MAA6617 COURSE NOTES
SPRING 2014

19. Normed vector spaces

Let $\mathcal{X}$ be a vector space over a field $\mathbb{K}$ (in this course we always have either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$).

Definition 19.1. A *norm* on $\mathcal{X}$ is a function $\|\cdot\| : \mathcal{X} \to \mathbb{K}$ satisfying:

(i) (postivity) $\|x\| \geq 0$ for all $x \in \mathcal{X}$, and $\|x\| = 0$ if and only if $x = 0$;
(ii) (homogeneity) $\|kx\| = |k|\|x\|$ for all $x \in \mathcal{X}$ and $k \in \mathbb{K}$, and
(iii) (triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathcal{X}$.

Using these three properties it is straightforward to check that the quantity

$$d(x, y) := \|x - y\| \tag{19.1}$$

defines a metric on $\mathcal{X}$. The resulting topology is called the *norm topology.* The next proposition is simple but fundamental; it says that the norm and the vector space operations are continuous in the norm topology.

Proposition 19.2 (Continuity of vector space operations)**.** *Let $\mathcal{X}$ be a normed vector space over $\mathbb{K}$.*

a) *If $x_n \to x$ in $\mathcal{X}$, then $\|x_n\| \to \|x\|$ in $\mathbb{R}$.*
b) *If $k_n \to k$ in $\mathbb{K}$ and $x_n \to x$ in $\mathcal{X}$, then $k_n x_n \to kx$ in $\mathcal{X}$.*
c) *If $x_n \to x$ and $y_n \to y$ in $\mathcal{X}$, then $x_n + y_n \to x + y$ in $\mathcal{X}$.*

Proof. The proofs follow readily from the properties of the norm, and are left as exercises. □

We say that two norms $\|\cdot\|_1, \|\cdot\|_2$ on $\mathcal{X}$ are *equivalent* if there exist absolute constants $C, c > 0$ such that

$$c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1 \quad \text{for all } x \in \mathcal{X}. \tag{19.2}$$

Equivalent norms defined the same topology on $\mathcal{X}$, and the same Cauchy sequences (Problem 20.2). A normed space is called a *Banach space* if it is complete in the norm topology, that is, every Cauchy sequence in $\mathcal{X}$ converges in $\mathcal{X}$. It follows that if $\mathcal{X}$ is equipped with two equivalent norms $\|\cdot\|_1, \|\cdot\|_2$ then it is complete in one norm if and only if it is complete in the other.

We will want to prove the completeness of the first examples we consider, so we begin with a useful proposition. Say a series $\sum_{n=1}^\infty x_n$ in $\mathcal{X}$ is *absolutely convergent* if $\sum_{n=1}^\infty \|x_n\| < \infty$. Say that the series *converges* in $\mathcal{X}$ if the limit $\lim_{N\to\infty} \sum_{n=1}^N x_n$ exists in $\mathcal{X}$ (in the norm topology). (Quite explicitly, we say that the series $\sum_{n=1}^\infty x_n$ converges to $x \in \mathcal{X}$ if $\lim_{N\to\infty} \left\| x - \sum_{n=1}^N x_n \right\| = 0$.)

Proposition 19.3. *A normed space $(\mathcal{X}, \|\cdot\|)$ is complete if and only if every absolutely convergent series in $\mathcal{X}$ is convergent.*

Proof. First suppose $\mathcal{X}$ is complete and let $\sum_{n=1}^{\infty} x_n$ be absolutely convergent. Write $s_N = \sum_{n=1}^{N} x_n$ for the N^{th} partial sum and let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} \|x_n\|$ is convergent, we can choose L such that $\sum_{n=L}^{\infty} \|x_n\| < \epsilon$. Then for all $N > M \geq L$,

$$\|s_N - s_M\| = \left\| \sum_{n=M+1}^{N} x_n \right\| \leq \sum_{n=M+1}^{N} \|x_n\| < \epsilon \tag{19.3}$$

so the sequence (s_N) is Cauchy in $\mathcal{X}$, hence convergent by hypothesis.

Conversely, suppose every absolutely convergent series in $\mathcal{X}$ is convergent. Let (x_n) be a Cauchy sequence in $\mathcal{X}$. The idea is to arrange an absolutely convergent series whose partial sums form a subsequence of (x_n). To do this, first choose N_1 such that $\|x_n - x_m\| < 2^{-1}$ for all $n, m \geq N_1$ and put $y_1 = x_{N_1}$. Next choose $N_2 > N_1$ such that $\|x_n - x_m\| < 2^{-2}$ for all $n, m \geq N_2$ and put $y_2 = x_{N_2} - x_{N_1}$. Continuing, we may choose inductively an increasing sequence of integers $(N_k)_{k=1}^{\infty}$ such that $\|x_n - x_m\| < 2^{-k}$ for all $n, m \geq N_k$ and define $y_k = x_{N_k} - x_{N_{k-1}}$. We have $\sum_{k=1}^{\infty} \|y_k\| < \sum_{k=1}^{\infty} 2^{-(k-1)} < \infty$ by construction, and the K^{th} partial sum of $\sum_{k=1}^{\infty} y_k$ is x_{N_K}. Thus by hypothesis, the series $\sum_{k=1}^{\infty} y_k$ is convergent in $\mathcal{X}$, which means that the subsequence (x_{N_k}) of (x_n) is convergent in $\mathcal{X}$.

The proof is finished by invoking a standard fact about convergence in metric spaces: if (x_n) is a Cauchy sequence which has a subsequence converging to x, then the full sequence converges to x. □

19.1. **Examples.**

a) Of course, $\mathbb{R}^n$ with the usual Euclicean norm $\|(x_1, \dots x_n)\| = (\sum_{k=1}^{n} |x_k|^2)^{1/2}$ is a Banach space. (Likewise $\mathbb{C}^n$ with the Euclidean norm.) Besides these, $\mathbb{K}^n$ can be equipped with the ℓ^p-*norms*

$$\|(x_1, \dots x_n)\|_p := \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} \tag{19.4}$$

for $1 \leq p < \infty$, and the ℓ^∞-*norm*

$$\|(x_1, \dots x_n)\|_\infty := \max(|x_1|, \dots |x_n|). \tag{19.5}$$

It is not too hard to show that all of the ℓ^p norms $(1 \leq p \leq \infty)$ are equivalent on $\mathbb{K}^n$ (though the constants c, C must depend on the dimension n). It turns out that *any* two norms on a finite-dimensional vector space are equivalent. As a corollary, every finite-dimensional normed space is a Banach space. See Problem 20.3.

b) (Sequence spaces) Define

$$c_0 := \{f : \mathbb{N} \to \mathbb{K} | \lim_{n \to \infty} |f(n)| = 0\}$$
$$\ell^\infty := \{f : \mathbb{N} \to \mathbb{K} | \sup_{n \in \mathbb{N}} |f(n)| < \infty\}$$
$$\ell^1 := \{f : \mathbb{N} \to \mathbb{K} | \sum_{n=0}^{\infty} |f(n)| < \infty\}$$

It is a simple exercise to check that each of these is a vector space. Define for functions $f : \mathbb{N} \to \mathbb{K}$

$$\|f\|_\infty := \sup_n |f(n)|$$

$$\|f\|_1 := \sum_{n=1}^{\infty} |f(n)|.$$

Then $\|f\|_\infty$ defines a norm on both c_0 and ℓ^∞, and $\|f\|_1$ is a norm on ℓ^1. Equipped with these respective norms, each is a Banach space. We sketch the proof for c_0, the others are left as exercises (Problem 20.4).

The key observation is that $f_n \to f$ in the $\|\cdot\|_\infty$ norm if and only if $f_n \to f$ uniformly as functions on $\mathbb{N}$. Suppose (f_n) is a Cauchy sequence in c_0. Then the sequence of functions f_n is uniformly Cauchy on $\mathbb{N}$, and in particular converges pointwise to a function f. To check completeness it will suffice to show that also $f \in c_0$, but this is a straightforward consequence of uniform convergence on $\mathbb{N}$. The details are left as an exercise.

Along with these spaces it is also helpful to consider the vector space

$$c_{00} := \{f : \mathbb{N} \to \mathbb{K} | f(n) = 0 \text{ for all but finitely many } n\} \tag{19.6}$$

Notice that c_{00} is a vector subspace of each of c_0, ℓ^1 and ℓ^∞. Thus it can be equipped with either the $\|\cdot\|_\infty$ or $\|\cdot\|_1$ norms. It is not complete in either of these norms, however. What is true is that c_{00} is *dense* in c_0 and ℓ^1 (but not in ℓ^∞). (See Problem 20.9).

c) (L^1 spaces) Let $(X, \mathscr{M}, m)$ be a measure space. The quantity

$$\|f\|_1 := \int_X |f| \, dm \tag{19.7}$$

defines a norm on $L^1(m)$, provided we agree to identify f and g when $f = g$ a.e. (Indeed the chief motivation for making this identification is that it makes $\|\cdot\|_1$ into a norm. Note that ℓ^1 from the last example is a special case of this (what is the measure space?))

