminus \{0\}$ with $(0, +\infty) \times S^{n-1}$. Precisely, the map $\Phi(x) = (\|x\|, \frac{x}{\|x\|})$ is a continuous bijection of $\mathbb{R}^n \setminus \{0\}$ and $(0, +\infty) \times S^{n-1}$. Using the map Φ we can define the push-forward of Lebesgue measure to $(0, +\infty) \times S^{n-1}$; namely $m_*(E) = m(\Phi^{-1}(E))$. Let $\rho = \rho_n$ denote the measure on $(0, +\infty)$ defined by $\rho(E) = \int_E r^{n-1}\, dr$.

Theorem 16.9 (Integration in polar coordinates)**.** *There is a unique finite Borel measure $\sigma = \sigma_{n-1}$ on S^{n-1} such that $m_* = \rho \times \sigma$. If f is an unsigned or L^1 Borel measurable function on $\mathbb{R}^n$ then*

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty \int_{S^{n-1}} f(r\xi) r^{n-1}\, d\sigma(\xi) dr.$$

Proof. □

17. Differentiation theorems

One version of the fundamental theorem of calculus says that if f is continuous on a closed interval $[a, b] \subset \mathbb{R}$, if we define the function

$$F(x) := \int_a^x f(t)\, dt$$

then F is differentiable on (a, b) and $F'(x) = f(x)$ for all $x \in (a, b)$. Using the definition of derivative, this can be reformulated as

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt = f(x)$$

for all $x \in (a, b)$. If we let $I(x, h)$ denote the open interval $(x, x + h)$, then, re-expressing in terms of the Lebesgue integral, we have

$$\lim_{h \to 0} \frac{1}{m(I_h)} \int_{I_h} f\, dm = f(x).$$

It is not hard to show that we can replace I_h with the interval $B(x,h)$ centered on x with radius h; in this case $m(B(x,h)) = 1/2h$ and we still have

$$\lim_{h\to 0} \frac{1}{m(B(x,h))} \int_{B(x,h)} f\, dm = f(x).$$

This can be interpreted to say that the average values of f over small intervals centered on x converge to $f(x)$, as one might expect from continuity. Perhaps surprisingly, the result remains true, at least for (Lebesgue) almost every x, when we drop the continuity hypothesis.

The goal of this section is to prove the *Lebesgue differentiation theorem*. To state it we introduce the notation $B(x,r)$ for the open ball of radius $r > 0$ centered at a point $x \in \mathbb{R}^n$. We also write $\int f(y)\, dy$ for integrals against Lebesgue measure. For the rest of this section L^1 refers to Lebesgue measure on $\mathbb{R}^n$ unless stated otherwise.

Definition 17.1. [Locally integrable functions] A Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{C}$ is called *locally integrable* if $\int_K |f(y)|\, dy < \infty$ for every compact set $K \subset \mathbb{R}^n$. The collection of all locally integrable functions on $\mathbb{R}^n$ is denoted $L^1_{loc}(\mathbb{R}^n)$. ◁

Since every compact set in $\mathbb{R}^n$ is contained in a closed ball, it suffices in the above definition to require only $\int_B |f(y)|\, dy < \infty$ for every ball B.

Theorem 17.2 (Lebesgue Differentiation Theorem)**.** *If $f \in L^1_{loc}(\mathbb{R}^n)$, then, for almost every $x \in \mathbb{R}^n$,*

$$a) \quad \lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, dy = 0$$

and

$$b) \quad \lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\, dy = f(x).$$

Notice that the second statement follows from the first. One can interpret the theorem as follows. Given $f \in L^1$, define for each $r > 0$ the function

$$A_{f,r}(x) := \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\, dy,$$

the average value of f over the ball of radius r centered at x. The second statement says that the functions $A_{r,f}$ converge to f almost everywhere as $r \to 0$. (It is not hard to show, using density of continuous functions of compact support in L^1 that if $f \in L^1$, then $A_{r,f} \to f$ in the L^1 norm as $r \to 0$. See Problem 19.15.)

To begin with, it is easy to prove Theorem 17.2 in the continuous case:

Lemma 17.3. *If $f : \mathbb{R}^n \to \mathbb{C}$ is continuous and compactly supported, then for all $x \in \mathbb{R}^n$,*

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, dy = 0.$$

†

Proof of Lemma 17.3. Since f is continuous and compactly supported, it is absolutely integrable. Fix $x \in \mathbb{R}^n$ and let $\epsilon > 0$ be given. By uniform continuity there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $|y - x| < \delta$. For $0 < r < \delta$,

$$\begin{aligned}\frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y) - f(x)|\,dy &< \frac{1}{m(B(x,r))}\int_{B(x,r)} \epsilon\,dy \\ &= \epsilon.\end{aligned}$$

□

To move from continuous, compactly supported f to absolutely integrable f we need the following estimate, which is quite important in its own right. It is an estimate on the *Hardy-Littlewood maximal function*, which is defined for $f \in L^1(\mathbb{R}^n)$ by

$$M_f(x) = \sup_{r>0} \frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y)|\,dy \tag{41}$$

Theorem 17.4 (Hardy Littlewood Maximal Theorem)**.** *If $f : \mathbb{R}^n \to \mathbb{C}$ is in L^1 and $t > 0$, then*

$$m(\{x \in \mathbb{R}^n : M_f(x) > t\}) \le C\frac{\|f\|_1}{t}.$$

for some absolute constant $C > 0$ that depends only on the dimension n.

Remark 17.5. It turns out that C can be chosen as 3^n. If it were the case that M_f were in $L^1(\mathbb{R}^n)$, then a similar estimate would be an immediate consequence of Markov's inequality. However M_f is essentially never in L^1, even in the simplest case of the indicator function of an interval. ◇

Before proving Theorem 17.4, we will see how it is used to prove the Lebesgue Differentiation Theorem.

Proof of Therem 17.2. First note that we may assume $f \in L^1(\mathbb{R}^n)$ (not just L^1_{loc}); to see this just replace f by $\mathbf{1}_{B(0,N)}f$ for $N \in \mathbb{N}$. So, let $f \in L^1(\mathbb{R}^n)$ and fix $\epsilon, t > 0$. We first prove (b) and then use this to deduce (a). First, by Theorem 16.6 there exists a continuous, compactly supported g such that

$$\int_{\mathbb{R}^n} |f(x) - g(x)|\,dx < \epsilon.$$

Applying the Hardy-Littlewood maximal inequality to $|f - g|$, we have

$$m\big(\{x \in \mathbb{R}^n : \sup_{r>0} \frac{1}{m(B(x,r))}\int_{\mathbb{R}^n} |f(x) - g(x)|\,dx > t\}\big) \le \frac{C\epsilon}{t}.$$

