INVARIANT SUBSPACES FOR A CLASS OF COMPLETE PICK KERNELS

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Abstract. Motivated by the work of McCullough and Trent, we investigate the z–invariant subspaces of the Hilbert function spaces associated to the Szegő kernels on the open unit disk. In particular, we characterize those kernels for which the the z–invariant subspaces are hyperinvariant, and (partially) those for which the so-called BLH subspaces are cyclic, obtaining counterexamples to two questions posed by McCullough and Trent.

1. Introduction

Fix a set Ω and a point $\omega \in \Omega$. Let $k(y,x)$ be a positive definite kernel on Ω, normalized so that $k(\cdot,\omega) \equiv 1$. We say that k is a *complete Pick kernel* if there exists a positive semidefinite function $b : \Omega \times \Omega \to \mathbb{C}$ with $|b(x,y)| < 1$ such that

$$1 - \frac{1}{k(y,x)} = b(y,x)$$

for all x and y in Ω. Since $b(y,x)$ is positive semidefinite, there exists an index set $\mathcal{B}$ and functions $b_j : \Omega \to \mathbb{C}$, $j \in \mathcal{B}$, such that

$$b(y,x) = \sum_{j\in\mathcal{B}} b_j(y)\overline{b_j(x)}.$$

McCullough and Trent show in [10] that in this case each function b_j defines a multiplier M_{b_j} of the associated Hilbert function space $H(k)$ and they prove the following analogue of the Beurling-Lax-Halmos theorem:

Theorem 1. *Let k be a complete Pick kernel on Ω, and for a Hilbert space $\mathcal{E}$ let $H_{\mathcal{E}}(k)$ denote the Hilbert space of $\mathcal{E}$-valued functions $H(k)\otimes\mathcal{E}$. Let $\mathcal{M}$ be a closed subspace of $H_{\mathcal{E}}(k)$. Then the following are equivalent:*

- *(i) $\mathcal{M}$ is invariant for each M_{b_j}.*
- *(ii) There exists an auxiliary Hilbert space $\mathcal{F}$ and an inner multiplier $\Phi : \Omega \to \mathcal{L}(\mathcal{F},\mathcal{E})$ such that*
$$\mathcal{M} = \Phi H_{\mathcal{F}}(k).$$
- *(iii) $\mathcal{M}$ is invariant for every multiplier M_ϕ of $H(k)$.*

Consider now $\Omega = \mathbb{D}$ (the open unit disk) and let s be the Szegő kernel

$$s(z,w) = \frac{1}{1-z\overline{w}}.$$

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The kernel s is a complete Pick kernel, and the space $H(s)$ is the usual Hardy space H^2, the space of analytic functions on $\mathbb{D}$ with square-summable power series. In this case it is possible to show that one may always choose $\mathcal{F}$ to be a subspace of $\mathcal{E}$, recovering the usual Beurling-Lax-Halmos theorem. Additionally, in the Hardy space the multiplier $\Phi(\zeta)$ is a coisometry almost everywhere on the unit circle. W. Arveson proved a special case of Theorem 1 in [4], and showed that in quite general circumstances $dim\ \mathcal{F}$ must be infinite even when $dim\ \mathcal{E} = 1$; nonetheless if $dim\ \mathcal{E}$ is finite and $\mathcal{M}$ has finite codimension, we can choose $\mathcal{F}$ to be finite dimensional. See [4] for details. Greene, Richter, and Sundberg [7] have shown that when $\Omega = \mathbb{B}^d$ (the unit ball of $\mathbb{C}^d$) and the kernel k satisfies some mild additional assumptions, the vector-valued multiplier $\Phi(\zeta)$ is a coisometry almost everywhere on the boundary of $\mathbb{B}^d$, strengthening the analogy with the usual Beurling theorem.