Proposition 19.4. *$L^1(m)$ is a Banach space.*

Proof. We use Proposition 19.3. Suppose $\sum_{n=1}^\infty f_n$ is absolutely convergent. Then the function $g := \sum_{n=1}^\infty |f_n|$ belongs to L^1 and is thus finite m-a.e. In particular the series $\sum_{n=1}^\infty f_n(x)$ is absolutely convergent in $\mathbb{K}$ for a.e. $x \in X$. Define $f(x) = \sum_{n=1}^\infty f_n(x)$. Then f is an a.e.-defined measurable function, and belongs to $L^1(m)$ since $|f| \leq g$. We claim that the partial sums $\sum_{n=1}^N f_n$ converge to f in the $L^1(m)$ norm: indeed,

$$\left\| f - \sum_{n=1}^N f_n \right\|_1 = \int_X \left| f - \sum_{n=1}^N f_n \right| dm \tag{19.8}$$

$$\leq \int_X \sum_{n=N+1}^\infty |f_n| \, dm \tag{19.9}$$

$$= \sum_{n=N+1} \|f_n\|_1 \to 0 \tag{19.10}$$

as $N \to \infty$. (What justifies the equality in the last line?) $\square$

d) (L^p spaces) Again let $(X, \mathscr{M}, m)$ be a measure space. For $1 \le p < \infty$ let $L^p(m)$ denote the set of measurable functions f for which

$$\|f\|_p := \left(\int_X |f|^p \, dm \right)^{1/p} < \infty \tag{19.11}$$

(again we identify f and g when $f = g$ a.e.). It turns out that this quantity is a norm on $L^p(m)$, and $L^p(m)$ is complete, though we will not prove this yet (it is not immediately obvious that the triangle inequality holds when $p > 1$). The sequence space ℓ^p is defined analagously: it is the set of $f : \mathbb{N} \to \mathbb{K}$ for which

$$\|f\|_p := \left(\sum_{n=1}^{\infty} |f(n)|^p \right)^{1/p} < \infty \tag{19.12}$$

and this quantity is a norm making ℓ^p into a Banach space.

When $p = \infty$, we define $L^\infty(m)$ to be the set of all functions $f : X \to \mathbb{K}$ with the following property: there exists $M > 0$ such that

$$|f(x)| \le M \qquad \text{for } m-\text{a.e. } x \in X; \tag{19.13}$$

as for the other L^p spaces we identify f and g when there are equal a.e. When $f \in L^\infty$, let $\|f\|_\infty$ be the smallest M for which (19.13) holds. Then $\|\cdot\|_\infty$ is a norm making $L^\infty(m)$ into a Banach space.

e) ($C(X)$ spaces) Let X be a compact metric space and let $C(X)$ denote the set of continuous functions $f : X \to \mathbb{K}$. It is a standard fact from advanced calculus that the quantity $\|f\|_\infty := \sup_{x \in X} |f(x)|$ is a norm on $C(X)$. A sequence is Cauchy in this norm if and only if it is uniformly Cauchy. It is thus also a standard fact that $C(X)$ is complete in this norm—completeness just means that a uniformly Cauchy sequence of continuous functions on X converges uniformly to a continuous function.

This example can be generalized somewhat: let X be a locally compact metric space. Say a function $f : X \to \mathbb{K}$ *vanishes at infinity* if for every $\epsilon > 0$, there exists a compact set $K \subset X$ such that $\sup_{x \notin K} |f(x)| < \epsilon$. Let $C_0(X)$ denote the set of continuous functions $f : X \to \mathbb{K}$ that vanish at inifinity. Then $C_0(X)$ is a vector space, the quantity $\|f\|_\infty := \sup_{x \in X} |f(x)|$ is a norm on $C_0(X)$, and $C_0(X)$ is complete in this norm. (Note that c_0 from above is a special case.)

f) (Subspaces and direct sums) If $(\mathcal{X}, \|\cdot\|)$ is a normed vector space and $\mathcal{Y} \subset \mathcal{X}$ is a vector subspace, then the restriction of $\|\cdot\|$ to $\mathcal{Y}$ is clearly a norm on $\mathcal{Y}$. If $\mathcal{X}$ is a Banach space, then $(\mathcal{Y}, \|\cdot\|)$ is a Banach space if and only if $\mathcal{Y}$ is *closed* in the norm topology of $\mathcal{X}$. (This is just a standard fact about metric spaces—a subspace of a complete metric space is complete in the restricted metric if and only if it is closed.)

If $\mathcal{X}, \mathcal{Y}$ are vector spaces then the *algebraic direct sum* is the vector space of ordered pairs

$$\mathcal{X} \oplus \mathcal{Y} := \{(x, y) : x \in \mathcal{X}, y \in \mathcal{Y}\} \tag{19.14}$$

with entrywise operations. If $\mathcal{X}$, $\mathcal{Y}$ are equipped with norms $\|\cdot\|_{\mathcal{X}}, \|\cdot\|_{\mathcal{Y}}$, then each of the quanitites

$$\|(x,y)\|_\infty := \max(\|x\|_{\mathcal{X}}, \|y\|_{\mathcal{Y}}),$$
$$\|(x,y)\|_1 := \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$$

is a norm on $\mathcal{X} \oplus \mathcal{Y}$. These two norms are equivalent; indeed it follows from the definitions that

$$\|(x,y)\|_\infty \leq \|(x,y)\|_1 \leq 2\|(x,y)\|_\infty. \tag{19.15}$$

If $\mathcal{X}$ and $\mathcal{Y}$ are both complete, then $\mathcal{X} \oplus \mathcal{Y}$ is complete in both of these norms. The resulting Banach spaces are denoted $\mathcal{X} \oplus_\infty \mathcal{Y}$, $\mathcal{X} \oplus_1 \mathcal{Y}$ respectively.

g) (Quotient spaces) If $\mathcal{X}$ is a normed vector space and $\mathcal{M}$ is a proper subspace, then one can form the *algebraic quotient* $\mathcal{X}/\mathcal{M}$, defined as the collection of distinct cosets $\{x + \mathcal{M} : x \in \mathcal{X}\}$. From linear algebra, $\mathcal{X}/\mathcal{M}$ is a vector space under the standard operations. If $\mathcal{M}$ is a *closed* subspace of $\mathcal{X}$, then the quantity

$$\|x + \mathcal{M}\| := \inf_{y \in \mathcal{M}} \|x - y\| \tag{19.16}$$

is a norm on $\mathcal{X}/\mathcal{M}$, called the *quotient norm.* (Geometrically, $\|x+\mathcal{M}\|$ is the distance in $\mathcal{X}$ from x to the closed set $\mathcal{M}$.) It turns out that if $\mathcal{X}$ is complete, so is $\mathcal{X}/\mathcal{M}$. See Problem 20.20.

More examples are given in the exercises. Shortly we will construct further examples from linear transformations $T : \mathcal{X} \to \mathcal{Y}$; to do this we first need to build up a few facts.

19.2. **Linear transformations between normed spaces.**

Definition 19.5. Let $\mathcal{X}, \mathcal{Y}$ be normed vector spaces. A linear transformation $T : \mathcal{X} \to \mathcal{Y}$ is called *bounded* if there exists a constant $C > 0$ such that $\|Tx\|_{\mathcal{Y}} \leq C\|x\|_{\mathcal{X}}$ for all $x \in \mathcal{X}$.

Remark: Note that in this definition it would suffice to require that $\|Tx\|_{\mathcal{Y}} \leq C\|x\|_{\mathcal{X}}$ just for all $x \neq 0$, or for all x with $\|x\|_{\mathcal{X}} = 1$ (why?)

The importance of boundedness is hard to overstate; the following proposition explains its importance.

Proposition 19.6. *Let $T : \mathcal{X} \to \mathcal{Y}$ be a linear transformation between normed spaces. Then the following are equivalent:*

(i) *T is bounded.*
(ii) *T is continuous.*
(iii) *T is continuous at 0.*

Proof. Suppose T is bounded and $\|Tx\| \leq C\|x\|$ for all $x \in \mathcal{X}$; let $\epsilon > 0$. Then if $\|x_1 - x_2\| < \delta := \epsilon/C$, we have

$$\|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \leq C\|x_1 - x_2\| < \epsilon \tag{19.17}$$

so T is continuous, and (i) implies (ii).