In addition, by Markov's inequality applied to $|f - g|$ we have

$$m(\{x \in \mathbb{R}^n : |f(x) - g(x)| > t\}) \le \frac{\epsilon}{t}.$$

Thus there is a set $E \subset \mathbb{R}^n$ of measure less than $\frac{(C+1)\epsilon}{t}$ such that, *outside* of E both

$$\sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - g(y)| \, dy \leq t \tag{42}$$

and

$$|f(x) - g(x)| \leq t. \tag{43}$$

Now consider $x \in E^c$. By the result for continuous, compactly supported functions (Lemma 17.3), we have for all sufficiently small $r > 0$

$$\left| \frac{1}{m(B(x,r))} \int_{B(x,r)} g(y) \, dy - g(x) \right| \leq t.$$

In the left-hand side of this inequality, we add and subtract $f(x)$ and the average value of f over $B(x,r)$. Then by (42), (43), and the triangle inequality, we have

$$\left| \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) \, dy - f(x) \right| \leq 3t$$

for all sufficiently small $r > 0$. Keeping t fixed, for each n there is a set E_n with $m(E_n) < \frac{1}{n}$ such that for each $x \in E_n^c$ there exists an $\eta > 0$ such that for $0 < r < \eta$,

$$|A_{r,f}(x) - f(x)| \leq 3t.$$

Let $E = \cap E_n$. Thus $m(E) = 0$ and for each $x \in E^c$ there exists an $\eta > 0$ such that the inequality above holds for $0 < r < \eta$. For each $m \in \mathbb{N}^+$ there exists a set F_m of measure zero such that for each $x \in F^c$ there is an $\eta > 0$ such that for $0 < r < \eta$

$$|A_{r,f}(x) - f(x)| \leq \frac{1}{m}.$$

Finally, let $F = \cup F_n$ and note that $m(F) = 0$ and if $x \in F^c$ then, for every $m \in \mathbb{N}^+$ there exists an $\eta > 0$ such that for all $0 < r < \eta$ the inequality above holds completing the second part of the Lebesgue Differentiation Theorem.

For part (a), note that if f is locally integrable and $c \in \mathbb{C}$, then $|f(x) - c|$ is locally integrable. Thus for each $c \in \mathbb{C}$ we can apply part (b) to conclude that

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - c| \, dy = |f(x) - c|$$

for all x outside an exceptional set E_c with $m(E_c) = 0$. Fix a countable dense subset $Q \subset \mathbb{C}$ and let $E = \bigcup_{c \in Q} E_c$; then $m(E) = 0$ and for fixed $x \notin E$ there exists $c \in Q$ with $|f(x) - c| < \epsilon$, so $|f(y) - f(x)| < |f(y) - c| + \epsilon$, and

$$\limsup_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, dy \leq |f(x) - c| + \epsilon < 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, this proves (a). $\square$

It remains to prove the Hardy-Littlewood maximal inequality, Theorem 17.4. The proof we give is based on the following lemma, known as the *Wiener covering lemma*. Let B denote an open ball in $\mathbb{R}^n$ and for $a > 0$ let aB denote the open ball with the same center as B, whose radius is a times the radius of B.

Lemma 17.6 (Wiener's covering lemma)**.** *Let $\mathcal{B}$ be a collection of open balls in $\mathbb{R}^n$, and let $U = \bigcup_{B\in\mathcal{B}} B$. If $c < m(U)$, then there exists finitely many disjoint balls $B_1, \dots B_k \in \mathcal{B}$ such that $m(\bigcup_{j=1}^k B_j) > 3^{-n}c$.* †

Proof. There exists a compact set $K \subset U$ such that $m(K) > c$. The collection of open balls $\mathcal{B}$ covers K, so there are finitely many balls $A_1, \dots A_m$ whose union covers K. From these we select a disjoint subcollection by a *greedy algorithm*: from $A_1, \dots A_m$ choose a ball with maximal radius. Call this B_1. Now discard all the balls that intersect B_1. From the balls that remain, choose one of maximal radius, necessarily disjoint from B_1, call this B_2. Continue inductively, at each stage choosing a ball of maximal radius disjoint from the balls that have already been picked. The process halts after a finite number of steps. We claim that the balls $B_1, \dots B_k$ have the desired property. By construction the B_j are pairwise disjoint. The claimed lower bound on the measure of the union follows from a geometric observation. If A, A' are open balls with radii $r \geq r'$ respectively and if $A \cap A' \neq \varnothing$, then $A' \subset 3 \cdot A$ (draw a picture and note the diameter of A' is at most twice the radius of A). From this observation, it follows that each ball A_j that was *not* picked during the construction is contained in $3 \cdot B_i$ for some i. In particular, the balls $3 \cdot B_1, \dots 3 \cdot B_k$ cover K. From the scaling property of Lebesgue measure (Theorem 16.3), $m(3 \cdot B) = 3^n m(B)$. Thus

$$c < m(K) \leq \sum m(3 \cdot B_j) = 3^n \sum m(B_j) = 3^n m(\bigcup_{j=1}^k B_j).$$

□

Proof of Theorem 17.4. Let $f \in L^1(\mathbb{R}^n)$ and fix $\lambda > 0$. Let $E_\lambda = \{x \in \mathbb{R}^n : M_f(x) > \lambda\}$. If $x \in E_\lambda$, then by definition of M_f there is an $r_x > 0$ such that $A_{r_x,|f|}(x) > \lambda$. The open balls $B(x, r_x)$ then cover E_λ. Fix c with $m(E_\lambda) > c$. Then $m(\bigcup_{x\in E_\lambda} B(x, r_x)) > c$, so by the Wiener covering lemma there are finitely many $x_1, \dots x_k \in E_\lambda$ so that the balls $B_k := B(x_k, r_{x_k})$ are disjoint and $m(\bigcup_{j=1}^k B_j) > 3^{-n}c$. From the way the radii r_x were chosen, for each $1 \leq j \leq k$,

$$\lambda < A_{r_{x_j},|f|}(x_j) = \frac{1}{m(B(x_j, r_{x_j}))} \int_{B(x_j,r_{x_j})} |f(y)|\, dy$$

so

$$m(B_j) < \frac{1}{\lambda} \int_{B_j} |f(y)|\, dy.$$

It follows that

$$c < 3^n m(\bigcup_{j=1}^k B_j) = 3^n \sum_j m(B_j) < \frac{3^n}{\lambda} \sum_j \int_{B_j} |f(y)|\, dy \leq \frac{3^n}{\lambda} \int_{\mathbb{R}^n} |f(y)|\, dy = \frac{3^n}{\lambda} \|f\|_1.$$

This holds for all $c < m(E_\lambda)$, so taking the supremum over such c we get finally

$$m(E_\lambda) \le 3^n \frac{\|f\|_1}{\lambda}.$$

□

18. Signed measures and the Lebesgue-Radon-Nikodym Theorem

A second form of the fundamental theorem of calculus says that if $f : [a,b] \to \mathbb{R}$ is differentiable at each point in $[a,b]$ and if f is in $L^1([a,b])$, then

$$f(x) - f(a) = \int_a^x f'(t)\,dt, \tag{44}$$

for all $a \le x \le b$. Suppose f is increasing on $[a,b]$. From our construction of Lebesgue-Stieltjes measures, the formula $\mu([c,d]) := f(d) - f(c)$, defined for all subintervals $[c,d] \subset [a,b]$, determines a unique Borel measure on $[a,b]$. On the other hand, from (44), this measure can equivalently be defined by the formula

$$\mu(E) = \int_E f'(x)\,dx. \tag{45}$$

From Problem 13.18, for an unsigned measurable $g : [a,b] \to \mathbb{R}$,

$$\int_a^b g\,d\mu = \int_a^b gf'dm,$$

where m is Lebesgue measure on $[a,b]$. It is tempting to write $d\mu = f'dt$ or even more suggestively, $\frac{d\mu}{dm} = f'$. As will be seen in this section, f' is the *Radon-Nikodym derivative* of μ with respect to m.

