Geometric properties of Schur class mappings of the unit ball in $\mathbb{C}^n$

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Theorem

Let f be holomorphic in the disk. TFAE:

1) $|f(z)| \leq 1$ *for all* $z \in \mathbb{D}$.
2) *The Hermitian kernel*

$$\frac{1 - f(z)\overline{f(w)}}{1 - z\overline{w}}$$

   is positive semidefinite. (Schur, Nevanlinna)

3) $\|f(T)\| \leq 1$ *for all* $n \times n$ *matrices* T *with* $\|T\| < 1$, *for all* n. *(von Neumann)*

Fix $d \geq 1$.
A (strict) $n \times n$ row contraction is a d-tuple of commuting matrices

$$T = (T_1, \dots, T_d)$$

such that

$$\|T_1 v_1 + \cdots T_d v_d\|^2 < \|v_1\|^2 + \cdots \|v_d\|^2$$

for all d-tuples of vectors $v_1, \dots v_d$ in $\mathbb{C}^n$.

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for all d-tuples of vectors $v_1, \dots v_d$ in $\mathbb{C}^n$. That is, the mapping

$$(T_1 \ T_2 \ \dots T_d) : \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} \to T_1 v_1 + \cdots T_d v_d$$

is contractive from $\mathbb{C}^{dn}$ to $\mathbb{C}^n$.

Theorem (Drury, 1978)

Let f be holomorphic in the ball $\mathbb{B}^d$. TFAE:

1') ~~$|f(z)| \le 1$ *for all* $z \in \mathbb{B}^d$.~~

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Definition

Say f belongs to the <u>Schur class</u> if it satisfies the conditions of the theorem.

Why are (2') and (3') equivalent?

(2') implies (3'): Write the kernel as a "sum of squares" (Aronszajn/Bergman):

$$\frac{1 - f(z)\overline{f(w)}}{1 - \langle z, w \rangle} = \sum_j g_j(z)\overline{g_j(w)}$$

Now

$$1 - f(z)\overline{f(w)} = \sum_j g_j(z)(1 - \langle z, w \rangle)\overline{g_j(w)}$$

Functional calculus:

$$\begin{aligned} I - f(T)f(T)^* &= \sum_j g_j(T)\left[1 - \sum T_i T_i^*\right] g_j(T)^* \\ &\geq 0 \end{aligned}$$

(3') implies (2'): Hahn-Banach theorem and GNS construction.

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A: Being a "sum of squares" is a much more rigid (and tractable) condition than mere pointwise positivity (cf. Hilbert's seventeenth problem).

Example:

Theorem (Uniqueness in the Schwarz lemma in $\mathbb{B}^d$)

If f is Schur class, $f(0) = 0$ and $Df(0) = \zeta$ with $|\zeta| = 1$, then $f(z) = \sum z_j \zeta_j$.

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all have $f(0) = 0, Df(0) = (1, 0)$.

Let $\mathcal{H}(\varphi)$ denote the reproducing kernel Hilbert space in $\mathbb{B}^d$ whose kernel is $k^\varphi(z,w)$. So each $f \in \mathcal{H}(\varphi)$ is holomorphic in $\mathbb{D}$ and

$$\langle f, k_z^\varphi \rangle_{\mathcal{H}(\varphi)} = f(z)$$

(the de Branges-Rovnyak spaces)

Idea:

[Sarason, 1990s, $d = 1$] Use the Hilbert space geometry of $\mathcal{H}(\varphi)$ to study the complex geometry of φ.

Example: The norm of k_w^φ is

$$\|k_w^\varphi\|^2 = \langle k_w^\varphi, k_w(\varphi)\rangle = \frac{1-|\varphi(w)|^2}{1-|w|^2}$$

Cauchy-Schwarz on k_w, k_z gives

$$\left|\frac{1-\varphi(z)\overline{\varphi(w)}}{1-z\overline{w}}\right|^2 \leq \frac{1-|\varphi(z)|^2}{1-|z|^2}\frac{1-|\varphi(w)^2}{1-|w|^2}$$

...precisely the invariant form of the Schwartz lemma.

A number of classical results that are true in the disk, but false in the ball, become true again if we restrict to the Schur class:

Example 1: Littlewood subordination (L^2 case):

Theorem (J., 2007)

Let φ be a Schur class mapping of $\mathbb{B}^d$ and $\varphi(0) = 0$. Then for all f holomorphic in $\mathbb{B}^d$ and all $r < 1$,

$$\int_{S_r} |f \circ \varphi|^2 \, d\sigma \leq \int_{S_r} |f|^2 \, d\sigma$$

(σ = surface measure on $\partial\mathbb{B}^d$)

This can fail if φ is not Schur class.
E.g. $\varphi(z_1, z_2) = (2z_1z_2, 0)$ (More examples: Cima-Wogen et al.)