(ii) implies (iii) is trivial. For (iii) implies (i), we exploit homogeneity of the norm and the linearity of T. By hypothesis, given $\epsilon > 0$ there exists $\delta > 0$ such that if $\|x\| < \delta$, then

$\|Tx\| < \epsilon$. Fix a nonzero vector $x \in \mathcal{X}$ and a real number $0 < \lambda < \delta$. The vector $\lambda x/\|x\|$ has norm less than δ, so

$$\left\| T\left(\frac{\lambda x}{\|x\|}\right) \right\| = \lambda \frac{\|Tx\|}{\|x\|} < \epsilon. \tag{19.18}$$

Rearranging this we find $\|Tx\| \leq (\epsilon/\lambda)\|x\|$ for all $x \neq 0$, which shows T is bounded; in fact we can take $C = \epsilon/\delta$. □

Definition 19.7 (The operator norm)**.** Let $\mathcal{X}, \mathcal{Y}$ be normed vector spaces and $T : \mathcal{X} \to \mathcal{Y}$ a bounded linear transformation. Define

$$\|T\| := \sup\{\|Tx\| : \|x\| = 1\} \tag{19.19}$$

$$= \sup_{x \neq 0}\{\frac{\|Tx\|}{\|x\|}\} \tag{19.20}$$

$$= \inf\{C : \|Tx\| \leq C\|x\| \text{ for all } x \in \mathcal{X}\} \tag{19.21}$$

(As an exercise, verify that the three quantities on the right-hand side are equal.) The quantity $\|T\|$ is called the *operator norm* of T. We write $B(\mathcal{X}, \mathcal{Y})$ for the set of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. Note that immediately we have the inequality

$$\|Tx\| \leq \|T\|\|x\| \tag{19.22}$$

for all $x \in \mathcal{X}$. If $T \in B(\mathcal{X}, \mathcal{Y})$ and $S \in B(\mathcal{Y}, \mathcal{Z})$ then from two applications of the the inequality (19.22) we have for all $x \in \mathcal{X}$

$$\|STx\| \leq \|S\|\|Tx\| \leq \|S\|\|T\|\|x\| \tag{19.23}$$

and it follows that $ST \in B(\mathcal{X}, \mathcal{Z})$ and $\|ST\| \leq \|S\|\|T\|$. If $\mathcal{X}$ is a Banach space, then by the next proposition $B(\mathcal{X}, \mathcal{X})$ is complete, and this inequality says that $B(\mathcal{X}, \mathcal{X})$ is a *Banach algebra.*

Proposition 19.8. *The operator norm makes $B(\mathcal{X}, \mathcal{Y})$ into a normed vector space, which is complete if $\mathcal{Y}$ is complete.*

Proof. That $B(\mathcal{X}, \mathcal{Y})$ is a normed vector space follows readily from the definitions and is left as an exercise. Suppose now $\mathcal{Y}$ is complete, and let T_n be a Cauchy sequence in $B(\mathcal{X}, \mathcal{Y})$. For each $x \in \mathcal{X}$, we have

$$\|T_n x - T_m x\| = \|(T_n - T_m)x\| \leq \|T_n - T_m\|\|x\| \tag{19.24}$$

which shows that $(T_n x)$ is a Cauchy sequence in $\mathcal{Y}$. By hypothesis, $T_n x$ converges in $\mathcal{Y}$. Define $T : \mathcal{X} \to \mathcal{Y}$ by setting $Tx := y$. It is straightforward to check that T is linear. We must show that T is bounded and $\|T_n - T\| \to 0$.

To see that T is bounded, note first that since (T_n) is a Cauchy sequence in the metric space $B(\mathcal{X}, \mathcal{Y})$, it is bounded as a subset of $B(\mathcal{X}, \mathcal{Y})$; that is, $M := \sup_n \|T_n\| = \sup_n d(0, T_n) < \infty$. But then $\|T_n x\| \leq M\|x\|$ for every x, and since $Tx = \lim T_n x$, we have $\|Tx\| \leq M\|x\|$ for every x also.

Finally, given $\epsilon > 0$ choose N so that $\|T_n - T_m\| < \epsilon$ for all $n, m \geq N$. Then for all $x \in \mathcal{X}$ with $\|x\| = 1$, we have $\|T_n x - T_m x\| < \epsilon$. Taking the limit as $m \to \infty$, we see that $\|T_n x - Tx\| \leq \epsilon$ for all $\|x\| = 1$; thus $\|T_n - T\| \leq \epsilon$ for $n \geq N$. □

The following proposition is very useful in constructing bounded operators—at least when the codomain is complete, it suffices to define the operator (and show that it is bounded) on a dense subspace:

Proposition 19.9 (Extending bounded operators)**.** *Let $\mathcal{X}$, $\mathcal{Y}$ be normed vector spaces with $\mathcal{Y}$ complete, and $\mathcal{E} \subset \mathcal{X}$ a dense linear subspace. If $T : \mathcal{E} \to \mathcal{Y}$ is a bounded linear operator, then there exists a unique bounded linear operator $\widetilde{T} : \mathcal{X} \to \mathcal{Y}$ extending T (so $\widetilde{T}|_{\mathcal{E}} = T$), and $\|\widetilde{T}\| = \|T\|$.*

Proof. Let $x \in \mathcal{X}$. By hypothesis there is a sequence (x_n) in $\mathcal{E}$ converging to x; in particular this sequence is Cauchy. Since T is bounded on $\mathcal{E}$, the sequence (Tx_n) is Cauchy in $\mathcal{Y}$, hence convergent by hypothesis. Define $\widetilde{T}x = \lim_n Tx_n$; using the fact that T is bounded on $\mathcal{E}$ it is not hard to see that $\widetilde{T}$ is well-defined and agrees with T on $\mathcal{E}$ (the proof is an exercise). It also follows readily that $\widetilde{T}$ is linear. To see that $\widetilde{T}$ is bounded (and prove the equality of norms), fix $x \in \mathcal{X}$ with $\|x\| = 1$ and let (x_n) be a sequence in $\mathcal{E}$ converging to x. Then $\|x_n\| \to \|x\|$; in particular for any ϵ we have $\|x_n\| < 1 + \epsilon$ for all n sufficiently large. By enlarging n if necessary, we may also assume that $\|\widetilde{T}x - Tx_n\| < \epsilon$. But then using the triangle inequality,

$$\|\widetilde{T}x\| < \epsilon + \|Tx_n\| < \epsilon + (1+\epsilon)\|T\|. \tag{19.25}$$

Since ϵ was arbitrary, we have $\|\widetilde{T}\| \leq \|T\|$. As the reverse inequality is trivial, the proof is finished.

□

Remark: The completeness of $\mathcal{Y}$ is essential in the above proposition; Problem 20.11 suggests a counterexample.

A bounded linear transformation $T \in B(\mathcal{X}, \mathcal{Y})$ is said to be *invertible* if it is bijective (so T^{-1} exists as a linear transformation) and T^{-1} is bounded from $\mathcal{Y}$ to $\mathcal{X}$. We can now define two useful notions of equivalence for normed vector spaces. Two normed spaces $\mathcal{X}, \mathcal{Y}$ are said to be *(boundedly) isomorphic* if there exists an invertible linear transformation $T : \mathcal{X} \to \mathcal{Y}$. The spaces are *isometrically isomorphic* if additionally $\|Tx\| = \|x\|$ for all x. A T with this property is called an *isometry*; note that an isometry automatically injective; if it is also surjective then it is automatically invertible, in which case T^{-1} is also an isometry. An isometry need not be surjective, however.

19.3. **Examples.**

a) If $\mathcal{X}$ is a finite-dimensional normed space and $\mathcal{Y}$ is any normed space, then every linear transformation $T : \mathcal{X} \to \mathcal{Y}$ is bounded.
b) Let $\mathcal{X}$ be c_{00} equipped with the $\|\cdot\|_1$ norm, and $\mathcal{Y}$ be c_{00} equipped with the $\|\cdot\|_\infty$ norm. Then the identity map $\mathrm{id} : c_{00} \to c_{00}$ is bounded as an operator from $\mathcal{X}$ to $\mathcal{Y}$ (in fact its norm is equal to 1), but is unbounded as an operator from $\mathcal{Y}$ to $\mathcal{X}$.
c) Consider c_{00} with the $\|\cdot\|_\infty$ norm. Let $a : \mathbb{N} \to \mathbb{K}$ be any function and define a linear transformation $T_a : c_{00} \to c_{00}$ by

$$T_a f(n) = a(n) f(n). \tag{19.26}$$

Then T_a is bounded if and only if $M = \sup_{n \in \mathbb{N}} |a(n)| < \infty$, in which case $\|T_a\| = M$. When this happens, T_a extends uniquely to a bounded operator from c_0 to c_0, and one may check that the formula (19.26) defines the extension. All of these claims remain true if we use the $\|\cdot\|_1$ norm instead of the $\|\cdot\|_\infty$ norm; we then get a bounded operator from ℓ^1 to itself.

d) For $f \in \ell^1$, define the *shift operator* by $Sf(1) = 0$ and $Sf(n) = f(n-1)$ for $n \geq 1$. (Viewing f as a sequence, S shifts the sequence one place to the right and fills in a 0 in the first position). This S is an isometry, but is not surjective. In contrast, if $\mathcal{X}$ is finite-dimensional, then the rank-nullity theorem from linear algebra guarantees that every injective linear map $T : \mathcal{X} \to \mathcal{X}$ is also surjective.

e) Let $C^\infty[0,1]$ denote the space of functions on $[0,1]$ with continuous derivatives of all orders. The differentiation map $f \to \frac{df}{dx}$ is a linear transformation from $C^\infty[0,1]$ to itself. Since $\frac{d}{dx}e^{tx} = te^{tx}$ for all real t, we find that there is *no* norm on $C^\infty[0,1]$ for which $\frac{d}{dx}$ is bounded.

20. Problems

Problem 20.1. Prove Proposition 19.2.

Problem 20.2. Prove that equivalent norms define the same topology and the same Cauchy sequences.