18.1. **Signed measures; the Hahn and Jordan decomposition theorems.**

If μ, ν are measures on a common measurable space $(X, \mathscr{M})$, then we have already seen that we can form new measures $c\mu$ (for $c \ge 0$) and $\mu + \nu$. We would like to extend these operations to allow negative constants and subtraction. The obvious thing to do is to define the difference of two measures to be

$$(\mu - \nu)(E) = \mu(E) - \nu(E). \tag{46}$$

The only difficulty is that the right-hand side may take the form $\infty - \infty$ and is therefore undefined. We deal with this problem by avoiding it: the measure $\mu - \nu$ will be defined only when at least one of μ, ν is a finite measure, in which case the formula (46) always makes sense. It is straightforward to check that, under this assumption, the set function $\mu - \nu$ is countably additive, and $(\mu - \nu)(\varnothing) = 0$. It is of course *not* monotone. From these observations we extract the definition of a *signed measure*:

Definition 18.1. Let $(X, \mathscr{M})$ be a measurable space. A *signed measure* is a function $\rho : \mathscr{M} \to \overline{\mathbb{R}}$ satisfying:

a) $\rho(\varnothing) = 0$,
b) ρ takes at most one of the values $+\infty, -\infty$,

c) if $(E_n)_{n=1}^\infty$ is a disjoint sequence of measurable sets, then $\sum_{n=1}^\infty \rho(E_n)$ converges to $\rho(\bigcup_{n=1}^\infty E_n)$, and the sum is absolutely convergent if $\rho(\bigcup_{n=1}^\infty E_n)$ is finite.

$\triangleleft$

Remark 18.2. Actually, the statement about absolute convergence in (c) is an immediate consequence of the Riemann rearrangement theorem. $\diamond$

The main result of this section is the Jordan decomposition theorem, which says that every signed measure is canonically the difference of two (unsigned) measures μ and ν. There is a natural partial order on the set of finite measures on $(X, \mathscr{M})$, determined by $\mu \geq \nu$ if and only if $\mu - \nu$ is a positive measure.

Example 18.3. Consider a measure space $(X, \mathscr{M}, \mu)$ and let $f : X \to \mathbb{R}$ belong to $L^1(\mu)$. The quantity

$$\mu_f(E) := \int_E f \, d\mu \tag{47}$$

then defines a signed measure on $(X, \mathscr{M})$. Indeed, decomposing $f = f^+ - f^-$ into its positive and negative parts $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$, the signed measure μ_f can be written as $\rho = \mu_{f^+} - \mu_{f^-}$ where $\mu_{f^\pm}$ denotes the measure

$$\mu_{f^\pm}(E) = \int_E f^\pm \, d\mu. \tag{48}$$

In fact this construction will work as long as f is *semi-integrable* (that is, at least one of f^+, f^- is integrable). $\triangle$

It is not hard to show that monotone and dominated convergence for sets still hold for signed measures.

Proposition 18.4. *Let ρ be a signed measure. If $(E_n)_{n=1}^\infty$ is an increasing sequence of measurable sets, then $\rho(\bigcup_{n=1}^\infty E_n) = \lim_{n\to\infty} \rho(E_n)$. If E_n is a decreasing sequence of measurable sets and $\rho(E_1)$ is finite, then $\rho(\bigcap_{n=1}^\infty E_n) = \lim_{n\to\infty} \rho(E_n)$.* $\dagger$

Proof. The proof is essentially the same as in the unsigned case (using the disjointification trick) and is left as an exercise (Problem 19.18). $\square$

Before going further we introduce some notation and a couple of definitions. If ρ is a signed measure on the measurable space $(X, \mathscr{M})$ and $Y \subset X$ is a measurable set, we let $\rho|_Y$ denote the measure on $\mathscr{M}$ defined by $\rho|_Y(E) := \rho(Y \cap E)$. A set E *totally positive* for ρ if $\rho|_E \geq 0$. As is easily verified, E is totally positive for ρ if and only if $\rho(F) \geq 0$ for all $F \subset E$. In the case ρ omits the value $+\infty$, it is totally positive $\rho(F) \leq \rho(E)$ for all measurable $F \subset E$. (Consider $E \setminus F$). The set E *totally negative* for ρ if $\rho|_E \leq 0$ and *totally null* if $\rho|_E = 0$. It is immediate that E is totally null for ρ if and only if it is both totally positive and totally negative. Finally, if $(E_n)_n$ is a sequence of totally positive sets, then $\cup E_n$ is also totally positive.

Note that when we decompose a real-valued function f into its positive and negative parts $f = f^+ - f^-$, the sets $X_+ := \{x : f^+(x) > 0\}$ and $X_- := \{x : f^-(x) > 0\}$ are disjoint, and $f|_{X_+} \geq 0$, $f|_{X_-} \leq 0$. (Compare with Example (18.3).) A similar statement holds for signed measures.

Theorem 18.5 (Hahn Decomposition Theorem)**.** *Let ρ be a signed measure. Then there exists a partition of X into disjoint measurable sets $X = X_+ \cup X_-$ such that $\rho|_{X_+} \geq 0$ and $\rho|_{X_-} \leq 0$. Moreover if X'_+, X'_- is another such pair, then $X_+ \Delta X'_+$ and $X_- \Delta X'_-$ are totally null for ρ.*

The following lemma will be used in the proof of Theorem 18.5

Lemma 18.6. *Let ρ be a signed measure that omits the value $+\infty$. If $\rho(G) > 0$, then there exists a subset $E \subset G$ such that E is totally positive and $\rho(E) > 0$.* †

Proof. For notational convenience, let $E_1 = G$. If E_1 is totally positive, then there is nothing to prove. Accordingly, suppose E_1 is not totally positive. Thus E_1 contains a subset F of strictly larger positive measure. In particular, the set

$$J_1 = \{n \in \mathbb{N}^+ : \text{ there is an } F \subset E_1 \text{ such that } \rho(F) \geq \rho(E_1) + \frac{1}{n}\}$$

is nonempty and thus has a smallest element n_1. In particular, if $m < n_1$ and $F \subset E_1$ is measurable, then $\rho(F) < \rho(E_1) + \frac{1}{m}$. Choose any $E_2 \subset E_1$ such that $\rho(E_2) \geq \rho(E_1) + 1/n_1$. Now, if E_2 were totally positive, then the proof is complete. Otherwise, let n_2 denote the smallest element of

$$J_2 = \{n \in \mathbb{N}^+ : \text{ there is an } F \subset E_2 \text{ such that } \rho(F) \geq \rho(E_2) + \frac{1}{n}\}$$

and choose $E_3 \subset E_2$ such that $\rho(E_3) \geq \rho(E_2) + 1/n_2$. Continuing by induction produces a totally positive subset E of G with $\rho(E) > 0$ or a decreasing sequence of measurable sets $E_{j+1} \subset E_j$ and a sequence of integers n_j such that $\rho(E_j) > 0$ for all j and

$$\rho(E_{j+1}) \geq \rho(E_j) + \frac{1}{n_j};$$

and

$$n_j = \min\{n \in \mathbb{N}^+ : \text{ there is an } F \subset E_j \text{ such that } \rho(F) \geq \rho(E_j) + \frac{1}{n}\}. \tag{49}$$