In the case of the Szegő kernel, the function b is $b(z,w) = z\overline{w}$. Thus the theorem says in particular that every M_z-invariant subspace of H^2 is hyperinvariant (*i.e.* invariant for every bounded operator on H^2 that commutes with M_z—here, the multipliers M_ϕ, $\phi \in H^\infty$). McCullough and Trent ask in [10] if every M_z-invariant subspace of $H(k)$ is hyperinvariant whenever k is a complete analytic Pick kernel on $\mathbb{D}$ and multiplication by z is bounded on $H(k)$. We show that this is not true in general, and give necessary and sufficient conditions for this to hold when the function b has the form

$$b(z,w) = f(z)\overline{f(w)}$$

for a univalent analytic function $f : \mathbb{D} \to \mathbb{D}$.

McCullough and Trent also investigate the cyclicity of the subspaces described by Theorem 1 (hereafter called *BLH subspaces*). They show that if k is a complete Pick kernel on $\mathbb{D}$, M_z is bounded above and below, and $\mathcal{M}$ is a BLH subspace, then the space $\mathcal{M} \ominus z\mathcal{M}$ is one–dimensional (*i.e.* M_z has the *codimension one property*). When k is the Szegő kernel, a nonzero vector in $\mathcal{M} \ominus z\mathcal{M}$ is cyclic for M_z (the unilateral shift) restricted to $\mathcal{M}$. They ask if M_z is cyclic on the BLH subspaces for more general k, at least when k is a *total* Pick kernel (defined later) and M_z is bounded above and below. Again, by appropriate choice of the function f, we provide a counterexample. However, a complete description of the kernels for which M_z is cyclic on the BLH subspaces, even for this special case, seems elusive.

Let Ω be a set. We say a function $k : \Omega \times \Omega \to \mathbb{C}$ is a *positive definite kernel on* Ω if for each finite set $\{x_1, \dots x_n\} \subseteq \Omega$, the matrix

$$(k(x_i, x_j))_{i,j=1}^n$$

is positive definite. For each $x \in \Omega$, define a function $k(\cdot, x)$ on Ω by $k(\cdot, x)(y) = k(y, x)$. Define an inner product on the linear span of these functions by

$$\langle \sum_i a_i k(\cdot, x_i), \sum_j b_j k(\cdot, x_j) \rangle = \sum_{i,j} a_i \overline{b_j} k(x_j, x_i)$$

Let $H(k)$ denote the Hilbert space obtained by completing the linear span of the functions $k(\cdot, x)$ with respect to the inner product $\langle \cdot, \cdot \rangle$. We may regard vectors f in $H(k)$ as functions on Ω, with $f(x) = \langle f, k(\cdot, x) \rangle$.

A function ϕ in $H(k)$ is called a *multiplier* of $H(k)$ if $\phi g \in H(k)$ for every $g \in H(k)$. We then define the operator $M_\phi : H(k) \to H(k)$ by $M_\phi g = \phi g, g \in H(k)$; boundedness of M_ϕ follows from the closed graph theorem. $Mult(H(k))$ will denote the algebra of multipliers $\{M_\phi\colon\ \phi \text{ is a multiplier of } H(k)\}$. The Pick problem is to determine, given points $x_1, \dots x_n$ in Ω and complex numbers $\lambda_1, \dots \lambda_n$, if there exists a multiplier ϕ on $H(k)$ with $\|M_\phi\| \le 1$ such that $\phi(x_i) = \lambda_i$ for each i. We may also formulate a matrix-valued version of this problem: for a positive integer m, a multiplier on the Hilbert space $H(k) \otimes \mathbb{C}^m$ is an $m \times m$-matrix valued function Φ on Ω such that

$$\Phi \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} \in H(k) \otimes \mathbb{C}^m$$

As in the scalar case, we define the operator M_Φ of "multiplication by Φ". Now, given points $x_1, \dots x_n$ and $m \times m$ matrices $\Lambda_1, \dots \Lambda_n$, we ask whether or not there exists a multiplier Φ on $H(k) \otimes \mathbb{C}^m$ with $\|M_\Phi\| \le 1$ such that $\Phi(x_i) = \Lambda_i$.

