# EXTREMAL MULTIPLIERS OF THE DRURY-ARVESON SPACE

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ABSTRACT. We give a new characterization of the so-called *quasi-extreme* multipliers of the Drury-Arveson space $H^2_d$, and show that every quasi-extreme multiplier is an extreme point of the unit ball of the multiplier algebra of $H^2_d$.

## 1. INTRODUCTION

In [8] and [9] we introduced the notion of a *quasi-extreme* multiplier of the Drury-Arveson space $H^2_d$, and gave a number of equivalent formulations of this property. (The relevant definitions are recalled in Section 2.) The main motivation is that in one variable, each of these conditions is equivalent to $b$ being an extreme point of the unit ball of $H^\infty$ (the space of bounded analytic functions in the unit disk $\mathbb{D} \subset \mathbb{C}$). The purpose of this paper is to give one further characterization of quasi-extremity in the general case, from which it will follow that every quasi-extreme multiplier of $H^2_d$ is in fact an extreme point of the unit ball of the multiplier algebra $\mathcal{M}(H^2_d)$. (The converse statement, namely whether or not every extreme point is quasi-extreme in our sense, remains open.) In particular we will prove the following theorem:

**Theorem 1.1.** *A contractive mulitplier $b$ of $H^2_d$ is quasi-extreme if and only if the only multiplier $a$ satisfying*

$$M_a^* M_a + M_b^* M_b \leq I \tag{1.1}$$

*is $a \equiv 0$.*

Since the corollary concerning extreme points follows immediately, we prove it here:

**Corollary 1.2.** *If $b \in ball(\mathcal{M}(H^2_d))$ is quasi-extreme, then $b$ is an extreme point of $ball(\mathcal{M}(H^2_d))$.*

*Proof.* We prove the contrapositive. If $b$ is not extreme, then there exists a nonzero $a \in ball(\mathcal{M}(H^2_d))$ such that both $b \pm a$ lie in $ball(\mathcal{M}(H^2_d))$, that is, are contractive multipliers of $H^2_d$. We then have the operator inequalities

$$M_{b+a}^* M_{b+a} \leq I, \quad M_{b-a}^* M_{b-a} \leq I$$

averaging these inequalities gives

$$M_a^* M_a + M_b^* M_b \leq I$$

so by Theorem 1.1, $b$ is not quasi-extreme. □

The remainder of the paper is devoted to proving Theorem 1.1. Since the techniques required are rather different, the two implications of the theorem will be proved as two separate propositions, Propositions 4.1 and 5.1. In Section 2 we recall the Drury-Arveson space and

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its multipliers, and review the necessary results concerning the de-Branges Rovnyak type spaces $\mathcal{H}(b)$ conctractively contained in $H^2_d$, and in particular the solutions to the Gleason problem in these spaces. We define the *quasi-extreme* multipliers and review some equivalent formulations of this property that will be used later. In Section 3 we study the non-quasi-extreme multipliers in more detail, and extend to this class of functions some of the results proved by Sarason [11] in the one-variable case. Our results rely heavily on the the construction of a particular solution to the Gleason problem with good extremal properties, which is carried out in this section. In Sections 4 and 5 respectively we prove Propositions 4.1 and 5.1.

## 2. The Drury-Arveson space, multipliers, and quasi-extremity

The *Drury-Arveson space* is the Hilbert space of holomorphic functions defined on the unit ball $\mathbb{B}^d \subset \mathbb{C}^d$ with reproducing kernel

$$k_w(z) = k(z,w) = \frac{1}{1 - zw^*}; \qquad z, w \in \mathbb{B}^d$$

(Here we use the notation $z = (z_1, z_2, \dots, z_d)$, so that $zw^* = \sum_{j=1}^d z_j \overline{w_j}$.) General facts about the $H^2_d$ spaces may be found in the recent survey [13].

A holomorphic function $b$ on $\mathbb{B}^d$ will be called a *multiplier* if $bf \in H^2_d$ whenever $f \in H^2_d$. In this case the operator $M_b : f \to bf$ is bounded, and we let $\mathcal{M}(H^2_d)$ denote the Banach algebra of multipliers, equipped with the operator norm. (Warning: the multiplier norm always dominates the supremum norm of $b$ over the unit ball, but the two are in general unequal. Also, not every bounded function $b$ is a multiplier, see *e.g.* [13, 1].) For the reproducing kernel $k_w$ we have $M_b^* k_w = b(w)^* k_w$. It follows that $\|M_b\| \leq 1$ if and only if the expression

$$k^b_w(z) = k^b(z,w) := \frac{1 - b(z)b(w)^*}{1 - zw^*},$$

defines a positive kernel on $\mathbb{B}^d$. When this is the case we let $\mathcal{H}(b)$ denote the associated reproducing kernel Hilbert space, called the deBranges-Rovnyak space of $b$. The space $\mathcal{H}(b)$ is a space of holomorphic functions on $\mathbb{B}^d$, contained in $H^2_d$, and the inclusion map $\mathcal{H}(b) \subset H^2_d$ is contractive for the respective Hilbert space norms. We write $\|\cdot\|_b$ and $\langle \cdot, \cdot \rangle_b$ for the norm and inner product in $\mathcal{H}(b)$ respectively.