Example 2: Julia-Caratheodory theorem

Definition

Given a point $\zeta \in \partial\mathbb{B}^d$ and a real number $c > 0$, the Koranyi region $D_c(\zeta)$ is the set

$$D_c(\zeta) = \left\{ z \in \mathbb{B}^d : |1 - \langle z, \zeta\rangle| \le \frac{c}{2}(1 - |z|^2) \right\}$$

A function f has K-limit equal to L at ζ if

$$\lim_{z \to \zeta} f(z) = L$$

whenever $z \to \zeta$ within a Koranyi region.

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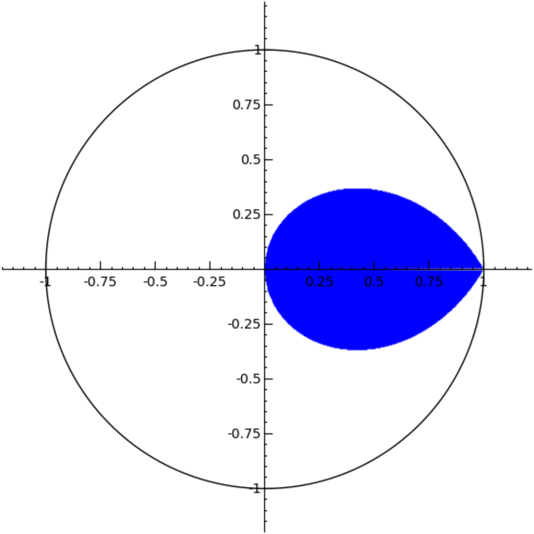
When $d = 1$ (the disk), K-limit is the same as non-tangential limit. Not so in the ball...

Slice of a Koranyi region with vertex at $(1,0)$ in $\mathbb{B}^2$:



$z_2 = 0$

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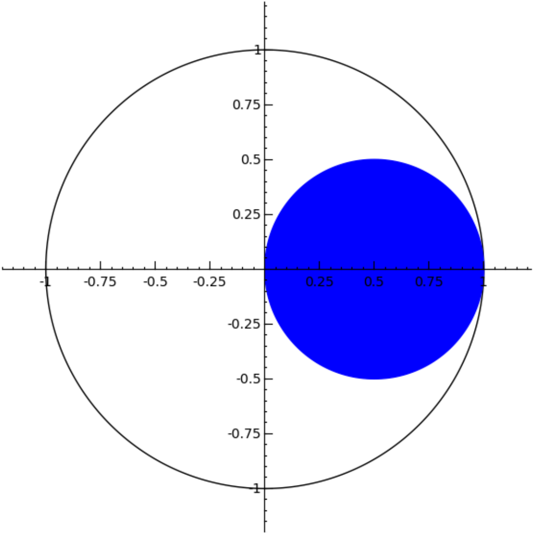

$\mathrm{Im} z_1 = \mathrm{Im} z_2 = 0$

$$\alpha = \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} < \infty. \qquad \text{(C)}$$

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Theorem (Rudin, 1980)

Suppose $\varphi = (\varphi_1, \dots \varphi_d)$ is a holomorphic mapping from $\mathbb{B}^d$ to itself satisfying condition (C)at e_1. The following functions are then bounded in every Koranyi region with vertex at e_1:

(i) $\frac{1-\varphi_1(z)}{1-z_1}$

(ii) $(D_1\varphi_1)(z)$

(iii) $\frac{1-|\varphi_1(z)|^2}{1-|z_1|^2}$

(iv) $\frac{1-|\varphi(z)|^2}{1-|z|^2}$

Moreover, each of these functions has restricted K-limit α at e_1.

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What is a restricted K-limit?

Fix a point $\zeta \in \partial\mathbb{B}^d$ and consider a curve $\Gamma : [0,1) \to \mathbb{B}^n$ such that $\Gamma(t) \to \zeta$ as $t \to 1$. Let $\gamma(t) = \langle \Gamma(t), \zeta \rangle \zeta$ be the projection of Γ onto the complex line through ζ. The curve Γ is called special if

$$\lim_{t \to 1} \frac{|\Gamma - \gamma|^2}{1 - |\gamma|^2} = 0 \tag{1}$$

and restricted if it is special and in addition

$$\frac{|\zeta - \gamma|}{1 - |\gamma|^2} \leq A \tag{2}$$

for some constant $A > 0$.

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Definition

We say that a function $f : \mathbb{B}^d \to \mathbb{C}$ has restricted K-limit L at ζ if $\lim_{z \to \zeta} f(z) = L$ along every restricted curve.