Definition 1. A kernel k on Ω has the $m \times m$ *Pick property* if, for any finite set of points $x_1, \dots x_n$ in Ω and any choice of $m \times m$ matrices $\Lambda_1, \dots \Lambda_n$, the following are equivalent:

(1) There exists a multiplier Φ on $H(k) \otimes \mathbb{C}^m$ such that $\|M_\Phi\| \le 1$ and $\Phi(x_i) = \Lambda_i$ for each i.
(2) The $mn \times mn$ matrix $(1 - \Lambda_j^* \Lambda_i) k(x_j, x_i)$ is positive semidefinite.

A kernel k has the *complete Pick property* if it has the $m \times m$ Pick property for every m.

Fix a set Ω and a point $\omega \in \Omega$, as before we assume k is normalized so that $k(\cdot, \omega) \equiv 1$. k has the complete Pick property if and only if there exists a positive semidefinite function $b : \Omega \times \Omega \to \mathbb{C}$ with $|b(x,y)| < 1$ such that

$$1 - \frac{1}{k(y,x)} = b(y,x) \tag{1}$$

for all x and y in Ω. (See [11], [12], [9]; see also [2]). Since $b(y,x)$ is positive semidefinite, there is an index set $\mathcal{B}$ and functions $b_j : \Omega \to \mathbb{C}$, $j \in \mathcal{B}$, such that

$$b(y,x) = \sum b_j(y) \overline{b_j(x)}.$$

Following [10], a complete Pick kernel k is called *total* if whenever $\mathcal{M}$ and $\mathcal{N}$ are BLH subspaces with $\mathcal{M} \subseteq \mathcal{N} \subseteq H(k)$, the kernel

$$\frac{P_{\mathcal{M}} k(z,w)}{P_{\mathcal{N}} k(z,w)}$$

is positive semidefinite. (Here $P_{\mathcal{M}}$ denotes orthogonal projection onto $\mathcal{M}$.) For example, the Szegő kernel is a total Pick kernel, as is shown in the next section using Beurling's theorem and the divisibility properties of inner functions.

2. Main Results

From now on we restrict ourselves to the following situation: f will be a univalent analytic function from $\mathbb{D}$ into itself, and we will assume for convenience that $f(0) =$

0. Let G denote the image of $\mathbb{D}$ under the conformal mapping f. Define a positive definite kernel $k_f : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ by

$$k_f(z,w) = \frac{1}{1 - f(z)\overline{f(w)}}.$$

Choosing the base point $\omega = 0$ (so that $k_f(\cdot, 0) \equiv 1$), (1) is easily seen to hold with $b(z,w) = f(z)\overline{f(w)}$. The kernels k_f are called *Szegő kernels*, because they may be regarded as restrictions of the Szegő kernel s to the subset $G = f(\mathbb{D}) \subseteq \mathbb{D}$ (see [2]).

In this situation, we show that M_z is a multiplier of $H(k_f)$ if and only if there exists a function $\psi \in H^\infty(\mathbb{D})$ such that $\psi|_G = f^{-1}$, and that every M_z-invariant subspace of $H(k_f)$ is hyperinvariant (*i.e.* left invariant by every bounded operator on $H(k_f)$ that commutes with M_z) if and only if this function ψ is a weak-star generator of $H^\infty(\mathbb{D})$. (A function ϕ is called a *weak-star generator of* $H^\infty(\mathbb{D})$ if the polynomials in ϕ are weak-star dense in $H^\infty(\mathbb{D})$.) Furthermore, we show that M_z is cyclic when restricted to the BLH subspaces if and only if the analytic Toeplitz operator T_ψ is cyclic on H^2.

To prove these results, we make use of the following propositions which describe the multipliers of $H(k_f)$:

Proposition 1. *A bounded analytic function ϕ on $\mathbb{D}$ is a multiplier of $H(k_f)$ if and only if there exists a function $\psi \in H^\infty$ such that $\phi \circ f^{-1} = \psi|_G$.*