Assuming this latter case, let $E = \bigcap_{j=1}^{\infty} E_j$. We will show that $\rho(E) > 0$ and E is totally positive. By Proposition 18.4, $\rho(E_j)$ increases to $\rho(E)$ and in particular the set E has finite positive measure (recall ρ omits the value $+\infty$). Since $\rho(E)$ is finite, the n_j go to infinity. To show that E must be totally positive, suppose, by way of contradiction, there exists an $F \subset E$ such that $\rho(F) > \rho(E)$ and it can be assumed that $\rho(F) > \rho(E) + \frac{1}{m}$ where

$$m = \min\{n \in \mathbb{N}^+ : \text{ there is an } F \subset E \text{ such that } \rho(F) \geq \rho(E) + \frac{1}{n}\}.$$

Thus $F \subset E_j$ for every j and $\rho(F) \geq \rho(E) + 1/m \geq \rho(E_j) + 1/m$, which, since the n_j go to infinity, contradicts (49) once j is large enough. □

Proof of Theorem 18.5. We may assume ρ avoids the value $+\infty$. The idea of the proof is to select X_+ to be a maximal totally positive set for ρ, and then show that $X_- := X \setminus X_+$ is totally negative. The set X_+ is obtained by a *greedy algorithm.* Let M denote the supremum of $\rho(E)$ over all totally positive sets E. Choose a sequence of totally positive sets E_n so that $M = \lim \rho(E_n)$. Since each E_n is totally positive, the union $X_+ := \bigcup_{n=1}^\infty E_n$ is also totally positive, and by Proposition 18.4 $\rho(X_+) = M$. (In particular, M is finite.)

The proof is finished if we can show that $X_- := X \setminus X_+$ is totally negative. By way of contradiction, suppose it is not. In this case there exists a $G \subset X_-$ with $\rho(G) > 0$. By Lemma 18.6, there exists a set $E \subset G$ such that E is totally positive and $\rho(E) > 0$. Now $X_+ \cup E$ is totally positive and $\rho(X_+ \cup E) > \rho(X_+)$, contradicting the choice of X_+.

The uniqueness statement in the theorem is left as an exercise (Problem 19.19). □

A set E is a *support set* for a signed measure ρ if E^c is totally null for ρ. Two signed measures ρ, σ are *mutually singular*, denoted $\rho \perp \sigma$, if they have disjoint support sets; i.e., there exists disjoint measurable sets E and F such that E^c is totally null for ρ and F^c is totally null for σ. In the case ρ, σ are *unsigned* measures, they are mutually singular if and only if there exists disjoint (measurable) sets E and F such that $\rho(E^c) = 0 = \sigma(F^c)$ (in which case it can be assumed that $F = E^c$ if desired).

Theorem 18.7 (Jordan Decomposition)**.** *If ρ is a signed measure on $(X, \mathscr{M})$, then there exist unique positive measures ρ_+, ρ_- such that $\rho_+ \perp \rho_-$ and $\rho = \rho_+ - \rho_-$.*

Proof. Let $X = X_+ \cup X_-$ be a Hahn decomposition for ρ and put $\rho_+ = \rho|_{X_+}$, $\rho_- = -\rho|_{X_-}$. It is immediate from the properties of the Hahn decomposition that ρ_+, ρ_- have the desired properties; uniqueness is left as an exercise (Problem 19.20). □

Example 18.8. Referring to Example 18.3, it is now immediate that the decomposition $m_f = m_{f^+} - m_{f^-}$ *is* the Jordan decomposition of m_f; i.e., $m_f^+ = m_{f^+}$ and likewise $m_f^- = m_{f^-}$. Thus the Jordan decomposition theorem should be seen as a generalization of the decomposition of a real-valued function into its positive and negative parts. △

Let ρ be a signed measure and $\rho = \rho_+ - \rho_-$ its Jordan decomposition. By analogy with the identity $|f| = f^+ + f^-$, we can define a measure $|\rho| := \rho_+ + \rho_-$; this is called the *absolute value* or *total variation* of ρ. The latter name is explained by the following proposition.

Proposition 18.9. *Let ρ be a signed measure on the measure space $(X, \mathscr{M})$. If $E \in \mathscr{M}$, then $|\rho|(E) = \sup \sum_{n=1}^\infty |\rho(E_n)|$, where the supremum is taken over all measurable partitions $E = \bigcup_{n=1}^\infty E_n$.* †

Proof. The proof is an exercise (Problem 19.22). □

Warning: A moment's thought shows that in general $|\rho(E)| \neq |\rho|(E)$. As an exercise, prove that $|\rho(E)| \leq |\rho|(E)$ always, with equality if and only if E is either totally positive, totally negative, or totally null for ρ.

We can now define a signed measure ρ to be *finite* or σ*-finite* according as $|\rho|$ is finite or σ-finite. It is not hard to show that ρ is finite if and only if $\rho(E)$ is finite for every E, if and only if ρ_+, ρ_- are finite unsigned measures. It is evident from this that the space of finite signed measures on $(X, \mathscr{M})$ is a real vector space, denoted $M(X)$. (We will see later in the course that the quantity $\|\rho\| := |\rho|(X)$ defines a norm on $M(X)$, called the *total variation norm.*)

A few remarks about integration against signed measures are in order. If ρ is a signed measure, then $L^1(\rho)$ is defined to be $L^1(|\rho|)$; note that $L^1(|\rho|) = L^1(\rho_+) \cap L^1(\rho_-)$. For $f \in L^1(\rho)$ define

$$\int f\, d\rho := \int f\, d\rho_+ - \int f\, d\rho_-. \tag{50}$$

Proposition 18.10. *Let ρ be a signed measure on $(X, \mathscr{M})$.*

(a) If $f \in L^1(\rho)$, then $\left|\int f\, d\rho\right| \leq \int |f|\, d|\rho|$.
(b) If $E \in \mathscr{M}$, then $|\rho|(E) = \sup\{\left|\int_E f\, d\rho\right| : |f| \leq 1\}$.