We have

K-limit $\implies$ restricted K-limit $\implies$ non-tangential limit

and each implication is strict when $d > 1$.

$$\alpha = \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} < \infty. \tag{C}$$

Theorem (J., 2008)

Let φ be a Schur class map and $\zeta \in \partial\mathbb{B}^d$. Then the following are equivalent:

1. *Condition (C).*
2. *There exists $\xi \in \partial\mathbb{B}^d$ such that the function*

$$h(z) = \frac{1 - \langle \varphi(z), \xi \rangle}{1 - \langle z, \zeta \rangle}$$

   belongs to $\mathcal{H}(\varphi)$.
3. *Every $f \in \mathcal{H}(\varphi)$ has a finite K-limit at ζ.*

$d = 1$ case: Sarason 1994

Theorem (J., 2008)

Suppose $\varphi = (\varphi_1, \ldots \varphi_d)$ is a ~~holomorphic~~ Schur class mapping from $\mathbb{B}^d$ to itself satisfying condition (C) at e_1. The following functions are then bounded in every Koranyi region with vertex at e_1:

(i) $\frac{1-\varphi_1(z)}{1-z_1}$

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(iii) $\frac{1-|\varphi_1(z)|^2}{1-|z_1|^2}$

(iv) $\frac{1-|\varphi(z)|^2}{1-|z|^2}$

Moreover, each of these functions (i)-(iii) has ~~restricted~~ K-limit α at e_1.

Theorem (MacCluer, 1983)

Let φ be a holomorphic self-map of $\mathbb{B}^d$. Then:

1. *There exists a unique point $\zeta \in \overline{\mathbb{B}^d}$ (the Denjoy-Wolff point) such that*

$$\varphi_n(z) \to \zeta$$

   locally uniformly in $\mathbb{B}^d$.

2. *If $\zeta \in \partial\mathbb{B}^d$, then*

$$0 < \liminf_{z\to\zeta} \frac{1-|\varphi(z)|^2}{1-|z|^2} = \alpha \leq 1$$

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If the Denjoy-Wolff point ζ lies in $\mathbb{B}^d$, then φ is called elliptic.

If $\zeta \in \partial\mathbb{B}^d$, the number α is called the dilatation coefficient of φ.

The map φ is called parabolic if $\alpha = 1$, and hyperbolic if $\alpha < 1$.

Example 3: Valiron's theorem

For $0 < \alpha < 1$, let θ_α denote the disk automorphism

$$\theta_\alpha(z) = \frac{z + \left(\frac{1-\alpha}{1+\alpha}\right)}{1 + \left(\frac{1-\alpha}{1+\alpha}\right) z}$$

Theorem (J., 2009)

Let φ be a hyperbolic Schur class self-map of $\mathbb{B}^d$ with dilatation coefficient α. Then there exists a nonconstant Schur class map $\sigma : \mathbb{B}^d \to \mathbb{D}$ such that

$$\sigma \circ \varphi = \theta_\alpha \circ \sigma$$

Proof uses strengthened Julia-Caratheodory theorem.

($d = 1$: Valiron, 1931; also Pommerenke 1979, C. Cowen 1981)
($d > 1$, under different assumptions:
Bracci, Gentili, Poggi-Corradini, 2007)

Corollary

If φ is a hyperbolic Schur class self-map of $\mathbb{B}^d$ with dilatation coefficient α, then for each $z_0 \in \mathbb{B}^d$

$$\lim_{n\to\infty} (1 - |\varphi(z_0)|)^{1/n} = \alpha$$

C. Cowen 1983 ($d = 1$)

Question: Is this true without the Schur class assumption?

Theorem

Let φ be a hyperbolic, holomorphic self-map of $\mathbb{B}^d$ with Denjoy-Wolff point ζ. Then

$$\varphi_n(z_0) \to \zeta$$

within a Koranyi region for every $z_0 \in \mathbb{B}^d$.

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If we knew that some orbit $\{\varphi_n(z_0)\}$ approached the Denjoy-Wolff point restrictedly, this combined with Rudin's Julia-Caratheodory theorem would imply

$$\lim_{n\to\infty} (1 - |\varphi_n(z_0)|)^{-1/2n} = \alpha^{-1/2}$$

It is not known if such orbits always exist. (Yes, if φ is an LFT.)

Example 4: Cowen-Pommerenke inequalites

Say $\zeta \in \mathbb{B}^d$ is a repelling boudnary fixed point (RBFP) if it is a boundary fixed point with dilatation coefficient $\tau > 1$.

Theorem

Suppose $\varphi : \mathbb{B}^d \to \mathbb{B}^d$ *is Schur class and* $\varphi(0) = 0$*. Suppose* $\zeta_1, \dots \zeta_n$ *are RBFP with dilatation coefficients* $\tau_1, \dots \tau_n$*. Then for every unit vector* $\eta \in \mathbb{C}^d$*,*

$$\sum_{j=1}^{n} \frac{|\langle (I - \varphi'(0))\eta, \zeta_j \rangle|^2}{\tau_j - 1} \leq 1 - |\varphi'(0)\eta|^2, \tag{3}$$

and thus

$$\sum_{j=1}^{n} \frac{\|(I - \varphi'(0)^*)\zeta_j\|^2}{\tau_j - 1} \leq \mathrm{Tr}\left(I - \varphi'(0)^*\varphi'(0)\right). \tag{4}$$

Cowen-Pommernke 1982 (d=1)
K.Y. Li 1990 (d=1, Hilbert space proof)

Corollary: φ has at most countably many RBFPs.

Corollary can fail without Schur class assumption (Ostapyuk)