Properties of the spaces $\mathcal{H}(b)$ when $d > 1$ were studied in [8], inspired among other things by the results of Sarason in the one variable case [11],[12]. In one variable, the $\mathcal{H}(b)$ spaces are invariant under the backward shift; in several variables we instead (following Ball, Bolotnikov, and Fang [2]) consider solutions to the *Gleason problem*: given a function $f \in \mathcal{H}(b)$, we seek functions $f_1, \dots f_d \in \mathcal{H}(b)$ such that

$$f(z) - f(0) = \sum_{j=1}^d z_j f_j(z). \tag{2.1}$$

From [2] we know that this problem always has a solution; in fact there exist (not necessarily unique) bounded operators $X_1, \dots X_d$ acting on $\mathcal{H}(b)$ such that the functions $f_j := X_j f$ solve (2.1 for any $f \in \mathcal{H}(b)$. Moreover these $X_j$ can be chosen to be *contractive* in the following sense: for every $f \in \mathcal{H}(b)$,

$$\sum_{j=1}^{d} \|X_j f\|_b^2 \leq \|f\|_b^2 - |f(0)|^2. \tag{2.2}$$

These contractive solutions were studied further in [8], where we proved the following (see also [9] for the vector-valued case):

**Proposition 2.1.** *A set of bounded operators $(X_1, \dots X_d)$ is a contractive solution to the Gleason problem in $\mathcal{H}(b)$ if and only if the $X_j$ act on reproducing kernels by the formula*

$$X_j k_w^b = w_j^* k_w^b - b(w)^* b_j \tag{2.3}$$

*for some choice of functions $b_1, \dots b_d \in \mathcal{H}(b)$ which satisfy*

(i) $\sum_{j=1}^d z_j b_j(z) = b(z) - b(0)$,
(ii) $\sum_{j=1}^d \|b_j\|_b^2 \leq 1 - |b(0)|^2$.

The set all contractive solutions $X$ is in one-to-one correspondence with the set of all tuples $b_1, \dots b_d$ satisfying these conditions [9, Theorem 4.10]. We will call such sets of $b_j$ *admissable*, or say that such a set is a *contractive Gleason solution for $b$*.

In turns out that for some contractive multipliers $b$, the operators $X_j$ of the proposition are unique, that is, there is only one admissible tuple. When this happens we will call the multiplier $b$ *quasi-extreme*. (The original definition of quasi-extreme in [8] is different, involving the so-called noncommutative Aleksandrov-Clark state for $b$, but this definition will be easier to work with for the present purposes.) In [8] and [9] we gave a number of equivalent formulations of quasi-extremity, we recall only a few of them here.

**Proposition 2.2.** *Let $b$ be a contractive multiplier of $H_d^2$. The following are equivalent:*

i) *$b$ is quasi-extreme.*
ii) *There is a unique contractive solution $(X_1, \dots X_d)$ to the Gleason problem in $\mathcal{H}(b)$.*
iii) *There exists a contractive solution $(X_1, \dots X_d)$ such that the equality $\sum_{j=1}^d \|X_j f\|_b^2 = \|f\|_b^2 - |f(0)|^2$ holds for every $f \in \mathcal{H}(b)$.*
iv) *There is a unique admissible tuple $(b_1, \dots b_d)$ satisfying the conditions of Proposition 2.1.*
v) *All admissible tuples $(b_1, \dots b_d)$ are* extremal, i.e.

$$\sum_{j=1}^{d} \|b_j\|_b^2 = 1 - |b(0)|^2,$$

*for any admissable tuple.*

vi) *$\mathcal{H}(b)$ does not contain the function $b$.*
vii) *$\mathcal{H}(b)$ does not contain the constant functions.*

In [9] these equivalences were extended to the case of operator-valued $b$.

What will be most useful in what follows is item (v); in particular $b$ is *not* quasi-extreme if and only if there exists an admissible tuple $(b_1, \dots b_d)$ which obeys the strict inequality

$$\sum_{j=1}^{d} \|b_j\|_b^2 < 1 - |b(0)|^2.$$

## 3. Non-quasi-extreme $b$

Let $b$ denote a contractive multiplier of the Drury-Arveson space $H^2_d$ on the unit ball $\mathbb{B}^d \subset \mathbb{C}^d$. We assume throughout that $b(z)$ is not constant. We let $G_b(z)$ denote the Cayley transform or Herglotz-Schur function of $b$:

$$G_b(z) = \frac{1+b(z)}{1-b(z)} \tag{3.1}$$

and we contstruct the reproducing kernel Hilbert spaces $\mathcal{H}(b), \mathscr{L}(b)$, the deBranges-Rovnyak and Herglotz spaces of $b$, respectively, with the kernels

$$k^b_w(z) := \frac{1-b(z)b(w)^*}{1-zw^*}, \tag{3.2}$$

and,

$$K^b_w(z) := \frac{1}{2}\frac{G_b(z)+G_b(w)^*}{1-zw^*} = (1-b(z))^{-1}(1-b(w)^*)^{-1}k^b_w(z). \tag{3.3}$$

The map $f \to (1-b)f$ is thus a unitary multiplier from $\mathscr{L}(b)$ onto $\mathcal{H}(b)$.

We define an operator $V : \mathscr{L}(b)^d \to \mathscr{L}(b)$ by declaring

$$V\begin{pmatrix} z_1^* K^b_z \\ \vdots \\ z_d^* K^b_z \end{pmatrix} := K^b_z - K^b_0 \tag{3.4}$$

on the span of the the columns appearing in the definition, and defining $V$ to be 0 on the orthogonal complement of this span. A quick calculation using the formula for the reproducing kernel (3.3) shows that

$$zw^* K^b_w(z) = \langle K^b_w - K^b_0, K^b_z - K^b_0\rangle_{\mathscr{L}(b)} \quad \text{for all } z, w \in \mathbb{B}^d$$

and hence that $V$ is a partial isometry. It follows that $V^*$ is 0 on the orthogonal complement of the set $\{K^b_z - K^b_0 : z \in \mathbb{B}^d\}$. Note that a vector $f \in \mathscr{L}(b)$ is orthogonal to this set if and only if $f(z) = f(0)$ for all $z$; that is, if and only if $f$ is constant. (In particular $V$ is a coisometry if and only if the only constant function in $\mathscr{L}(b)$ is 0.) We next observe:

**Lemma 3.1.** *The space $\mathscr{L}(b)$ contains the constants if and only if $\mathcal{H}(b)$ contains $b$; that is, by Proposition 2.2, if and only if $b$ is not quasi-extreme.*

*Proof.* $1 \in \mathscr{L}(b)$ if and only if $1 - b \in \mathcal{H}(b)$; since $k^b_0 = 1 - b(z)b(0)^* \in \mathcal{H}(b)$ always, the lemma follows. □

For the remainder of this section we assume that $b$ is not quasi-extreme, so by the lemma, $\mathscr{L}(b)$ contains the constant functions. By construction the tuple $(V_1, \dots V_d)$ is a row contraction and $\sum_{j=1}^d V_j V_j^*$ is the projection in $\mathscr{L}(b)$ orthogonal to the constants; so that $V_j^* 1 = 0$ for all $j$. We first record some facts about the $V_j$ that will be of use later.