†

Proof. Problem 19.23. □

18.2. **The Lebesgue-Radon-Nikodym theorem.** Fix for reference a measurable space $(X, \mathscr{M})$ and an unsigned measure m on this space. (In this section all measures are defined on the same σ-algebra $\mathscr{M}$.) For an unsigned measurable function f, we have the measure

$$m_f(E) = \int_E f\, dm. \tag{51}$$

The map $f \to m_f$ is thus a map from the space of unsigned measurable functions into the space of nonnegative measures on $(X, \mathscr{M})$. Likewise the mapping $f \to m_f$ maps $L^1(\mu)$ into the space of finite signed measures on X. One may ask if every finite measure μ on X may be expressed as m_f for some f, but one can quickly see this is not the case in general. Indeed, if $\mu = m_f$ then $\mu(E) = 0$ whenever $m(E) = 0$, which need not always be the case (e.g., m is Lebesgue measure on $\mathbb{R}$ and μ is the point mass at 0.) However, when the measures involved are σ-finite, it turns out this is the only obstruction.

Given unsigned measures μ and ν on a measure space $(X, \mathscr{M})$, we say ν is *absolutely continuous* with respect to μ if (for measurable sets E) $\mu(E) = 0$ implies $\nu(E) = 0$. The following proposition explains the reason for the terminology.

Proposition 18.11. *If μ and ν are unsigned finite measures on the measurable space $(X, \mathscr{M})$, then the following are equivalent.*

(i) $\nu << \mu$;
(ii) for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\nu(E) < \epsilon$ whenever $\mu(E) < \delta$.

†

Proof. Suppose (ii) hold, $\nu(A) > 0$ and $\nu(A) > \epsilon > 0$. By assumption there is $\delta > 0$ such that $\mu(E) < \delta$ implies $\nu(E) < epsilon$. Hence $\mu(A) > \delta > 0$ and (i) holds (we did not need to know the measures were finite here).

Now suppose (ii) fails. There is an $\epsilon > 0$ such that for each $n \in \mathbb{N}^+$ there exists a set E_n such that $\mu(E_n) < 2^{-n}$, but $\nu(E_n) \geq \epsilon$. Let $F_n = \cup_{k \geq n} E_n$. Thus $\mu(F_n) < 2^{-n+1}$. Hence, with $F = \cap F_n$, we have, by dominated convergence for sets twice (here is where finiteness of the measures is used),

$$\mu(F) = \lim_n \mu(F_n) = 0, \quad \nu(F) = \lim_n \mu(F_n) \geq \epsilon.$$

□

To see that the σ-finiteness assumption is needed in the implication (ii) implies (i) of Proposition 18.11, let μ and ν be the measures on $(\mathbb{N}, 2^{\mathbb{N}})$ determined by

$$\mu(\{n\}) = 2^{-n}, \quad \nu(\{n\}) = 1.$$

The only set for either measure is $\emptyset$. Hence $\nu << \mu$. But with $0 < \epsilon < 1$, there is no δ such that $\mu(E) < \delta$ implies $\nu(E) < \epsilon$.

Theorem 18.12 (Radon-Nikodym). *Suppose m and μ are unsigned σ-finite measures on $(X, \mathscr{M})$. If $\mu << m$, then there is a unique measurable $f : X \to [0, \infty)$ such that $\mu = m_f$. Further, if μ is finite, then $f \in L^1(m)$.*

Remark 18.13. In the case μ is unsigned, apply Theorem 18.12 to the $\mu^\pm$ from its Jordan decomposition. In this case f is semi-integrable.

To see that the σ-finiteness hypothesis is needed, let m denote Lebesgue measure on $(\mathbb{R}, \mathcal{L})$ and c counting measure. If $c(E) = 0$, then $E = \emptyset$ and $m(E) = 0$ too. Hence $m << c$. On the other hand, if f is unsigned and measurable, then, for each $x \in \mathbb{R}$,

$$c_f(\{x\}) = \int_{\{x\}} f \, dc = f(x).$$

Hence, $c_f \neq m$. ◇

Lemma 18.14. *If μ and ν are finite positive measures on $(X, \mathscr{M})$, then either $\mu \perp \nu$, or else there exist $\epsilon > 0$ and a measurable set E such that $\mu(E) > 0$ and $\nu(F) \geq \epsilon\mu(F)$ for $F \subset E$ measurable (that is, E is totally positive for $\nu - \epsilon\mu$).*

In particular, if $\nu << \mu$ and $\nu \neq 0$, then there exist $\epsilon > 0$ and a measurable set E such that $\mu(E) > 0$ and $\nu(F) \geq \epsilon\mu(F)$ for $F \subset E$ measurable. †

Proof. For each $n \geq 1$, let $X = X_+^n \cup X_-^n$ be a Hahn decomposition for $\nu - \frac{1}{n}\mu$. Let $P = \bigcup_{n=1}^\infty X_+^n$ and $N = \bigcap_{n=1}^\infty X_-^n$. In particular $N = P^c$. Since N is totally negative for $\nu - \frac{1}{n}\mu$ for all n, it follows that $\nu(N) = 0$. If $\mu(P) = 0$, then $\mu \perp \nu$. Otherwise, $\mu(X_+^n) > 0$ for some n, and by construction X_+^n is totally positive for $\nu - \frac{1}{n}\mu$ (thus we take $\epsilon = \frac{1}{n}$, $E = X_+^n$). □

Proof of Theorem 18.12. We prove this only for the case that μ, m are finite; and only outline the extension to the σ-finite case.

As before, f is selected by a '*greedy algorithm.* Let $\mathscr{S}$ denote the set of unsigned f such that $m_f \le \mu$ and observe $0 \in \mathscr{S}$. Let M be the supremum of the set $\{\int_X f\,dm : f \in \mathscr{S}\}$. Note that M is finite, since μ is. Choose a sequence f_n so that $\int_X f_n\,dm \to M$. Define $g_n = \max_{1\le k\le n} f_k$ and note that $f_n \le g_n$ and the g_n are increasing. An exercise shows if $g, h \in \mathscr{S}$, then $m_\varphi \le \mu$ where $\varphi = \max\{g, h\}$ from which $m_{g_n} \le \mu$ follows. Hence each $g_n \in \mathscr{S}$. Since (g_n) is pointwise increasing, it converges pointwise (in $[0, \infty]$) to some unsigned measurable g. Since, $\int_E g_n\,dm \le \mu(E)$ for each E and n, the MCT implies $g \in \mathscr{S}$ (and in particular g is finite a.e.). Since $\int_X f_n\,dm \le \int_X g_n\,dm \le M$ for each n, it follows that the sequence $(\int_X g_n\,dm)$ converges to M and hence

$$M = \int_X g\,dm.$$

Now choose any $f \in \mathscr{S}$ that achieves this maximum M. In particular, f is unsigned and in $L^1(m)$ (its integral is finite). By construction $\nu = \mu - m_f$ is totally positive since $f \in \mathscr{S}$ and ν is absolutely continuous with respect to m since both μ and m_f are. If $\nu \ne 0$, then then by Lemma 18.14 there is an $\epsilon > 0$ and an E such that $m(E) > 0$ and $\nu - \epsilon m|_E \ge 0$, equivalently $\nu \ge \epsilon m|_E$. Hence $\mu \ge m_{f+\epsilon \mathbf{1}_E}$, contradicting the maximality of f since $\int_X (f + \epsilon \mathbf{1}_E)\,dm = \int_X f\,dm + \epsilon m(E) > M$.