Problem 20.3. a) Prove that all norms on a finite dimensional vector space $\mathcal{X}$ are equivalent. (Hint: fix a basis $e_1, \dots e_n$ for $\mathcal{X}$ and define $\|\sum a_k e_k\|_1 := \sum |a_k|$. Compare any given norm $\|\cdot\|$ to this one. Begin by proving that the "unit sphere" $S = \{x \in \mathcal{X} : \|x\|_1 = 1\}$ is compact in the $\|\cdot\|_1$ topology.)

b) Combine the result of part (a) with the result of Problem 20.2 to conclude that every finite-dimensional normed vector space is complete.

c) Let $\mathcal{X}$ be a normed vector space and $\mathcal{M} \subset \mathcal{X}$ a finite-dimensional subspace. Prove that $\mathcal{M}$ is closed in $\mathcal{X}$.

Problem 20.4. Finish the proofs from Example 19.1(b).

Problem 20.5. A function $f : [0,1] \to \mathbb{K}$ is called *Lipschitz continuous* if there exists a constant C such that

$$|f(x) - f(y)| \leq C|x - y| \tag{20.1}$$

for all $x, y \in [0,1]$. Define $\|f\|_{Lip}$ to be the best possible constant in this inequality. That is,

$$\|f\|_{Lip} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \tag{20.2}$$

Let $\mathrm{Lip}[0,1]$ denote the set of all Lipschitz continuous functions on $[0,1]$. Prove that $\|f\| := |f(0)| + \|f\|_{Lip}$ is a norm on $\mathrm{Lip}[0,1]$, and that $\mathrm{Lip}[0,1]$ is complete in this norm.

Problem 20.6. Let $C^1[0,1]$ denote the space of all functions $f : [0,1] \to \mathbb{R}$ such that f is differentiable in $(0,1)$ and f' extends continuously to $[0,1]$. Prove that

$$\|f\| := \|f\|_\infty + \|f'\|_\infty \tag{20.3}$$

is a norm on $C^1[0,1]$ and that C^1 is complete in this norm. Do the same for the norm $\|f\| := |f(0)| + \|f'\|_\infty$. (Is $\|f'\|_\infty$ a norm on C^1?)

Problem 20.7. Let $(X, \mathscr{M})$ be a measurable space. Let $M(X)$ denote the (real) vector space of all signed measures on $(X, \mathscr{M})$. Prove that the *total variation norm* $\|\mu\| := |\mu|(X)$ is a norm on $M(X)$, and $M(X)$ is complete in this norm.

Problem 20.8. Prove that if $\mathcal{X}, \mathcal{Y}$ are normed spaces, then the operator norm is a norm on $B(\mathcal{X}, \mathcal{Y})$.

Problem 20.9. Prove that c_{00} is dense in c_0 and ℓ^1. (That is, given $f \in c_0$ there is a sequence f_n in c_{00} such that $\|f_n - f\|_\infty \to 0$, and the analogous statement for ℓ^1.) Using these facts, or otherwise, prove that c_{00} is *not* dense in ℓ^∞. (In fact there exists $f \in \ell^\infty$ with $\|f\|_\infty = 1$ such that $\|f - g\|_\infty \geq 1$ for all $g \in c_{00}$.)

Problem 20.10. Prove that c_{00} is not complete in the $\|\cdot\|_1$ or $\|\cdot\|_\infty$ norms. (After we have studied the Baire Category theorem, you will be asked to prove that there is *no* norm on c_{00} making it complete.)

Problem 20.11. Consider c_0 and c_{00} equipped with the $\|\cdot\|_\infty$ norm. Prove that there is no bounded operator $T : c_0 \to c_{00}$ such that $T|_{c_{00}}$ is the identity map. (Thus the conclusion of Proposotion 19.9 can fail if $\mathcal{Y}$ is not complete.)

Problem 20.12. Prove that the $\|\cdot\|_1$ and $\|\cdot\|_\infty$ norms on c_{00} are not equivalent. Conclude from your proof that the identity map on c_{00} is bounded from the $\|\cdot\|_1$ norm to the $\|\cdot\|_\infty$ norm, but not the other way around.

Problem 20.13. a) Prove that $f \in C_0(\mathbb{R}^n)$ if and only if f is continuous and $\lim_{|x|\to\infty} |f(x)| = 0$. b) Let $C_c(\mathbb{R}^n)$ denote the set of continuous, compactly supported functions on $\mathbb{R}^n$. Prove that $C_c(\mathbb{R}^n)$ is dense in $C_0(\mathbb{R}^n)$ (where $C_0(\mathbb{R}^n)$ is equipped with sup norm).

Problem 20.14. Prove that if $\mathcal{X}, \mathcal{Y}$ are normed spaces and $\mathcal{X}$ is finite dimensional, then every linear transformation $T : \mathcal{X} \to \mathcal{Y}$ is bounded.

Problem 20.15. Prove the claims in Example 19.3(c).

Problem 20.16. Let $g : \mathbb{R} \to \mathbb{K}$ be a (Lebesgue) measurable function. The map $Mg : f \to gf$ is a linear transformation on the space of measurable functions. Prove that M_g is bounded from $L^1(\mathbb{R})$ to itself if and only if $g \in L^\infty(\mathbb{R})$, in which case $\|M_g\| = \|g\|_\infty$.

Problem 20.17. Prove the claims about direct sums in Example 19.1(f).

Problem 20.18. Let $\mathcal{X}$ be a normed vector space and $\mathcal{M}$ a proper *closed* subspace. Prove that for every $\epsilon > 0$, there exists $x \in \mathcal{X}$ such that $\|x\| = 1$ and $\inf_{y \in \mathcal{M}} \|x - y\| > 1 - \epsilon$. (Hint: take any $u \in \mathcal{X} \setminus \mathcal{M}$ and let $a = \inf_{y \in \mathcal{M}} \|u - y\|$. Choose $\delta > 0$ small enough so that $\frac{a}{a+\delta} > 1 - \epsilon$, and then choose $v \in \mathcal{M}$ so that $\|u - v\| < a + \delta$. Finally let $x = \frac{u-v}{\|u-v\|}$.)

Problem 20.19. Prove that if $\mathcal{X}$ is an infinite-dimensional normed space, then the unit ball $ball(\mathcal{X}) := \{x \in \mathcal{X} : \|x\| \leq 1\}$ is not compact in the norm topology. (Hint: use the result of Problem 20.18 to construct inductively a sequence of vectors $x_n \in \mathcal{X}$ such that $\|x_n\| = 1$ for all n and $\|x_n - x_m\| \geq \frac{1}{2}$ for all $m < n$.)

Problem 20.20. (The quotient norm) Let $\mathcal{X}$ be a normed space and $\mathcal{M}$ a proper closed subspace.

a) Prove that the quotient norm is a norm (see Example 19.1(g)).
b) Show that the quotient map $x \to x + \mathcal{M}$ has norm 1. (Use Problem 20.18.)
c) Prove that if $\mathcal{X}$ is complete, so is $\mathcal{X}/\mathcal{M}$.

Problem 20.21. A normed vector space $\mathcal{X}$ is called *separable* if it is separable as a metric space (that is, there is a countable subset of $\mathcal{X}$ which is dense in the norm topology). Prove that c_0 and ℓ^1 are separable, but ℓ^∞ is not. (Hint: for ℓ^∞, show that there is an uncountable collection of elements $\{f_\alpha\}$ such that $\|f_\alpha - f_\beta\| = 1$ for $\alpha \neq \beta$.)

21. Linear functionals and the Hahn-Banach theorem

If there is a "fundamental theorem of functional analysis," it is the Hahn-Banach theorem. The particular version of it we will prove is somewhat abstract-looking at first, but its importance will be clear after studying some of its corollaries.

Let $\mathcal{X}$ be a normed vector space over the field $\mathbb{K}$. A *linear functional* on $\mathcal{X}$ is a linear map $L : \mathcal{X} \to \mathbb{K}$. As one might expect, we are especially interested in bounded linear functionals. Since $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is complete, the vector space of bounded linear functionals $B(\mathcal{X}, \mathbb{K})$ is itself a normed vector space, and is always complete (even if $\mathcal{X}$ is not). This space is called the *dual space* of $\mathcal{X}$ and is denoted $\mathcal{X}^*$. It is not yet obvious that $\mathcal{X}^*$ need be non-trivial (that is, that there are any bounded linear functionals on $\mathcal{X}$ besides 0). One corollary of the Hahn-Banach theorem will be that there always exist many linear functionals on any normed space $\mathcal{X}$ (in particular, enough to separate points of $\mathcal{X}$).

21.1. **Examples.**

a) For each of the sequence spaces c_0, ℓ^1, ℓ^∞, for each n the map $f \to f(n)$ is a bounded linear functional. If we fix $g \in \ell^1$, then the functional $L_g : c_0 \to \mathbb{K}$ defined by

$$L_g(f) := \sum_{n=0}^{\infty} f(n)g(n) \tag{21.1}$$

is bounded, since

$$|L_g(f)| \leq \sum_{n=0}^{\infty} |f(n)g(n)| \leq \|f\|_\infty \sum_{n=0}^{\infty} |g(n)| = \|g\|_1 \|f\|_\infty. \tag{21.2}$$

This shows that $\|L_g\| \leq \|g\|_1$. In fact, equality holds, and every bounded linear functional on c_0 is of this form:

Proposition 21.1. *The map $g \to L_g$ is an ismoetric isomorphism from ℓ^1 onto the dual space c_0^*.*

Proof. We have already seen that each $g \in \ell^1$ gives rise to a bounded linear functional $L_g \in c_0^*$ via

$$L_g(f) := \sum_{n=0}^{\infty} g(n)f(n) \tag{21.3}$$

and that $\|L_g\| \leq \|g\|_1$. We will prove simultaneously that this map is onto and that $\|L_g\| \geq \|g\|_1$.