From the definition of $V$ we have

$$V_j^*(K_z - K_0) = z_j^* K^b_z. \tag{3.5}$$

We also record the following chain of equalities for later use; these use only the fact that $V_j^* 1 = 0$: For each $j$,

$$
\begin{aligned}
V_j^*(K_0) &= V_j^* \frac{1}{2}\left(\frac{2}{1-b} - 1 + \frac{1+b(0)^*}{1-b(0)}^*\right) \\
&= V_j^* \left(\frac{1}{1-b} - 1\right) \\
&= V_j^* \left(\frac{b}{1-b}\right). \qquad (3.6)
\end{aligned}
$$

We next observe that the $V_j$ solve the Gleason problem in $\mathscr{L}(b)$; indeed for $f \in \mathscr{L}(b)$ we take the inner product of $f$ with the identity

$$K_z^b - K_0^b = \sum_{j=1}^{d} z_j^* V_j K_z^b \qquad (3.7)$$

we get

$$f(z) - f(0) = \sum_{j=1}^{d} z_j (V_j^* f)(z). \qquad (3.8)$$

We can now define operators $S_j$ on $\mathcal{H}(b)$ conjugate to the $V_j$ via the unitary $g \to \frac{1}{\sqrt{2}}(1-b)g$; specifically for $g \in \mathcal{H}(b)$ we define

$$(S_j^* g)(z) = (1-b) V_j^* \frac{g}{1-b}. \qquad (3.9)$$

Again the row $S = (S_1, \dots, S_d)$ is a row contraction; in fact a row partial isometry whose final space $\operatorname{ran}(\sum_{j=1}^d S_j S_j^*)$ is the orthognal complement of the one-dimensional space spanned by $1 - b$.

We now use the operators $S_j$ to define an admissible tuple $b_1, \dots b_d$ and construct the associated solution to the Gleason problem in $\mathcal{H}(b)$ with good extremal properties. In particular put

$$b_j = (1 - b(0)) S_j^* b \in \mathcal{H}(b) \qquad (3.10)$$

and define operators $X_j$ as in (2.3).

**Proposition 3.2.** *The tuple $X = (X_1, \dots X_d)$ is a contractive solution to the Gleason problem in $\mathcal{H}(b)$. Moreover it is the unique solution with the property such that*

$$X_j b = b_j \qquad (3.11)$$

*where the $b_j$ are those belonging to $X_j$.*

*Proof.* The fact that $V^b$ can be used to define a contractive Gleason solution for $K(b)$ in this way is a special case of [9, Theorem 4.4, Lemma 4.6]. We include a proof below for completeness.

We first verify that the $b_j$ defined by (3.10) are admissible. Sine the $V_j^*$ solve the Gleason problem in $\mathscr{L}(b)$, we have

$$\begin{aligned}\sum_{j=1}^d z_j V_j^*(\frac{b}{1-b}) &= \frac{b(z)}{1-b(z)} - \frac{b(0)}{1-b(0)} \\ &= \frac{b(z)-b(0)}{(1-b(z))(1-b(0))}\end{aligned}$$

so by the definition of $S_j^*$ and $b_j$

$$\begin{aligned}b(z)-b(0) &= (1-b(0))(1-b(z))\sum_{j=1}^d z_j(V_j^*\frac{b}{1-b})(z) \\ &= (1-b(0))\sum_{j=1}^d z_j S_j^* b(z) \\ &= \sum_{j=1}^d z_j b_j(z).\end{aligned}$$

To prove the norm inequality, observe that

$$\begin{aligned}\sum_{j=1}^d \|b_j\|_b^2 = \sum_{j=1}^d \|S_j^* b\|_b^2 & \\ &= |1-b(0)|^2 \sum_{j=1}^d \|V_j^*\frac{b}{1-b}\|^2_{\mathscr{L}(b)} \\ &= |1-b(0)|^2 \sum_{j=1}^d \|V_j^* K_0\|^2_{\mathscr{L}(b)} \quad \text{by (3.6)} \\ &\leq 1-|b(0)|^2.\end{aligned}$$

where the last inequality holds since $V^*$ is a column contraction and $\|K_0\|^2 = \frac{1-|b(0)|^2}{|1-b(0)|^2}$. Moreover, we observe that, since $V^*$ is a partial isometry, equality holds in the above chain if and only if $K_0$ is orthogonal to the scalars, but this obviously never happens, so the inequality is always strict in this case when $V^b$ is not a co-isometry. This also shows that this choice of admissible $b_j$ minimizes the sum $\sum_{j=1}^d \|b_j\|_b$ over all choices of admissible $b_j$ (see also [9, Corollary 4.8, Remark 4.9]).

To show that $X_j b = b_j$, we first show that

$$X_j = S_j^* - (1-b(0))^{-1} b_j \otimes k_0^b. \tag{3.12}$$

This equation follows from Clark-type intertwining formulas of [9, Section 4.15].