Still assuming the measures are finite, to see that f is unique, suppose $\mu = m_f = m_g$ for some unsigned g. It follows that $g \in L^1(m)$ and hence, for all measurable sets E,

$$0 = m_f(E) - m_g(E) = \int_E (f - g)\,dm$$

and thus $f = g$ a.e. m.

Now suppose μ and m are sigma-finite. In this case there exists set $X_n \subset X_{n+1}$ such that $X = \cup X_n$ and $\mu(X), m(X) < \infty$. Letting $\mu_n(E) = \mu(E \cap X_n)$ and $m_n(E) = m(E \cap X_n)$, it follows that $\mu_n << m_n$ and hence there exists a unique unsigned measurable function f_n supported on X_n such that if $E \subset X_n$ is measurable, then

$$\mu(E) = \int_E f_n\,dm.$$

Since also, for $E \subset X_n \subset X_{n+1}$,

$$\mu(E) = \int_E f_{n+1}\,dm,$$

it follows that $f_{n+1} = f_n$ a.e. m on X_n (by uniqueness). Hence the sequence (f_n) is pointwise increasing. Choose f to be its limit. By the Monotone Convergence Theorem, for measurable sets $E \subset X$,

$$\begin{aligned}\mu(E) &= \lim \mu(E \cap X_n) = \lim \int_{E\cap X_n} f_n\, dm \\ &= \lim \int_E f_n\, dm = \int_E f\, dm = m_f(E).\end{aligned}$$

□

The function f is called the *Radon-Nikodym derivative* of μ with respect to m, denoted $\frac{d\mu}{dm} = f$. The basic manipulations suggested by the derivative notation are valid. For example it is easy to check that $d(\mu_1 + \mu_2)/dm = d\mu_1/dm + d\mu_2/dm$. We will see in a moment that the chain rule is valid.

Corollary 18.15. *Assuming μ and ν are (unsigned) σ-finite measures on the measurable space $(X, \mathscr{M})$, $\mu << \nu$ if and only if there is an unsigned measurable function f such that $\mu = \nu_f$.* †

One important corollary of the Lebesgue-Radon-Nikodym theorem is the existence of *conditional expectations.*

Proposition 18.16. *Let $(X, \mathscr{M}, \mu)$ be a σ-finite measure space (μ a positive measure), $\mathscr{N}$ a sub-σ-algebra of $\mathscr{M}$, and suppose $\nu = \mu|_{\mathscr{N}}$ is σ-finite. If $f \in L^1(\mu)$ then there exists $g \in L^1(\nu)$ (unique modulo ν-null sets) such that*

$$\int_E f\, d\mu = \int_E g\, d\nu$$

for all $E \in \mathscr{N}$ (g is called the conditional expectation of f on $\mathscr{N}$). †

Proof. Problem 19.27 □

Proposition 18.17 (Chain rule for Radon-Nikodym derivatives)**.** *Suppose μ, m, λ are σ-finite positive measures on the measure space $(X, \mathscr{M})$, and $\mu \ll \lambda \ll m$.*

(i) If $g \in L^1(|\mu|)$, then $g\frac{d\mu}{d\lambda} \in L^1(\lambda)$ and

$$\int_X g\, d\mu = \int_X g \frac{d\mu}{d\lambda}\, d\lambda. \tag{52}$$

(ii)

$$\frac{d\mu}{dm} = \frac{d\mu}{d\lambda}\frac{d\lambda}{dm} \quad m\text{-a.e.} \tag{53}$$

†

Proof. In this case the Radon-Nikodym derivatives $\frac{d\mu}{d\lambda}$ and $\frac{d\lambda}{dm}$ are unsigned. In particular, for measurable sets E,

$$\mu(E) = \int_E d\mu = \int_E \frac{d\mu}{d\lambda}\, d\lambda.$$

By Problem 13.18, if either g is unsigned or $g \in L^1(\mu)$, then

$$\int_X g\,d\mu = \int_X g\,\frac{d\mu}{d\lambda}\,d\lambda \tag{54}$$

proving (i). As a warm up for (ii), note that (i) gives, for h either unsigned or $h \in L^1(\lambda)$,

$$\int_X h\,d\lambda = \int_X h\frac{d\lambda}{dm}\,dm.$$

With $h = \mathbf{1_E}\frac{d\mu}{d\lambda}$ for a measurable set E,

$$\mu(E) = \int_E \frac{d\mu}{d\lambda}\,d\lambda = \int_E \frac{d\mu}{d\lambda}\frac{d\lambda}{dm}\,dm.$$

On the other hand, $\mu \ll m$, and $\mu(E) = \int_E \frac{d\mu}{dm}dm$ by the definition of $\frac{d\mu}{dm}$. Hence (ii) follows by uniqueness of the Radon-Nikodym derivative. □

Theorem 18.18 (Lebesgue Decomposition)**.** *If ν and μ are unsigned σ-finite measures on the measure space $(X, \mathscr{M})$, then there exist unique measures μ_a and μ_s such that $\mu_a << \nu$ and $\mu_s \perp \nu$ and $\mu = \mu_a + \mu_s$.*

As before, using the Jordan decomposition one can allow ν to be a signed measures in the Lebesgue decomposition.

Proof. Let $m = \nu + \mu$. In particular, m is σ-finite and $\mu << m$. Hence there is a uniquely (a.e. m) determined unsigned function f such that $\mu = m_f$. Thus, for all measurable E,

$$\mu(E) = \int_E f\,d(\mu+\nu) = \int_E f\,dm.$$

From here we prove the result under the added assumption that μ and ν are finite, leaving the proof of the general case to the interested reader.

Let $F_a = \{f < 1\}$ and $F_s = \{f = 1\}$ and define

$$\mu_t(E) = \mu(E \cap F_t),$$

for $t = a, s$. Both are measures

An easy argument shows $f \leq 1$ a.e. m. Hence $\mu = \mu_a + \mu_s$. Next,

$$\mu(F_s) = \int_{F_s} f\,d(\mu+\nu) = \mu(F_s) + \nu(F_s).$$

Thus $\nu(F_s) = 0$ and $\mu_a \perp \nu$. Now suppose E is measurable and $\nu(E) = 0$. Letting $F_n = \{f < 1 - \frac{1}{n}\}$ (for positive integers n),

$$\mu(E \cap F_n) \leq (1 - \frac{1}{n})\mu(E \cap F_n).$$

Hence $\mu(E \cap F_n) = 0$. Since $E \cap F_s = \cup(E \cap F_n)$ it follows that $\mu(E \cap F_s) = 0$ and therefore $\mu_s << \nu$.