Let $L \in c_0^*$, we will first show that there is unique $g \in \ell^1$ so that $L = L_g$. Let $e_n \in c_0$ be the indicator function of n, that is

$$e_n(m) = \delta_{nm}. \tag{21.4}$$

Define a function $g : \mathbb{N} \to \mathbb{K}$ by

$$g(n) = L(e_n). \tag{21.5}$$

We claim that $g \in \ell^1$ and $L = L_g$. To see this, fix an integer N and let $h \in c_{00}$ be the function

$$h(n) = \begin{cases} \overline{g(n)}/|g(n)| & \text{if } n \le N \text{ and } g(n) \neq 0 \\ 0 & \text{otherwise} \end{cases}. \tag{21.6}$$

By definition $h \in c_0$ and $\|h\|_\infty \le 1$. Note that $h = \sum_{n=0}^{N} h(n)e_n$. Now

$$\sum_{n=0}^{N} |g(n)| = \sum_{n=0}^{N} h(n)g(n) = L(h) = |L(h)| \le \|L\|\|h\| \le \|L\|. \tag{21.7}$$

It follows that $g \in \ell^1$ and $\|g\|_1 \le \|L\|$. Moreover, the same calculation shows that $L = L_g$ when restricted to c_{00}, so by the uniqueness of extensions of bounded operators, $L = L_g$. Uniqueness of g is clear from its construction, since if $L = L_g$, we must have $g(n) = L_g(e_n)$. Thus the map $g \to L_g$ is onto and $\|L_g\|_{c_0{}^*} = \|g\|_1$. □

Proposition 21.2. *$(\ell^1)^*$ is isometrically isomorphic to ℓ^∞.*

Proof. The proof follows the same lines as the proof of the previous proposition; the details are left as an exercise. □

The same mapping $g \to L_g$ also shows that every $g \in \ell^1$ gives a bounded linear functional on ℓ^∞, but it turns out these do not exhaust $(\ell^\infty)^*$ (see Problem 22.7).

b) If $f \in L^1(m)$ and g is a bounded measurable function with $\sup_{x\in X} |g(x)| = M$, then the map

$$L_g(f) := \int_X fg\, dm \tag{21.8}$$

is a bounded linear functional of norm at most M. We will prove in Section **??** that the norm is in fact equal to M, and every bounded linear functional on $L^1(m)$ is of this type (at least when m is σ-finite).

c) If X is a compact metric space and μ is a finite, signed Borel measure on X, then

$$L_\mu(f) := \int_X f\, d\mu \tag{21.9}$$

is a bounded linear functional on $C_{\mathbb{R}}(X)$ of norm at most $\|\mu\|$.

To state and prove the Hahn-Banach theorem, we first work in the setting $\mathbb{K} = \mathbb{R}$, then extend our results to the complex case.

Definition 21.3. Let $\mathcal{X}$ be a real vector space. A *Minkowski functional* is a function $p : \mathcal{X} \to \mathbb{R}$ such that $p(x + y) \le p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for all $x, y \in \mathcal{X}$ and nonnegative $\lambda \in \mathbb{R}$.

For example, if $L : \mathcal{X} \to \mathbb{R}$ is any linear functional, then the function $p(x) := |L(x)|$ is a Minkowski functional. Also $p(x) = \|x\|$ is a Minkowski functional. More generally, $p(x) = \|x\|$ is a Minkowski functional whenever $\|x\|$ is a *seminorm* on $\mathcal{X}$. (A seminorm is a function obeying all the requirements of a norm, except we allow $\|x\| = 0$ for nonzero x. Note that both of the given examples of Minkowski functionals come from seminorms.)

Theorem 21.4 (The Hahn-Banach Theorem, real version)**.** *Let $\mathcal{X}$ be a normed vector space over $\mathbb{R}$, p a Minkowski functional on $\mathcal{X}$, $\mathcal{M}$ a subspace of $\mathcal{X}$, and L a linear functional on $\mathcal{M}$ such that $L(x) \leq p(x)$ for all $x \in \mathcal{M}$. Then there exists a linear functional L' on $\mathcal{X}$ such that $L'(x) \leq p(x)$ for all $x \in \mathcal{X}$ and $L'|_{\mathcal{M}} = L$.*

Proof. The idea is to show that the extension can be done one dimension at a time, and we then infer the existence of an extension to the whole space by appeal to Zorn's lemma. So, fix a vector $x \in \mathcal{X} \setminus \mathcal{M}$ and consider the subspace $\mathcal{M} + \mathbb{R}x \subset \mathcal{X}$. For any $m_1, m_2 \in \mathcal{M}$, we have by hypothesis

$$L(m_1) + L(m_2) = L(m_1 + m_2) \leq p(m_1 + m_2) \leq p(m_1 - x) + p(m_2 + x). \tag{21.10}$$

which rearranges to

$$L(m_1) - p(m_1 - x) \leq p(m_2 + x) - L(m_2) \quad \text{for all } m_1, m_2 \in \mathcal{M} \tag{21.11}$$

so

$$\sup_{m \in \mathcal{M}} \{L(m) - p(m - x)\} \leq \inf_{m \in \mathcal{M}} \{p(m + x) - L(m)\}. \tag{21.12}$$

Now choose any real number λ satisfying

$$\sup_{m \in \mathcal{M}} \{L(m) - p(m - x)\} \leq \lambda \leq \inf_{m \in \mathcal{M}} \{p(m + x) - L(m)\}. \tag{21.13}$$

and define $L'(m + tx) = L(m) + t\lambda$. This L' is linear by definition and agrees with L on $\mathcal{M}$. We now check that $L'(y) \leq p(y)$ for all $y \in \mathcal{M} + \mathbb{R}x$. This is immediate if $y \in \mathcal{M}$. In general let $y = m + tx$ with $m \in \mathcal{M}$ and $t > 0$. Then

$$L'(m + tx) = t\left(L(\frac{m}{t}) + \lambda\right) \leq t\left(L(\frac{m}{t}) + p(\frac{m}{t} + x) - L(\frac{m}{t})\right) = p(m + tx) \tag{21.14}$$

and a similar estimate shows that $L'(m + tx) \leq p(m + tx)$ for $t < 0$.

We have thus successfully extended L to $\mathcal{M} + \mathbb{R}x$. To finish, let $\mathcal{L}$ denote the set of pairs $(L', \mathcal{N})$ where $\mathcal{N}$ is a subspace of $\mathcal{X}$ containing $\mathcal{M}$, and L' is an extension of L to $\mathcal{N}$ obeying $L'(y) \leq p(y)$ on $\mathcal{N}$. Declare $(L'_1, \mathcal{N}_1) \leq (L'_2, \mathcal{N}_2)$ if $\mathcal{N}_1 \subset \mathcal{N}_2$ and $L'_2|_{nn_1} = L'_1$. This is a partial order on $\mathcal{L}$. Given any increasing chain $(L'_\alpha, \mathcal{N}_\alpha)$ in $\mathcal{L}$, it has an upper bound $(L', \mathcal{N})$ in $\mathcal{L}$, where $\mathcal{N} := \bigcup_\alpha \mathcal{N}_\alpha$ and $L(n_\alpha) := L'_\alpha(n_\alpha)$ for $n_\alpha \in \mathcal{N}_\alpha$. By Zorn's lemma, then, the collection $\mathcal{L}$ has a maximal element $(L', \mathcal{N})$ with respect to the order $\leq$. Since it always possible to extend to a strictly larger subspace, the maximal element must have $\mathcal{N} = \mathcal{X}$, and the proof is finished. □

The proof is a typical application of Zorn's lemma—one knows how to carry out a construction one step a time, but there is no clear way to do it all at once.

In the special case that p is a seminorm, since $L(-x) = -L(x)$ and $p(-x) = p(x)$ the inequality $L \leq p$ is equivalent to $|L| \leq p$.

Corollary 21.5. *Let $\mathcal{X}$ be a normed vector space over $\mathbb{R}$, $\mathcal{M}$ a subspace, and L a bounded linear functional on $\mathcal{M}$ satisfying $|L(x)| \leq C\|x\|$ for all $x \in \mathcal{M}$. Then there exists a bounded linear functional L' on $\mathcal{X}$ extending L, with $\|L'\| \leq C$.*

Proof. Apply the Hahn-Banach theorem with the Minkowski functional $p(x) = C\|x\|$. □

Before obtaining further corollaries, we extend these results to the complex case. First, if $\mathcal{X}$ is a vector space over $\mathbb{C}$ then trivially it is also a vector space over $\mathbb{R}$, and there is a simple relationship between the $\mathbb{R}$- and $\mathbb{C}$-linear functionals.