Indeed, from (3.6) we have

$$V_j^* K_0 = V_j^*(\frac{1}{1-b}) = \frac{1}{1-b} S_j^* b = \frac{1}{(1-b)(1-b(0))} b_j$$

The formula (3.12) is then verified by checking it on kernels $k^b_w$, where we have from the definition of $S_j^*$

$$\begin{aligned}
S_j^* k^b_w &= (1-b)V_j^*\left(\frac{1-bb(w)^*}{(1-b)(1-zw^*)}\right) \\
&= (1-b)(1-b(w)^*)V_j^*(K^b_w) \\
&= (1-b)(1-b(w)^*)V_j^*(K_w - K_0 + K_0) \\
&= w_j^* k^b_w + \frac{1-b(w)^*}{1-b(0)} b_j
\end{aligned}$$

and so

$$\begin{aligned}
S_j^* k^b_w - [(1-b(0))^{-1} b_j \otimes k^b_0]k^b_w &= w_j^* k^b_w + \frac{1-b(w)^*}{1-b(0)} b_j - \frac{1}{1-b(0)}(1-b(0)b(w)^*)b_j \\
&= w_j^* k^b_w - b(w)^* b_j \\
&= X_j k^b_w
\end{aligned}$$

as desired.

Finally, the claim that $X_j b = b_j$ follows immediately from (3.12) and the definition of the $b_j$ in (3.10). $\square$

*Remark:* We observe in passing that these $X_j$ annihilate the scalars: indeed, from the definition of $X_j$ in (2.3) and the fact that $X_j b = b_j$, we have

$$X_j 1 = X_j(1 - b(0)^* b + b_0^* b) = X_j k^b_0 + b(0)^* X_j b = -b(0)^* b_j + b(0)^* b_j = 0.$$

We also have that the defect operator $I - \sum X_j^* X_j$ has rank two when $b$ is non-extreme:

**Proposition 3.3.** *Let $b$ be a non-extreme multiplier. If $X_j$ is a solution to the Gleason problem in $\mathcal{H}(b)$ with $\sum \|b_j\|^2 = 1 - |b(0)|^2 - |a_0|^2$, then*

$$I - \sum X_j^* X_j = k^b_0 \otimes k^b_0 + |a_0|^2 b \otimes b. \tag{3.13}$$

*Proof.* We first compute the inner product $\langle X_j^* X_j k^b_w, k^b_z\rangle$, using (2.3):

$$\begin{aligned}
\langle X_j^* X_j k^b_w, k^b_z\rangle &= \langle X_j k^b_w, X_j k^b_z\rangle \\
&= \langle w_j^* k^b_w - b(w)^* b_j, z_j^* k^b_z - b(z)^* b_j\rangle \\
&= z_j w_j^* k^b(z,w) - z_j b_j(z) b(w)^* - w_j^* b_j(w)^* b(z) + \|b_j\|_b^2 b(z) b(w)^*.
\end{aligned}$$

Summing over $j = 1, \dots d$ (and using the fact that the $b_j$ are admissible) gives

$$\sum_{j=1}^d \langle X_j^* X_j k^b_w, k^b_z\rangle = zw^* k^b(z,w) - (b(z)-b(0))b(w)^* - b(z)(b(w)^* - b(0)^*) + (1-|b(0)|^2 - |a_0|^2) b(z) b(w)^*.$$

Finally, we find

$$\begin{aligned}
\langle (I - \sum_{j=1}^{d} X_j X_j^*) k_w^b, k_z^b \rangle &= (1 - zw^*) k^b(z, w) + (b(z) - b(0)) b(w)^* + b(z)(b(w)^* - b(0)^*) \\
&\quad -(1 - |b(0)|^2 - |a_0|^2) b(z) b(w)^* \\
&= 1 - b(z)b(0)^* - b(0)b(w)^* + b(0)b(z)b(w)^* b(0)^* + |a_0|^2 b(z) b(w)^* \\
&= \langle (k_0^b \otimes k_0^b + |a_0|^2 b \otimes b) k_w^b, k_z^b \rangle .
\end{aligned}$$

This completes the proof. □

Since $b$ is assumed non-constant, this proposition shows that the range of $I - \sum_{j=1}^{d} X_j^* X_j$ is two dimensional, spanned by $k_0 = 1 - b(0)^* b$ and $b$, or equivalently, by $b$ and 1. We can use this fact and the foregoing identity to relate the "defect" $|a_0|^2 := 1 - |b(0)|^2 - \sum_{j=1}^{d} \|b_j\|_b^2$ of the admissible tuple $b_1, \dots b_d$ to the $\mathcal{H}(b)$-norm of the function $b$ (compare [11],[3] for the scalar and vector valued cases, respectively, in one variable).

**Lemma 3.4.** *Suppose $b$ is not quasi-extreme, and let $(X_1, \dots X_j)$ and $b_1, \dots b_j$ be as in Proposition 3.2. Define $a_0 > 0$ by $|a_0|^2 = 1 - |b(0)|^2 - \sum_{j=1}^{d} \|b_j\|_b^2$. Then*

$$|a_0|^2 = \frac{1}{1 + \|b\|_b^2}. \tag{3.14}$$

*Proof.* We compute $(I - \sum_{j=1}^{d} X_j^* X_j) b$ in two different ways. First, from its definition, and using the fact that $X_j b = b_j$,

$$(I - \sum_{j=1}^{d} X_j^* X_j) b = b - \sum_{j=1}^{d} X_j^* b_j .$$

Then observe that

$$\begin{aligned}
\sum_{j=1}^{d} (X_j^* b_j)(z) &= \sum_{j=1}^{d} \langle X_j^* b_j, k_z^b \rangle_b \\
&= \sum_{j=1}^{d} \langle b_j, z_j^* k_z^b - b_j b(z)^* \rangle_b \\
&= b(z) - b(0) - \sum_{j=1}^{d} \|b_j\|_b^2 b(z),
\end{aligned}$$

so that

$$(I - \sum_{j=1}^{d} X_j^* X_j) b = b(0) + b \sum_{j=1}^{d} \|b_j\|_b^2 . \tag{3.15}$$