To prove uniqueness, suppose $\mu = \rho_a + \rho_s$ with $\rho_a << \nu$ and $\rho_s \perp \nu$. In particular, there is a set G such that $\rho_s(G) = 0$ and $\nu(G^c) = 0$. In particular, $\rho_a(G^c) = 0 = \mu_a(G^c)$ and further

$$\rho_a(E) = \rho_a(E \cap G), \quad \rho_s(E) = \rho_s(E \cap G^c).$$

It follows that

$$\rho_a(E) = \rho_a(E \cap G \cap F) = \mu_a(E \cap F \cap G) = \mu_a(E).$$

Thus, if $\mu(E) < \infty$, then $\rho_s(E) = \mu_s(E)$ too. The general case now follows from the σ-finiteness assumption. □

Example 18.19. This example shows the σ-finiteness hypothesis is needed in the Lebesgue decomposition. On $(\mathbb{R}, 2^{\mathbb{R}})$, let c denote counting measure and let ν denote the measure defined by $\nu(E) = 1$ if E is uncountable and $\nu(E) = 0$ otherwise. Arguing by contradiction, suppose $c = c_a + c_s$ where $c_a << \nu$ and $c_s \perp \nu$. It follows that there is a set B such that $c_s(B^c) = 0$ and $\nu(B) = 0$. In particular, B is at most countable. There is an $x \in B^c$ (since $\mathbb{R}$ is not countable). Now $\nu(\{x\}) = 0$ and hence $c_a(\{x\}) = 0$ and also $c_s(\{x\}) = 0$ since $x \in B^c$. Hence $c_a(\{x\}) + c_s(\{x\}) = 0 \neq c(\{x\}) = 1$. △

18.3. **Lebesgue differentiation revisited.** Finally, we describe the connection between Radon-Nikodym derivatives and Lebesgue differentiation on $\mathbb{R}^n$. Recall a positive measure μ is *regular* if

i) $\mu(K) < \infty$ for every compact $K \subset \mathbb{R}^n$, and
ii) for every Borel set $E \subset \mathbb{R}^n$, we have $\mu(E) = \inf\{\mu(U) : U \text{ open }, E \subset U\}$.

Theorem 18.20. *Let μ be a regular Borel measure on $\mathbb{R}^n$ with Lebesgue decomposition*

$$\mu = m_f + \mu_s$$

with respect to Lebesgue measure m. For m-a.e. $x \in \mathbb{R}^n$.

$$\lim_{r \to 0} \frac{\mu(B_r(x))}{m(B_r(x))} = f(x) \tag{55}$$

Proof. By the regularity of μ, we see that the measure m_f is locally finite, so $f \in L^1_{loc}$. One may verify that the measure m_f is regular, and so μ_s is as well. Applying the Lebesgue differentiation theorem, (55) holds already with $\mu = m_f$, so it suffices to prove that

$$\lim_{r \to 0} \frac{\mu_s(B_r(x))}{m(B_r(x))} = 0 \quad m - \text{a.e.} \tag{56}$$

for the singular part μ_s.

Fix a Borel set E such that $\mu_s(E) = m(E^c) = 0$ and let

$$E_k = \left\{ x \in E : \forall t > 0 \ \ \exists\, 0 < r < t \text{ such that } \frac{\mu_s(B_r(x))}{m(B_r(x))} > \frac{1}{k} \right\}.$$

It will suffice to prove that $m(E_k) = 0$ for each integer $k \geq 1$.

By regularity, for given $\epsilon > 0$ there is an open set U containing E such that $\mu_s(U) < \epsilon$. By the definition of E_k, for each $x \in E_k$ there is a ball B_x centered at x such that $B_x \subset U$ and $\mu_s(B_x) > \frac{m(B_x)}{k}$. Let $V = \bigcup_{x\in E_k} B_x$ be the union of these balls. Fix a number $c < m(V)$ and apply Wiener's covering lemma 17.6 to obtain points $x_1, \dots x_m \in E_k$ such that the balls $B_1, \dots B_m$ are disjoint and

$$c < 3^n \sum_{j=1}^m m(B_j) \le 3^n k \sum_{j=1}^m \mu_s(B_j) \le 3^n k\mu_s(V) \le 3^n k\mu_s(U) < 3^n k\epsilon.$$

Thus $m(V) \le 3^n k\epsilon$, and since $E_k \subset V$ and ϵ was arbitrary, we conclude $m(E_k) = 0$. □

19. Problems

19.1. **Product measures.**

Problem 19.1. Let μ_X denote counting measure on X. Prove that if X, Y are both at most countable, then $2^X \otimes 2^Y = 2^{X\times Y}$ and $\mu_X \times \mu_Y = \mu_{X\times Y}$.

Problem 19.2. Prove that the product measure construction is associative: that is, if $(X_j, \mathscr{M}_j, \mu_j)$, $j = 1, 2, 3$ are σ-finite measure spaces, then $(\mathscr{M}_1 \otimes \mathscr{M}_2) \otimes \mathscr{M}_3 = \mathscr{M}_1 \otimes (\mathscr{M}_2 \otimes \mathscr{M}_3)$, and $(\mu_1 \times \mu_2) \times \mu_3 = \mu_1 \times (\mu_2 \times \mu_3)$.

Problem 19.3. Let $X = Y = [0, 1]$, $\mathscr{M} = \mathscr{B}_{[0,1]}$, $\mathscr{N} = 2^{\mathbb{R}}$, let μ Lebesgue measure on $\mathscr{M}$, and let ν counting measure on $\mathscr{N}$. Let Δ denote the diagonal $\Delta = \{(x, x) : x \in [0, 1]\} \subset [0, 1] \times [0, 1]$. Prove that $\Delta \in \mathscr{M} \otimes \mathscr{N}$ and

$$\int_X \left(\int_Y \mathbf{1}_E(x, y)\, d\nu(y)\right) d\mu(x), \quad \int_Y \left(\int_X \mathbf{1}_E(x, y)\, d\mu(x)\right) d\nu(y) \tag{57}$$

are unequal. Show that, for each $P \in \mathscr{M} \otimes \mathscr{N}$, the functions $f(x) = \nu(P_x)$ and $g(y) = \mu(P^y)$ are measurable (with respect to $\mathscr{M}$ and $\mathscr{N}$ respectively) and

$$\tau(P) = \int_X \nu(P_x)\, d\mu, \quad \rho(P) = \int_Y \mu(P^y)\, d\nu$$

are both measures $\mathscr{M} \otimes \mathscr{N}$. (Note that Theorem 15.7 does not (directly) apply).

Problem 19.4. Prove Proposition 15.11.

Problem 19.5. Prove Corollary 15.8.

Problem 19.6. Prove Corollary 15.14.

19.2. **Integration on $\mathbb{R}^n$.**

Problem 19.7. Compare the three integrals

$$\iint_{[0,1]^2} f\, dm^2, \quad \int_0^1 \left(\int_0^1 f(x, y)\, dx\right) dy, \quad \int_0^1 \left(\int_0^1 f(x, y)\, dy\right) dx \tag{58}$$

for the functions

a) $f(x, y) = \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$
b) $f(x, y) = (1 - xy)^{-s}$, $s > 0$

Problem 19.8. Prove that if $f \in L^1[0,1]$ and $g(x) = \int_x^1 t^{-1} f(t)\, dt$, then $g \in L^1[0,1]$ and $\int_0^1 g(x)\, dx = \int_0^1 f(x)\, dx$.