Proposition 21.6. *Let $\mathcal{X}$ be a vector space over $\mathbb{C}$. If $L : \mathcal{X} \to \mathbb{C}$ is a $\mathbb{C}$-linear functional, then $u(x) = ReL(x)$ defines an $\mathbb{R}$-linear functional on $\mathcal{X}$, and $L(x) = u(x) - iu(ix)$. Conversely if $u : \mathcal{X} \to \mathbb{R}$ is $\mathbb{R}$-linear then $L(x) := u(x) - iu(ix)$ is $\mathbb{C}$-linear. If in addition $\mathcal{X}$ is normed, then $\|u\| = \|L\|$.*

Proof. Problem **??** ☐

Theorem 21.7 (The Hahn-Banach Theorem, complex version)**.** *Let $\mathcal{X}$ be a normed vector space over $\mathbb{C}$, p a seminorm on $\mathcal{X}$, $\mathcal{M}$ a subspace of $\mathcal{X}$, and $L : \mathcal{M} \to \mathbb{C}$ a $\mathbb{C}$-linear functional satisfying $|L(x)| \le p(x)$ for all $x \in \mathcal{M}$. Then there exists a linear functional $L' : \mathcal{X} \to \mathbb{C}$ satisfying $|L'(x)| \le p(x)$ for all $x \in \mathcal{X}$.*

Proof. The proof consists of applying the real Hahn-Banach theorem to extend the $\mathbb{R}$-linear functional $u = \mathrm{Re}L$ to a functional $u' : \mathcal{X} \to \mathbb{R}$ and then defining L' from u' as in Proposition 21.6. The details are left as an exercise. ☐

The following corollaries are quite important, and when the Hahn-Banach theorem is applied it is usually in one of the following forms:

Corollary 21.8. *Let $\mathcal{X}$ be a normed vector space.*

(i) (Linear functionals detect norms) *If $x \in \mathcal{X}$ is nonzero, there exists $L \in \mathcal{X}^*$ with $\|L\| = 1$ such that $L(x) = \|x\|$.*

(ii) (Linear functionals separate points) *If $x \neq y$ in $\mathcal{X}$, there exists $L \in \mathcal{X}^*$ such that $L(x) \neq L(y)$.*

(iii) (Linear functionals detect distance to subspaces) *If $\mathcal{M} \subset \mathcal{X}$ is a closed subspace and $x \in \mathcal{X} \setminus \mathcal{M}$, there exists $L \in \mathcal{X}^*$ such that $L|_{\mathcal{M}} = 0$ and $L(x) = dist(x, \mathcal{M}) = \inf_{y \in \mathcal{M}} \|x - y\| > 0$.*

Proof. (i): Let $\mathcal{M}$ be the one-dimensional subspace of $\mathcal{X}$ spanned by x. Define a functional $L : \mathcal{M} \to \mathbb{K}$ by $L(t\frac{x}{\|x\|}) = t$. For $p(x) = \|x\|$, we have $|L(x)| \le p(x)$ on $\mathcal{M}$, and $L(x) = \|x\|$. By the Hahn-Banach theorem the functional L extends to a functional (still denoted L) on $\mathcal{X}$ and satisfies $|L(y)| \le \|y\|$ for all $y \in \mathcal{X}$; thus $\|L\| \le 1$; since $L(x) = \|x\|$ by construction we conclude $\|L\| = 1$.

(ii): Apply (i) to the vector $x - y$.

(iii): Let $\delta = \mathrm{dist}(x, \mathcal{M})$. Define a functional $L : \mathcal{M} + \mathbb{K}x \to \mathbb{K}$ by $L(y + tx) = t\delta$. Since for $t \neq 0$

$$\|y + tx\| = |t|\|t^{-1}y + x\| \ge |t|\delta = |L(y + tx)|, \tag{21.15}$$

by Hahn-Banach we can extend L to a functional $L \in \mathcal{X}^*$ with $\|L\| \le 1$. ☐

Needless to say, the proof of the Hahn-Banach theorem is thoroughly non-constructive, and in general it is an important (and often difficult) problem, given a normed space $\mathcal{X}$, to find some concrete description of the dual space $\mathcal{X}^*$. Usually this means finding a Banach space $\mathcal{Y}$ and a bounded (or, better, isometric) isomorphism $T : \mathcal{Y} \to \mathcal{X}^*$.

A final corollary, which is also quite important. Note that since $\mathcal{X}^*$ is a normed space, we can form its dual, denoted $\mathcal{X}^{**}$ and called the *bidual* of $\mathcal{X}$. There is a canonical relationship between $\mathcal{X}$ and $\mathcal{X}^{**}$. First observe that if $L \in \mathcal{X}^*$, each fixed $x \in \mathcal{X}$ gives rise to a linear functional $\hat{x} : \mathcal{X}^* \to \mathbb{K}$ via evaluation:

$$\hat{x}(L) := L(x). \tag{21.16}$$

Corollary 21.9. *(Embedding in the bidual) The map $x \to \hat{x}$ is an isometric linear map from $\mathcal{X}$ into $\mathcal{X}^{**}$.*

Proof. First, from the definition we see that

$$|\hat{x}(L)| = |L(x)| \leq \|L\|\|x\| \tag{21.17}$$

so $\hat{x} \in \mathcal{X}^{**}$ and $\|\hat{x}\| \leq \|x\|$. It is straightforward to check (recalling that the L's are linear) that the map $x \to \hat{x}$ is linear. Finally, to show that $\|\hat{x}\| = \|x\|$, fix a nonzero $x \in \mathcal{X}$. From Corollary 21.8(i) there exists $L \in \mathcal{X}^*$ with $\|L\| = 1$ and $L(x) = \|x\|$. But then for this x and L, we have $|\hat{x}(L)| = |L(x)| = \|x\|$ so $\|\hat{x}\| \geq \|x\|$, and the proof is complete. □

Definition 21.10. A Banach space $\mathcal{X}$ is called *reflexive* if the map $\hat{}: \mathcal{X} \to \mathcal{X}^{**}$ is surjective.

In other words, $\mathcal{X}$ is reflexive if the map $\hat{}$ is an isomorphism of $\mathcal{X}$ with $\mathcal{X}^{**}$. For example, every finite dimensional Banach space is reflexive (Problem **??**). On the other hand, we will see below that $c_0^{**} = \ell^\infty$, so c_0 is not reflexive. After we have studied the L^p and ℓ^p spaces in more detail, we will see that L^p is reflexive for $1 < p < \infty$.

The embedding into the bidual has many applications; one of the most basic is the following:

Proposition 21.11 (Completion of normed spaces)**.** *If $\mathcal{X}$ is a normed vector space, there is a Banach space $\overline{\mathcal{X}}$ and in isometric map $\iota : \mathcal{X} \to \overline{\mathcal{X}}$ such that the image $\iota(\mathcal{X})$ is dense in $\overline{\mathcal{X}}$.*

Proof. Embed $\mathcal{X}$ into $\mathcal{X}^{**}$ via the map $x \to \hat{x}$ and let $\overline{\mathcal{X}}$ be the closure of the image of $\mathcal{X}$ in $\mathcal{X}^{**}$. Since $\overline{\mathcal{X}}$ is a closed subspace of a complete space, it is complete. □

The space $\overline{\mathcal{X}}$ is called the *completion* of $\mathcal{X}$. It is unique in the sense that if $\mathcal{Y}$ is another Banach space and $j : \mathcal{X} \to \mathcal{Y}$ embeds $\mathcal{X}$ isometrically as a dense subspace of $\mathcal{Y}$, then $\mathcal{Y}$ is isometrically isomorphic to $\overline{\mathcal{X}}$. The proof of this fact is left as an exercise.

21.2. **Dual spaces and adjoint operators.** Let X, Y be normed spaces with duals X^*, Y^*. If $T : X \to Y$ is a linear transformation, then given any linear map $f : Y \to \mathbb{K}$ we can define another linear map $T^*f : X \to \mathbb{K}$ by the formula

$$(T^*f)(x) = f(Tx). \tag{21.18}$$

In fact, if f is bounded then so is T^*f, and more is true:

Theorem 21.12. *The formula (21.18) defines a bounded linear transformation $T^* : Y^* \to X^*$, and in fact $\|T^*\| = \|T\|$.*

Proof. Let $f \in Y^*$ and $x \in X$. Then

$$|T^*f(x)| = |f(Tx)| \leq \|f\|\|Tx\| \leq \|f\|\|T\|\|x\|. \tag{21.19}$$

Taking the supremum over $\|x\| = 1$, we find that T^*f is bounded and $\|T^*f\| \leq \|T\|\|f\|$, so $\|T^*\| \leq \|T\|$. For the reverse inequality, let $0 < \epsilon < 1$ be given and choose $x \in X$ with $\|x\| = 1$ and $\|Tx\| > (1-\epsilon)\|T\|$. Now consider Tx. By the Hahn-Banach theorem, there exists $f \in Y^*$ such that $\|f\| = 1$ and $f(Tx) = \|Tx\|$. For this f, we have

$$\|T^*f\| \geq |T^*f(x)| = |f(Tx)| = \|Tx\| > (1-\epsilon)\|T\|. \tag{21.20}$$

Since ϵ was arbitrary, we conclude that $\|T^*\| := \sup_{\|f\|=1} \|T^*f\| \geq \|T\|$. □

22. Problems

Problem 22.1. a) Prove that if $\mathcal{X}$ is a finite-dimensional normed space, then every linear functional $f : \mathcal{X} \to \mathbb{K}$ is bounded.

b) Prove that if $\mathcal{X}$ is any normed vector space, $\{x_1, \dots x_n\}$ is a linearly independent set in $\mathcal{X}$, and $\alpha_1, \dots \alpha_n$ are scalars, then there exists a bounded linear functional f on $\mathcal{X}$ such that $f(x_j) = \alpha_j$ for $j = 1, \dots n$.