(Here we have used the fact that the $X_j^*$ act by

$$(X_j^* f)(z) = z_j f(z) - \langle f, b_j \rangle_b b(z),$$

which follows easily from the definition of the $X_j$ and the reproducing formula $(X_j^* f)(z) = \langle X_j^* f, k_z^b\rangle_b = \langle f, X_j k_z^b\rangle_b$.) Second, using the defect formula (3.13),

$$(I - \sum_{j=1}^d X_j^* X_j)b = b(0)k_0^b + |a_0|^2\|b\|_b^2 b = b(0) + (-|b(0)|^2 + |a_0|^2\|b\|_b^2)b(z). \tag{3.16}$$

Equating (3.15) and (3.16) gives

$$b(0) + b(z)\sum_{j=1}^d \|b_j\|_b^2 = b(0) + (-|b(0)|^2 + |a_0|^2\|b\|_b^2)b(z)$$

Subtracting $b(0)$ from both sides leaves an equality between two constant multiples of $b(z)$; since $b$ is assumed nonzero we have

$$-|b(0)|^2 + |a_0|^2\|b\|_b^2 = \sum_{j=1}^d \|b_j\|_b^2 = 1 - |b(0)|^2 - |a_0|^2$$

and solving for $|a_0|^2$ gives (3.14). □

## 4. The $a$-function

In this section we prove the first half of Theorem 1.1:

**Proposition 4.1.** *If $b$ is not quasi-extreme, then there exists a nonzero multiplier $a$ such that*

$$M_a^* M_a + M_b^* M_b \le I.$$

In the one-variable case if $b$ is not extreme then there is an outer function $a$ defined by the property that

$$|a(\zeta)|^2 + |b(\zeta)|^2 = 1 \quad \text{a.e. on } \mathbb{T}; \tag{4.1}$$

we can assume that $a(0) > 0$. In the above $\mathbb{T}$ denotes the unit circle. For this $a$ we have immediately $M_a^* M_a + M_b^* M_b = I$. It is known in general that an equality of this sort cannot hold when $d > 1$ except in trivial cases (where the functions are constant), see [6]. In any case, when $d > 1$ we do not have any direct recourse to the theory of outer functions so different methods are required.

Nonetheless, the proof of (4.1) is in a sense constructive: $a$ will be given in terms of a *transfer function realization* [4],[2]. It is remarkable that the algebraic construction given here, if carried out in one variable, produces exactly the outer function in (4.1). This follows from our transfer function realization and Sarason's computation of the Taylor coefficients of $a$ [11]; we prove this at the end of the section.

We begin by recalling the relevant facts about transfer function realizations [4] and the *generalized functional models* of [2].

Let $\mathcal{X}, \mathcal{U}, \mathcal{Y}$ be Hilbert spaces and let $\mathcal{X}^d$ denote the direct sum of $d$ copies of $\mathcal{X}$. By a *d-colligation* we mean an operator $\mathbf{U} : \mathcal{X} \oplus \mathcal{U} \to \mathcal{X}^d \oplus \mathcal{Y}$ expressed in the block matrix form

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{bmatrix}$$

The colligation is called *contractive, isometric, unitary, etc.* if $\mathbf{U}$ is an operator of that type. For points $z = (z_1, \dots z_d) \in \mathbb{C}^d$, it will be convenient to identify $z$ with the row contraction:

$$z : \mathcal{X}^d \to \mathcal{X}; \qquad z \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} := z_1 x_1 + \dots + z_d x_d.$$

Observe that $\|z\|^2 = \|zz^*\|_{\mathcal{L}(\mathcal{X})} = \sum_{j=1}^d |z_j|^2$, so $\|z\| = |z|_{\mathbb{C}^d} < 1$ if and only if $z \in \mathbb{B}^d$. If $\mathbf{U}$ is a contractive colligation, the *transfer function* for $\mathbf{U}$ is

$$S(z) = D + C(I - zA)^{-1} zB.$$

The transfer function $S(z)$ is a holomorphic function in $\mathbb{B}^d$ taking values in the space of bounded operators from $\mathcal{U}$ to $\mathcal{Y}$. (For our purposes we will only need to consider finite-dimensional $\mathcal{U}$ and $\mathcal{Y}$). It is a theorem of Ball, Trent and Vinnikov [4] that $b$ is a contractive multiplier of $H^2_d \otimes \mathcal{U}$ into $H^2_d \otimes \mathcal{Y}$ if and only if it possesses a transfer function realization. In [2], it was shown that such a transfer function could always be chosen to be of a special form, called a *generalized functional model* realization. In particular, (in the case $\mathcal{U} = \mathcal{Y} = \mathbb{C}$) if $X = (X_1, \dots X_d)$ is a contractive solution to the Gleason problem in $\mathcal{H}(b)$, if we take $\mathcal{X} = \mathcal{H}(b)$ and define for all $f \in \mathcal{H}(b)$ and $\lambda \in \mathbb{C}$

- $A_j f = X_j f$, $j = 1, \dots d$
- $B_j \lambda = \lambda b_j$, $j = 1, \dots d$
- $Cf = f(0)$
- $D\lambda = b(0)\lambda$

then the corresponding colligation is contractive and its transfer function is $b(z)$. Since $Cf = f(0) = \langle f, k_0^b \rangle$, we will write $k_0^{b^*}$ for $C$ and express the colligation as

$$\mathbf{U} = \begin{bmatrix} X_1 & b_1 \\ \vdots & \vdots \\ X_d & b_d \\ k_0^{b*} & b(0) \end{bmatrix}$$

*Proof of Proposition 4.1.* Fix the admissible tuple $(b_1, \dots b_d)$ and corresponding operators $(X_1, \dots X_d)$ of Proposition 3.2. We can then consider the colligation acting between $\mathcal{H}(b) \oplus \mathbb{C}$ and $\mathcal{H}(b) \oplus \mathbb{C}^2$ given by

$$\widetilde{\mathbf{U}} = \begin{bmatrix} X_1 & b_1 \\ \vdots & \vdots \\ X_d & b_d \\ k_0^{b*} & b(0) \\ -a_0 b^* & a_0 \end{bmatrix}$$

We claim that $\widetilde{\mathbf{U}}$ is isometric. If this is so, then the colligation

$$\mathbf{V} = \begin{bmatrix} X_1 & b_1 \\ \vdots & \vdots \\ X_d & b_d \\ -a_0 b^* & a_0 \end{bmatrix}$$

is contractive, and hence the associated transfer function is a contractive multiplier $a(z)$. Moreover it is apparent that $a$ is nonzero, since $a(0) = a_0 \neq 0$.