Problem 19.9. Prove that $\int_0^\infty |\frac{\sin x}{x}|\, dx = +\infty$, but the limit $\lim_{b \to +\infty} \int_0^b \frac{\sin x}{x}\, dx$ exists and is finite. (For a bigger challenge, show that the value of the limit is $\frac{\pi}{2}$.)

Problem 19.10. Prove Theorem 16.1.

Problem 19.11. Complete the proof of Theorem 16.6.

Problem 19.12. [The Gamma function] Define

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt \tag{59}$$

a) Prove that the function $t \to t^{x-1} e^{-t}$ is absolutely integrable for all fixed $x > 0$ (thus $\Gamma(x)$ is defined for all $x > 0$).
b) Prove that $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$.
c) Compute $\Gamma(1/2)$. (Hint: if you haven't seen this before, first make the change of variables $u = \sqrt{t}$, then evaluate the *square* of the resulting integral using Tonelli's theorem and polar coordinates.)
d) Using (b) and (c), conclude that $\Gamma(n+\frac{1}{2}) = (n-\frac{1}{2})(n-\frac{3}{2}) \cdots (\frac{1}{2})\sqrt{\pi}$ for all $n \geq 1$.

Problem 19.13. Complete the following outline to prove that

$$\sigma(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{60}$$

a) Show that if $f \in L^1(\mathbb{R}^n)$ and f is a *radial* function (that is, $f(x) = g(|x|)$ for some function $g : [0, \infty) \to \mathbb{C}$), then

$$\int_{\mathbb{R}^n} f(x)\, dx = \sigma(S^{n-1}) \int_0^\infty g(r) r^{n-1}\, dr. \tag{61}$$

b) Show that for all $c > 0$,

$$\int_{\mathbb{R}^n} e^{-c|x|^2}\, dx = \left(\frac{\pi}{c}\right)^{n/2}. \tag{62}$$

(Hint: write $e^{-c|x|^2} = \prod_{j=1}^n e^{-c|x_j|^2}$ and use Tonelli's theorem.)
c) Finish by combining (a) and (b). (Using the results on the Gamma function from the previous exercise, one finds that $\sigma(S^{n-1})$ is always a rational multiple of an integer power of π.)

19.3. **Differentiation theorems.**

Problem 19.14. Prove that if $0 \neq f \in L^1(\mathbb{R})$, then there exist constants $C, R > 0$ (depending on f) such that

$$M_f(x) \geq \frac{C}{|x|} \quad \text{for all } |x| > R. \tag{63}$$

(Hint: reduce to the case $f = \mathbf{1}_E$ where E is a bounded set of positive measure.) Conclude that M_f never belongs to $L^1(\mathbb{R})$ if $f \in L^1$ is not a.e. 0.

Problem 19.15. The Lebesgue differentiation theorem says that for $f \in L^1(\mathbb{R}^n)$, we have $A_{r,f} \to f$ pointwise a.e. as $r \to 0$. Prove that also $A_{r,f} \to f$ in the L^1 norm. (Hint: the proof can be done in three steps: first prove this under the assumption that f is continuous with compact support. Then prove that for all $f \in L^1$ and $r > 0$, the functions $A_{r,f} \in L^1$; in fact $\|A_{r,f}\|_1 \leq \|f\|_1$ for all r. Tonelli's theorem will help. Finally, to pass to general L^1 functions, use a density argument.)

Problem 19.16. Let E be a Borel set in $\mathbb{R}$. Define the *density* of E at x to be

$$D_E(x) = \lim_{r \to 0} \frac{m(E \cap B(x,r))}{m(B(x,r))}$$

whenever the limit exists.

a) Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \notin E$.
b) Give examples of E and x for which $D_E(x) = \alpha$ $(0 < \alpha < 1)$ and for which $D_E(x)$ does not exist.

Problem 19.17. Define the *decentered* Hardy-Littlewood maximal function for $f \in L^1(\mathbb{R}^n)$ by

$$M_f^*(x) = \sup_B \frac{1}{m(B)} \int_B |f(x)|\, dx \tag{64}$$

where the supremum is taken over all open balls containing x (not just those centered at x). Prove that

$$M_f \leq M_f^* \leq 2^n M_f. \tag{65}$$

19.4. **Signed measures and the Lebesgue-Radon-Nikodym theorem.**

Problem 19.18. Prove Proposition 18.4.

Problem 19.19. Complete the proof of Theorem 18.5.

Problem 19.20. Prove the uniqueness statement in the Jordan decomposition theorem. (Hint: if also $\rho = \sigma_+ - \sigma_-$, use $\sigma_\pm$ to obtain another Hahn decomposition of X.)

Problem 19.21. Prove that if $\nu_j \perp \mu$ for $j \in \mathbb{N}$ then $(\sum_j \nu_j) \perp \mu$, and if $\nu_j \ll \mu$ for $j \in \mathbb{N}$ then $(\sum_j \nu_j) \ll \mu$.

Problem 19.22. Complete the proof of Proposition 18.9.

Problem 19.23. Prove Proposition 18.10.

Problem 19.24. Complete the proof of Theorem 18.12 in the σ-finite case.

Problem 19.25. Complete the proof of the (i) $\implies$ (iii) implication in Corollary 18.15

Problem 19.26. Suppose ρ is a signed measure on $(X, \mathscr{M})$ and $E \in \mathscr{M}$. Prove that

a) $\rho_+(E) = \sup\{\rho(F) : F \in \mathscr{M}, F \subset E\}$ and $\rho_-(E) = -\inf\{\rho(F) : F \in \mathscr{M}, F \subset E\}$
b) $|\rho|(E) = \sup\{\sum_1^n |\rho(E_j)| : E_1, \ldots E_n$ are disjoint and $\cup_1^n E_j = E\}$

Problem 19.27. a) Prove Proposition 18.16. b) In the case $\mu =$ Lebesgue measure on $[0,1)$, fix a positive integer k and let $\mathscr{N}$ be the sub-σ-algebra generated by the intervals $[\frac{j}{k}, \frac{j+1}{k})$ for $j = 0, \dots k-1$. Give an explicit formula for the conditional expectation g in terms of f. c) Show that the σ-finite hypothesis on ν is needed.

19.5. **The Riesz-Markov Theorem.**

Problem 19.28. Explain how to construct Lebesgue measure on $[0,1]$ from the Riemann integral and Theorem 14.2.

Problem 19.29. Suppose X is a locally compact abelian *topological group* (the definitions are available online). Given $y \in X$, let $t_y : X \to X$ denote translation by y so that $t_y(x) = x + y$ (the group is abelian so the group operation is written as $+$). A linear functional $\lambda : C_c(X) \to \mathbb{C}$ is translation invariant if $\lambda(f) = \lambda(f \circ t_y)$ for each $y \in X$ and $f \in C(X)$. Prove, if λ is a positive linear functional that is translation invariant, then the representing measure μ for λ from Theorem 14.2 is translation invariant.

Problem 19.30. Let X be a compact Hausdorff space. Fix $p \in X$ and consider the linear functional $E_p : C(X) \to \mathbb{C}$ defined by $E_p(f) = f(p)$. Show E_p is positive and determine the representing measure for E_p.

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