Problem 22.2. Let $\mathcal{X}, \mathcal{Y}$ be normed spaces and $T : \mathcal{X} \to \mathcal{Y}$ a linear transformation. Prove that T is bounded if and only if there exists a constant C such that for all $x \in \mathcal{X}$ and $f \in \mathcal{Y}^*$,

$$|f(Tx)| \leq C\|f\|\|x\|; \tag{22.1}$$

in which case $\|T\|$ is equal to the best possible C in (22.1).

Problem 22.3. Let $\mathcal{X}$ be a normed vector space. Show that if $\mathcal{M}$ is a closed subspace of $\mathcal{X}$ and $x \notin \mathcal{M}$, then $\mathcal{M} + \mathbb{C}x$ is closed. Use this to give another proof that every finite-dimensional subspace of $\mathcal{X}$ is closed.

Problem 22.4. Prove that if $\mathcal{M}$ is a *finite-dimensional* subspace of a Banach space $\mathcal{X}$, then there exists a closed subspace $\mathcal{N} \subset \mathcal{X}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N} = \mathcal{X}$. (In other words, every $x \in \mathcal{X}$ can be written uniquely as $x = y + z$ with $y \in \mathcal{M}$, $z \in \mathcal{N}$.) *Hint:* Choose a basis $x_1, \dots x_n$ for $\mathcal{M}$ and construct bounded linear functionals $f_1, \dots f_n$ on $\mathcal{X}$ such that $f_i(x_j) = \delta_{ij}$. Now let $\mathcal{N} = \cap_{i=1}^n \ker f_i$. (Warning: this conclusion can fail badly if $\mathcal{M}$ is not assumed finite dimensional, even if $\mathcal{M}$ is still assumed closed.)

Problem 22.5. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector spaces and $T \in L(\mathcal{X}, \mathcal{Y})$.

a) Consider $T^{**} : \mathcal{X}^{**} \to \mathcal{Y}^{**}$. Identifying $\mathcal{X}, \mathcal{Y}$ with their images in $\mathcal{X}^{**}$ and $\mathcal{Y}^{**}$, show that $T^{**}|_{\mathcal{X}} = T$.
b) Prove that T^* is injective if and only if the range of T is dense in $\mathcal{Y}$.
c) Prove that if the range of T^* is dense in $\mathcal{X}^*$, then T is injective; if $\mathcal{X}$ is reflexive then the converse is true.

Problem 22.6. Prove that if $\mathcal{X}$ is a Banach space and $\mathcal{X}^*$ is separable, then $\mathcal{X}$ is separable. *Hint:* let $\{f_n\}$ be a countable dense subset of $\mathcal{X}^*$. For each n choose x_n such that $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$. Show that the set of linear combinations of $\{x_n\}$ is dense in $\mathcal{X}$.

Problem 22.7. a) Prove that there exists a bounded linear functional $L \in (\ell^\infty)^*$ with the following property: whenever $f \in \ell^\infty$ and $\lim_{n\to\infty} f(n)$ exists, then $L(f)$ is equal to this limit. (Hint: first show that the set of such f forms a closed subspace $\mathcal{M} \subset \ell^\infty$).
b) Show that such a functional L is not equal to L_g for any $g \in \ell^1$; thus the map $T : \ell^1 \to (\ell^\infty)^*$ given by $T(g) = L_g$ is not surjective.
c) Give another proof that T is not surjective, using Problem 22.6.

23. The Baire Categegry Theorem and applications

A topological space X is called a *Baire space* if it has the following property: whenever $(U_n)_{n=1}^\infty$ is a countable sequence of open, dense subsets of X, the intersection $\cap_{n=1}^\infty U_n$ is dense in X. An equivalent formulation can be given in terms of nowhere dense sets: X is Baire if and only if it is *not* a countable union of nowhere dense sets. (Recall that a set $E \subset X$

is called *nowhere dense* if its closure has empty interior.) In particular, note that if E is nowhere dense, then $X \setminus \overline{E}$ is open and dense. The Baire property is used as a kind of pigeonhole principle: the "thick" Baire space X cannot be expressed as a countable union of the "thin" nowhere dense sets E_n. Equivalently, if X is Baire and $X = \bigcup_n E_n$, then at least one of the E_n is somewhere dense.

Theorem 23.1 (The Baire Category Theorem)**.** *Every complete metric space X is a Baire space.*

Proof. Let U_n be a sequence of open dense sets in X. Note that a set is dense if and only if it has nonempty intersection with every nonempty open set $W \subset X$. Fix such a W. Since U_1 is open and dense, there is a point $x_1 \in W \cap U_1$ and a radius $0 < r_1 < 1$ such that the $\overline{B(x_1, r_1)}$ is contained in $W \cap U_1$. Similarly, since U_2 is dense there is a point $x_2 \in B(x_1, r_1) \cap U_2$ and a radius $0 < r_2 < \frac{1}{2}$ such that

$$\overline{B(x_2, r_2)} \subset B(x_1, r_1) \cap U_2. \tag{23.1}$$

Continuing inductively, since each U_n is dense we obtain a sequence of points $(x_n)_{n=1}^{\infty}$ and radii $0 < r_n < \frac{1}{n}$ such that

$$\overline{B(x_n, r_n)} \subset B(x_{n-1}, r_{n-1}) \cap U_n. \tag{23.2}$$

Since $x_m \in B(x_n, r_n)$ for all $m \geq n$ and $r_n \to 0$, the sequence (x_n) is Cauchy, and thus convergent since X is complete. Moreover, since each the *closed* sets $\overline{B(x_n, r_n)}$ contains a tail of the sequence, each contains the limit x as well. Thus $x \in W \cap U_1 \cap \cdots \cap U_n$ for all n, and the proof is complete. ☐

Remark: In some books a countable intersection of open dense sets is called *residual* or *second category* and a countable union of nowhere dense sets is called *meager* or *first category*. In $\mathbb{R}$ with the usual topology, $\mathbb{Q}$ is first category and $\mathbb{R} \setminus \mathbb{Q}$ is second category.

We now give three important applications of the Baire category theorem in functional analysis: the Principle of Uniform boundedness (also known as the Banach-Steinhaus theorem), the Open Mapping Theorem, and the Closed Graph Theorem. (In learning these theorems, keep careful track of what completeness hypotheses are needed.)

Theorem 23.2 (The Principle of Uniform Boundedness (PUB))**.** *Let $\mathcal{X}, \mathcal{Y}$ be normed spaces with $\mathcal{X}$ complete, and let $\{T_\alpha : \alpha \in A\} \subset B(\mathcal{X}, \mathcal{Y})$ a collection of bounded linear transformations from $\mathcal{X}$ to $\mathcal{Y}$. If*

$$M(x) := \sup_\alpha \|T_\alpha x\| < \infty \tag{23.3}$$

for each $x \in \mathcal{X}$, then $\sup_\alpha \|T_\alpha\| < \infty$. (In other words, a pointwise bounded family of linear operators is uniformly bounded.)

Proof. For each integer $n \geq 1$ consider the set

$$V_n := \{x \in \mathcal{X} : M(x) > n\}. \tag{23.4}$$

Since each T_α is bounded, the sets V_n are open. (Indeed, for each α the map $x \to \|T_\alpha x\|$ is continuous from $\mathcal{X}$ to $\mathbb{R}$, so if $\|T_\alpha x\| > n$ for some α then also $\|T_\alpha y\| > n$ for all y sufficiently close to x.) If each V_n were dense, then since $\mathcal{X}$ is complete we would conclude from the Baire Category theorem that the intersection $V = \cap_{n=1}^{\infty} V_n$ is nonempty. But then for any

$x \in V$, we would have $M(x) > n$ for all n, which contradicts the pointwise boundedness hypothesis (23.3). Thus for some N, V_N is not dense.