With $a$ defined in this way, the transfer function associated to $\widetilde{\mathbf{U}}$ is the $2 \times 1$ multiplier

$$\begin{pmatrix} b \\ a \end{pmatrix}$$

which is contractive; this proves the proposition.

It remains to prove the claim that $\widetilde{\mathbf{U}}$ is isometric. Let us write out $\widetilde{\mathbf{U}}^*\widetilde{\mathbf{U}}$ explictly; we have

$$\widetilde{\mathbf{U}}^*\widetilde{\mathbf{U}} = \begin{bmatrix} \sum_{j=1}^d X_jX_j^* + k_0^b \otimes k_0^b + |a_0|^2 b \otimes b & \sum_{j=1}^d X_j^* b_j + b(0)k_0^b - |a_0|^2 b \\ * & \sum_{j=1}^d \|b_j\|_b^2 + |b(0)|^2 + |a_0|^2 \end{bmatrix}$$

where we note that $(2,1)$ entry is just the adjoint of the $(1,2)$ entry. We consider the entries of the right-hand side one at a time.

The $(1,1)$ entry is equal to the identity operator on $\mathcal{H}(b)$ by (3.13).

The $(2,2)$ entry is equal to 1 by the definition of $a_0$ in Proposition 3.3.

The $(1,2)$ (and by symmetry, $(2,1)$) entry is equal to 0. To see this we use again the fact that $X_j b = b_j$ and compute:

$$\begin{aligned} \sum_{j=1}^d X_j^* b_j &= \sum_{j=1}^d \left[z_j b_j(z) - b(z)\|b_j\|_b^2\right] \\ &= -b(0) + b(z)(1 - \sum_{j=1}^d \|b_j\|_b^2) \\ &= -b(0) + (|a_0|^2 + |b(0)|^2) b(z) \\ &= -b(0)(1 - b(0)^* b(z)) + |a_0|^2 b(z) \\ &= -b(0)k_0^b + |a_0|^2 b(z) \end{aligned}$$

Thus $\widetilde{\mathbf{U}}$ is isometric, which finishes the proof.

□

4.1. **The one-variable case.** We analyze the foregoing construction in the one-variable case. Here the Drury-Arveson space becomes the classical Hardy space $H^2(\mathbb{D})$ and its multiplier algebra is the space of bounded analytic functions $H^\infty(\mathbb{D})$, equipped with the supremum norm. In this case it is known [12] that $b \in ball(H^\infty)$ is quasi-extreme if and only if it is an extreme point of $ball(H^\infty)$, which is equivalent to the condition

$$\int_{\mathbb{T}} \log(1 - |b|^2)\, dm = -\infty. \tag{4.2}$$

(See [7, p.138]). Conversely, if $b$ is not (quasi)-extreme, this integral is finite, and hence there exists (as noted at the beginning of this section) an outer function $a \in ball(H^\infty)$ satisfying

$$|a(\zeta)|^2 + |b(\zeta)|^2 = 1 \tag{4.3}$$

for almost every $|\zeta| = 1$; this $a$ is unique if we impose the normalization $a(0) > 0$.

In this setting, there is of course ever only one solution to the Gleason problem in $\mathcal{H}(b)$, namely the usual backward shift operator on holomorphic functions

$$S^*f(z) = \frac{f(z) - f(0)}{z}.$$

Following Sarason [11] we denote the restriction

$$X = S^*|_b.$$

All of the above discussion of transfer function realizations applies here, so $b$ is realized by the colligation

$$\mathbf{U} = \begin{bmatrix} X & S^*b \\ k_0^{b*} & b(0) \end{bmatrix}$$

Let now $a$ be the outer function of (4.3) with $a(0) > 0$.

We expand $a$ as a power series

$$a(z) = \sum_{n=0}^{\infty} \hat{a}(n) z^n.$$

Sarason [11] proves the following formula for the Taylor coefficients $\hat{a}(n)$:

**Proposition 4.2.** *We have* $|a(0)|^2 = \frac{1}{1+\|b\|_b^2}$ *and for* $n \geq 1$

$$\langle X^n b, b\rangle_{\mathcal{H}(b)} = \frac{-\hat{a}(n)}{a(0)}.$$

Mutliplying by $z^n$ and summing we get

$$\begin{aligned} a(z) &= a(0) - a(0) \sum_{n=1}^{\infty} \langle X^n b, b\rangle_b z^n \\ &= a(0) - a(0)\langle (I - zX)^{-1} zXb, b\rangle_b \end{aligned}$$

Since $Xb = S^*b$, this shows that $a$ is a transfer function for the colligation

$$\begin{bmatrix} X & S^*b \\ -a(0)b^* & a(0) \end{bmatrix}$$

acting on $\mathcal{H}(b) \oplus \mathbb{C}$. Finally, since

$$a(0) = \sqrt{\frac{1}{1 + \|b\|_b^2}} = a_0$$

(the first equality by Proposition 4.2 and the second by Lemma 3.4) this is precisely the transfer function which is used to define $a$ in the proof of Proposition 4.1.

## 5. Conclusion of the Proof of Theorem 1.1

In this section we prove the second half of Theorem 1.1:

**Proposition 5.1.** *If $b$ is a multiplier of $H^2_d$ and there exists a nonzero multiplier $a$ such that*

$$M_a^* M_a + M_b^* M_b \leq I$$

*then $b$ is not quasi-extreme.*

The proof requires an elementary-seeming lemma, which nonetheless appears easiest to prove using the notion of a *free lifting* of a multiplier. We review the relevant results, prove the lemma, and finally prove Proposition 5.1.

We recall quickly the construction of the *free or non-commutative Toeplitz algebra* of Popescu. This is a canonical example of a *free semigroup algebra* as described by Davidson and Pitts [5], which contains proofs of all the claims made here. Fix an alphabet of $d$ letters $\{1, \dots d\}$ and let $\mathbb{F}_d^+$ denote the set of all words $w$ in these $d$ letters, including the empty word $\varnothing$. The set $\mathbb{F}_d^+$ is a saemigroup under concatenation: if $w = i_1, \dots i_n$ and $v = j_1 \dots j_m$, we define

$$wv = i_1 \cdots i_n j_1 \dots j_m.$$

Let $F_d^2$ denote the Hilbert space (called the *Fock space*) with orthonormal basis $\{\xi_w\}_{w\in\mathbb{F}_d^+}$. This space comes equipped with a system of isometric operators $L_1, \dots L_d$ which act on basis vectors $\xi_w$ by *left creation*:

$$L_i \xi_w = \xi_{iw}.$$

The operators $L_1, \dots L_d$ obey the relations

$$L_i^* L_j = \delta_{ji} I,$$

in other words they are isometric with orthogonal ranges. The *free semigroup algebra* $\mathcal{L}_d$ is the WOT-closed algebra of bounded operators on $F_d^2$ generated by $L_1, \dots L_d$. Each operator $F \in \mathcal{L}_d$ has Fourier-like expansion

$$F \sim \sum_{w\in\mathbb{F}_d^+} f_w L^w \tag{5.1}$$

where, for a word $w = i_1 \cdots i_n$, by $L^w$ we mean the product $L_{i_1} L_{i_2} \cdots L_{i_n}$. The coefficients $f_w$ are determined by the relation

$$f_w = \langle F\xi_\varnothing, \xi_w \rangle_{F_d^2}$$

and the Cesaro means of the series converge WOT to $F$. To each $F \in \mathcal{L}_d$ we can associated a $d$-variable holomorphic function $\lambda(F)$ as follows: to each word $w = i_1 \cdots i_n$ let $z^w$ denote the product

$$z^w = z_{i_1} z_{i_2} \cdots z_{i_n}.$$

(Observe that $z^w = z^v$ precisely when $w$ is obtained by permuting the letters of $v$). Then for $F \in \mathcal{L}_d$ we define $\lambda(F)$ by the series

$$\lambda(F)(z) = \sum_{w\in\mathbb{F}_d^+} f_w z^w.$$

The series converges uniformly on compact subsets of $\mathbb{B}^d$, and is always a multiplier of $H_d^2$. In fact, Davidson and Pitts prove that the map $\lambda$ is completely contractive from $\mathcal{L}_d$ to $\mathcal{M}(H_d^2)$. Conversely, if $f \in \mathcal{M}(H_d^2)$ and $\|f\| \leq 1$, then there exists (by commutant lifting) an $F \in \mathcal{L}_d$ (not necessarily unique) such that $\|F\| \leq 1$ and $\lambda(F) = f$. We call such an $F$ a *free lifting* of $f$. Free liftings also always exist for matrix-valued multipliers, so in particular if, say,

$$\begin{pmatrix} f \\ g \end{pmatrix}$$

is a contractive $2 \times 1$ multiplier, then there exist $F, G \in \mathcal{L}_d$ such that $\lambda(F) = f, \lambda(G) = g$, and

$$\begin{pmatrix} F \\ G \end{pmatrix}$$

is contractive.

We will need the following lemma, which we prove using free liftings:

**Lemma 5.2.** *If $b$ is a multiplier and there exists a nonzero multiplier $a$ satisfying $M_a^* M_a + M_b^* M_b \leq I$, then an $a$ can be chosen satisfying this inequality and such that $a(0) \neq 0$.*

*Proof.* By the above remarks there exist free liftings $A, B$ of $a$ and $b$ to the free semigroup algebra $\mathcal{L}_d$ such that the column $\begin{pmatrix} B \\ A \end{pmatrix}$ is contractive. The element $A$ has Fourier expansion

$$A \sim \sum a_w L^w$$

with $a_\varnothing = 0$ (since $a(0) = 0$). Choose a word $v$ of minimal length such that $c_v \neq 0$. It follows that

$$\widetilde{A} = L_v^* A = \sum_w c_w L_v^* L_w = \sum_u \widetilde{c}_u L_u$$

is a contractive free multiplier, and $\widetilde{A}(0) := \widetilde{c}_\varnothing = c_v \neq 0$, and we then have that

$$\begin{pmatrix} B \\ \widetilde{A} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & L_v^* \end{pmatrix} \begin{pmatrix} B \\ A \end{pmatrix}$$

is contractive. Since the Davidson-Pitts symmetrization map $\lambda$ is completely contractive, on putting $\widetilde{a} = \lambda(\widetilde{A})$ we have $\widetilde{a}(0) \neq 0$ and

$$\begin{pmatrix} b \\ \widetilde{a} \end{pmatrix}$$

is a contractive $2 \times 1$ multiplier, which proves the lemma. □

**Remark:** This is really the same proof that works in the disk (without the need for the free lifting step). In the disk we just get that $\widetilde{a}$ satisfies $a(z) = z^n \widetilde{a}(z)$ for some $n$, and hence

$$M_a^* M_a = M_{\widetilde{a}}^* M_{\widetilde{a}};$$

(since $M_z$ is an isometry). More generally we could let $a = \theta F$ be the inner-outer factorization of $a$; since $M_\theta$ is isometric we would have

$$M_a^* M_a = M_F^* M_\theta^* M_\theta M_f = M_F^* M_F.$$

*Proof of Proposition 5.1.* Suppose that $b$ is a contractive multiplier and there exists a nonzero multiplier $a$ so that

$$M_a^* M_a + M_b^* M_b \leq I$$

By the lemma we may assume that $a(0) \neq 0$. We will construct an admissible tuple $b_1, \ldots b_d$ such that

$$\sum_{j=1}^{d} \|b_j\|_b^2 \leq 1 - |b(0)|^2 - |a(0)|^2 < 1 - |b(0)|^2;$$

by the remark following Proposition 2.2 this proves that $b$ is not quasi-extreme.