Choose $x_0 \in \mathcal{X} \setminus \overline{V_N}$ and $r > 0$ so that $x_0 - x \in \mathcal{X} \setminus \overline{V_N}$ for all $\|x\| < r$. Then for every α and every $\|x\| < r$ we have

$$\|T_\alpha x\| \le \|T_\alpha (x - x_0)\| + \|T_\alpha x_0\| \le N + M(x_0) := R. \tag{23.5}$$

That is, $\|T_\alpha x\| \le R$ for all $\|x\| < r$, so by rescaling x we conclude that $\|T_\alpha x\| \le R/r$ for all $\|x\| < 1$, so $\sup_\alpha \|T_\alpha\| \le R/r < \infty$. □

Recall that if X, Y are topological spaces, a mapping $f : X \to Y$ is called *open* if $f(U)$ is open in Y whenever U is open in X. In the case of normed linear spaces the condition that a linear map be open can be refined somewhat:

Lemma 23.3 (Translation and Dilation lemma). *Let $\mathcal{X}, \mathcal{Y}$ be normed vector spaces, let B denote the open unit ball of $\mathcal{X}$, and let $T : \mathcal{X} \to \mathcal{Y}$ be a linear map. Then the map T is open if and only if $T(B)$ contains an open ball centered at 0.*

Proof. This is more or less immediate from the fact that the translation map $z \to z + z_0$, for fixed z_0, and dilation map $z \to rz, r \in \mathbb{K}$ are continuous in any normed vector space. The details are left as an exercise. □

Theorem 23.4 (The Open Mapping Theorem). *Suppose that $\mathcal{X}, \mathcal{Y}$ are Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ is a bounded,* surjective *linear map. Then T is open.*

Proof. Let B_r denote the open ball of radius r centered at 0 in $\mathcal{X}$. Trivially $\mathcal{X} = \bigcup_{n=1}^\infty B_n$ and since $\mathcal{Y}$ is surjective, we have $\mathcal{Y} = \bigcup_{n=1}^\infty T(B_n)$. Since $\mathcal{Y}$ is complete, from the Baire category theorem we have that $T(B_N)$ is somewhere dense, for some N. That is, $\overline{T(B_N)}$ has nonempty interior; by rescaling we conclude that $\overline{T(B_1)}$ has nonempty interior. We must deduce from this that in fact $T(B_1)$ contains an open ball centered at 0 in $\mathcal{Y}$. First we show that $\overline{T(B_1)}$ contains an open ball centered at 0.

There exists $y_0 \in \mathcal{Y}$ and $r > 0$ such that $\overline{T(B_1)}$ contains $B_r^{\mathcal{Y}}(y_0)$. Choose $y_1 = Tx_1$ in $T(B_1)$ such that $\|y_1 - y_0\| < r/2$, then by the triangle inequality

$$B_{r/2}^{\mathcal{Y}}(y_1) \subset B_r^{\mathcal{Y}}(y_0) \subset \overline{T(B_1)}. \tag{23.6}$$

It follows that for all $\|y\| < r/2$,

$$y = -y_1 + (y + y_1) = -Tx_1 + (y + y_1) \in \overline{T(-x_1 + B_1)} \subset \overline{T(B_2)}. \tag{23.7}$$

Halving the radius again, we conclude that if $\|y\| < r/4$, then $y \in \overline{T(B_1)}$.

Finally, we show that by shrinking the radius further we can find an open ball contained in $T(B_1)$. From the previous paragraph, by scaling again for each n, we see that if $\|y\| < r/2^{n+2}$ then $y \in \overline{T(B_{1/2^n})}$. If $\|y\| < r/8$, then there exists $x_1 \in B_{1/2}$ such that $\|y - Tx_1\| < r/16$. Inductively there exist $x_n \in B_{1/2^n}$ with $\|y - T\sum_{k=1}^n x_j\| < r2^{-n-3}$. The series $\sum_{k=1}^\infty x_k$ is thus absolutely convergent, and hence convergent, since $\mathcal{X}$ is complete. Call its sum x. By construction we have the estimate $\|x\| < \sum_{k=1}^\infty 2^{-k}$, so $x \in B_1$ and $y = Tx$. In conclusion, we have shown that $y \in T(B_1)$ whenever $\|y\| < r/8$, so the proof is finished. □

Corollary 23.5 (The Banach Isomorphism Theorem). *If $\mathcal{X}, \mathcal{Y}$ are Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ is a bounded bijection, then T^{-1} is also bounded (hence, T is an isomorphism).*

Proof. Note that when T is bijection, T is open if and only if T^{-1} is continuous. The result then follows from the Open Mapping Theorem and Proposition 19.6.

□

To state the final result of this section, we need a few more definitions. Let $\mathcal{X}, \mathcal{Y}$ be normed spaces. The Cartesian product $\mathcal{X} \times \mathcal{Y}$ is then a topological space in the product topology. (In fact the product topology can be realized by norming $\mathcal{X} \times \mathcal{Y}$, e.g. with the norm $\|(x, y)\| := \max(\|x\|, \|y\|)$.) The space $\mathcal{X} \times \mathcal{Y}$ is equipped with the coordinate projections $\pi_{\mathcal{X}}(x, y) = x, \pi_{\mathcal{Y}}(x, y) = y$; it is clear that these maps are continuous. Given a linear map $T : \mathcal{X} \to \mathcal{Y}$, its *graph* is the set

$$G(T) := \{(x, y) \in \mathcal{X} \times \mathcal{Y} : y = Tx\} \tag{23.8}$$

Observe that since T is a linear map, $G(T)$ is a linear subspace of $\mathcal{X} \times \mathcal{Y}$. The transformation T is called *closed* if $G(T)$ is a closed subset of $\mathcal{X} \times \mathcal{Y}$. This criterion is usually expressed in the following equivalent formulation: whenever $x_n \to x$ in $\mathcal{X}$ and $Tx_n \to y$ in $\mathcal{Y}$, it holds that $Tx = y$. Note that this is not the same thing as saying that T is continuous: T is continuous if and only if whenever $x_n \to x$ in $\mathcal{X}$, then $Tx_n \to Tx$ in $\mathcal{Y}$. Evidently, if T is continuous then T is closed. The converse is false in general (see Problem 24.2). However, the converse does hold when $\mathcal{X}$ and $\mathcal{Y}$ are complete:

Theorem 23.6 (The Closed Graph Theorem)**.** *If $\mathcal{X}, \mathcal{Y}$ are Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ is closed, then T is bounded.*

Proof. Let π_1, π_2 be the coordinate projections $\pi_{\mathcal{X}}, \pi_{\mathcal{Y}}$ restricted to $G(T)$; explicitly $\pi_1(x, Tx) = x$ and $\pi_2(x, Tx) = Tx$. Note that π_1 is a bijection between $G(T)$ and $\mathcal{X}$. Moreover, as maps between sets we have $T = \pi_2 \circ \pi_1^{-1}$. So, it suffices to prove that π_1^{-1} is bounded.

By the remarks above π_1 and π_2 are bounded (from $G(T)$ to $\mathcal{X}, \mathcal{Y}$ respectively). Since $\mathcal{X}$ and $\mathcal{Y}$ are complete, so is $\mathcal{X} \times \mathcal{Y}$, and therefore $G(T)$ is complete since it is assumed closed. Since π_1 is a bijection between $G(T)$ and $\mathcal{X}$, its inverse is also bounded (Corollary 23.5).

□

24. Problems

Problem 24.1. Show that there exists a sequence of open, dense subsets $U_n \subset \mathbb{R}$ such that $m(\bigcap_{n=1}^{\infty} U_n) = 0$.

Problem 24.2. Consider the linear subspace $\mathcal{D} \subset c_0$ defined by

$$\mathcal{D} = \{f \in c_0 : \lim_{n \to \infty} |nf(n)| = 0\} \tag{24.1}$$

and the linear transformation $T : \mathcal{D} \to c_0$ defined by $(Tf)(n) = nf(n)$.

a) Prove that T is closed, but not bounded. b) Prove that $T^{-1} : c_0 \to \mathcal{D}$ is bounded and surjective, but not open.

Problem 24.3. Suppose $\mathcal{X}$ is a vector space equipped with two norms $\|\cdot\|_1, \|\cdot\|_2$ such that $\|\cdot\|_1 \le \|\cdot\|_2$. Prove that if $\mathcal{X}$ is complete in both norms, then the two norms are equivalent.

Problem 24.4. Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. Provisionally, say that a linear transformation $T : \mathcal{X} \to \mathcal{Y}$ is *weakly bounded* if $f \circ T \in \mathcal{X}^*$ whenever $f \in \mathcal{Y}^*$. Prove that if T is weakly bounded, then T is bounded.

Problem 24.5. Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. Suppose (T_n) is a sequence in $B(\mathcal{X}, \mathcal{Y})$ and $\lim_n T_n x$ exists for every $x \in \mathcal{X}$. Prove that if T is defined by $Tx = \lim_n T_n x$, then T is bounded.

Problem 24.6. Suppose that $\mathcal{X}$ is a vector space with a countably infinite basis. (That is, there is a linearly independent set $\{x_n\} \subset \mathcal{X}$ such that every vector $x \in \mathcal{X}$ is expressed uniquely as a *finite* linear combination of the x_n's.) Prove that there is no norm on $\mathcal{X}$ under which it is complete. (Hint: consider the finite-dimensional subspaces $\mathcal{X}_n := \text{span}\{x_1, \ldots x_n\}$.)

Problem 24.7. The Baire Category Theorem can be used to prove the existence of (very many!) continuous, nowhere differentiable functions on $[0, 1]$. To see this, let F_n denote the set of all functions $f \in C[0, 1]$ for which there exists $x_0 \in [0, 1]$ (which may depend on f) such that $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0, 1]$. Prove that the sets E_n are nowhere dense in $C[0, 1]$; the Baire Category Theorem then shows that the set of nowhere differentiable functions is residual. (To see that E_n is nowhere dense, approximate an arbitrary continuous function f uniformly by piecewise linear functions g, whose pieces have slopes greater than $2n$ in absolute value. Any function sufficiently close to such a g will not lie in E_n.)