Let

$$c = \begin{pmatrix} b & 0 \\ a & 0 \end{pmatrix}.$$

Then $c$ is a $2 \times 2$ contractive multiplier, and

$$c(0)^* c(0) = \begin{pmatrix} |b(0)|^2 + |a(0)|^2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.2}$$

We form the deBranges-Rovnyak space $\mathscr{H}(c)$ of the function $c$, which has reproducing kernel

$$\begin{aligned} k^c(z,w) &= \frac{I - c(z)c(w)^*}{1 - zw^*} \\ &= \begin{bmatrix} k^b(z,w) & \frac{-b(z)a(w)^*}{1-zw^*} \\ \frac{-a(z)b(w)^*}{1-zw^*} & k^a(z,w) \end{bmatrix}. \end{aligned}$$

Now we apply the vector-valued generalization of a basic result from the theory of reproducing kernel Hilbert spaces: let $H(k)$ be a $\mathcal{H}$-valued RKHS of functions on a set $X$. An $\mathcal{H}$-valued function $F$ on $X$ belongs to $H(k)$ if and only if there is a $t \geq 0$ such that

$$F(x)F(y)^* \leq t^2 k(x,y),$$

as positive $\mathcal{L}(\mathcal{H})$-valued kernel functions on $X$. Moreover the least such $t$ that works is $t = \|F\|_{H(k)}$ [10, Theorem 10.17].

Note that in the above we view $F(x) : \mathbb{C} \to \mathcal{H}$ as a linear map for any fixed $x \in X$. It follows that $F(y)^* h = \langle F(y), h \rangle_{\mathcal{H}}$ for any $h \in \mathcal{H}$. For example, if (as in the case of $\mathscr{H}(c)$) $\mathcal{H} = \mathbb{C}^2$ then in the standard basis $F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}$ and

$$F(x)F(y)^* = \begin{bmatrix} F_1(x)\overline{F_1(y)} & F_1(x)\overline{F_2(y)} \\ F_2(x)\overline{F_1(y)} & F_2(x)\overline{F_2(y)} \end{bmatrix}.$$

So now let $C : \mathbb{C}^2 \to K(c) \otimes \mathbb{C}^d$ be a contractive Gleason solution for $c$, e.g., the one appearing in a generalized functional model realization for $c$ (which exists by [2]): that is, $C = \begin{bmatrix} c_1 \\ \vdots \\ c_d \end{bmatrix}$ obeys

$$zC(z) = z_1 c_1(z) + \ldots + z_d c_d(z) = c(z) - c(0),$$

and contractivity means:

$$C^* C \leq I - c(0)^* c(0).$$

So each $c_j(z) \in \mathbb{C}^{2\times 2}$ and we write

$$c_j(z) = \begin{bmatrix} b_j(z) & * \\ a_j(z) & * \end{bmatrix},$$

and observe that the $B = \begin{bmatrix} b_1 \\ \vdots \\ b_d \end{bmatrix}$, and the $A = \begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix}$ are Gleason solutions for $b, a$ in the sense that

$$b(z) - b(0) = \sum_{j=1}^{d} z_j b_j(z),$$

and similarly for $a$. Note that

$$c_j(z)e_1 = \begin{bmatrix} b_j(z) \\ a_j(z) \end{bmatrix}.$$

We need to check that $B$ actually belongs to $\mathcal{H}(b) \otimes \mathbb{C}^d$ and is a contractive Gleason solution for $b$: Let $\{e_1, e_2\}$ denote the standard orthonormal basis of $\mathbb{C}^2$ and let $t_j := \|c_k e_1\|_{\mathscr{H}(c)}$. Then by the vector-valued RKHS proposition discussed above, and the form of the reproducing kernel for $\mathscr{H}(c)$,

$$\begin{aligned}
(c_j(z)e_1)(c_j(w)e_1)^* &= \begin{bmatrix} b_j(z) \\ a_j(z) \end{bmatrix} \begin{bmatrix} b_j(w)^* & a_j(w)^* \end{bmatrix} \\
&= \begin{bmatrix} b_j(z)b_j(w)^* & b_j(z)a_j(w)^* \\ a_j(z)b_j(w)^* & a_j(z)a_j(w)^* \end{bmatrix} \\
&\le t_j^2 \begin{bmatrix} k^b(z,w) & \frac{-b(z)a(w)^*}{1-zw^*} \\ \frac{-a(z)b(w)^*}{1-zw^*} & k^a(z,w) \end{bmatrix},
\end{aligned}$$

as positive kernel functions. In particular the $(1,1)$ entry of the above equation must be a positive kernel function so that

$$b_j(z)b_j(w)^* \le t_j^2 k^b(z,w).$$

Again, by the scalar version of the RKHS result this implies that $b_j \in \mathcal{H}(b)$ and that

$$\|b_j\|_{\mathcal{H}(b)} \le t_J = \|c_j e_1\|_{\mathscr{H}(c)}.$$

This yields the inequalities

$$\begin{aligned}
\sum_{k=1}^{d} \|b_j\|^2_{\mathcal{H}(b)} &\le \sum_{k=1}^{d} t_j^2 \\
&= \sum_{k=1}^{d} \|c_j e_1\|^2_{\mathscr{H}(c)} \\
&= \sum_{k=1}^{d} \langle c_j^* c_j e_1, e_1 \rangle_{\mathbb{C}^2} \\
&= \langle C^* C e_1, e_1 \rangle_{\mathbb{C}^2} \\
&\le \langle (I - c(0)^* c(0)) e_1, e_1 \rangle_{\mathbb{C}^2} \\
&= 1 - |b(0)|^2 - |a(0)|^2, \\
&< 1 - |b(0)|^2,
\end{aligned}$$

and the proof is complete. □

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