Proof. We exploit the fact that the kernel k_f and the Szegő kernel s have the Pick property. Suppose ϕ is a multiplier of $H(k_f)$ with $\|M_\phi\| = M$, and let $z_1, \dots z_n$ be points in $\mathbb{D}$. By computing the Gramian of $M^2 \cdot I - M_\phi M_{\phi^*}$ with respect to the set of vectors $\{k_f(\cdot, z_j)\}$, we see that the matrix $(A_{ij})_{i,j=1}^n$ with (i,j) entry given by

$$A_{ij} = (M^2 - \phi(z_i)\overline{\phi(z_j)})k_f(z_i, z_j)$$

is positive semidefinite. Let $\zeta_i = f(z_i)$. Then

$$\begin{aligned} A_{ij} &= (M^2 - ((\phi \circ f^{-1})(\zeta_i)\overline{(\phi \circ f^{-1})(\zeta_j)})k_f(f^{-1}(\zeta_i), f^{-1}(\zeta_j)) \\ &= (M^2 - ((\phi \circ f^{-1})(\zeta_i)\overline{(\phi \circ f^{-1})(\zeta_j)})\frac{1}{(1 - \zeta_i\overline{\zeta_j})} \\ &= (M^2 - ((\phi \circ f^{-1})(\zeta_i)\overline{(\phi \circ f^{-1})(\zeta_j)})s(\zeta_i, \zeta_j) \end{aligned}$$

Thus, by the Pick property for s, there exists a function $h \in H^\infty$, $\|h\|_\infty \leq M$, such that $h(\zeta_i) = (\phi \circ f^{-1})(\zeta_i)$. Now let $\Lambda = \{\lambda_i : i \in \mathbb{N}\}$ be a set of uniqueness for analytic functions on G, and let h_N be the function obtained in the manner above with $\zeta_i = \lambda_i, i = 1, \dots N$. Since $\|h_N\|_\infty \leq M$ for each N, by the Banach-Alaoglu theorem there exists a subsequence $(h_{N_j})_{j=1}^\infty$ of $(h_N)_{N=1}^\infty$ which is weak-star convergent in H^∞ to a function ψ. We have $\|\psi\|_\infty \leq M$, and $\psi(\lambda_i) = (\phi \circ f^{-1})(\lambda_i)$ for each $\lambda_i \in \Lambda$. Since Λ is a set of uniqueness for G, $\psi|_G = \phi \circ f^{-1}$. The reverse implication is proved by similar reasoning, reversing the roles of k_f and s and using the fact that k_f is a Pick kernel.

□

The preceding argument is standard in spaces possessing complete Pick kernels; a more general form of this proposition can be found in [5].

For $\psi \in H^\infty(\mathbb{D})$, the operator $T_\psi : H^2 \to H^2$ defined by $T_\psi f = \psi f$ is called the *analytic Toeplitz operator with symbol* ψ. The map $\psi \to T_\psi$ is an isometric

Banach algebra isomorphism and a weak-star homeomorphism (see Hoffman [8]). We now show that each multiplier M_ϕ of $H(k_f)$ is unitarily equivalent to an analytic Toeplitz operator.

Proposition 2. *Let ϕ be a multiplier of $H(k_f)$ and let ψ be the analytic continuation of $\phi \circ f^{-1}$ to $\mathbb{D}$, as in Proposition 1. Then $M_\phi \cong T_\psi$.*

Proof. The linear span of the functions $\{s(\cdot, \lambda) : \lambda \in G\}$ is dense in H^2, for if $f(\lambda) = \langle f, s(\cdot, \lambda)\rangle_{H^2} = 0$ for all λ in G, then $f \equiv 0$ on G and hence on $\mathbb{D}$, since G is open.

Define a map V from $\vee\{s(\cdot, \lambda) : \lambda \in G\}$ to $\vee\{k_f(\cdot, w) : w \in \mathbb{D}\}$ by setting $Vs(\cdot, \lambda) = k_f(\cdot, f^{-1}(\lambda))$ and extending linearly. Since

$$\begin{aligned}\langle Vs(\cdot,\lambda), Vs(\cdot,\mu)\rangle_{H(k_f)} &= \langle k_f(\cdot, f^{-1}(\lambda)), k_f(\cdot, f^{-1}(\mu))\rangle \\ &= k_f(f^{-1}(\mu), f^{-1}(\lambda)) \\ &= \frac{1}{1-\mu\overline{\lambda}} \\ &= \langle s(\cdot,\lambda), s(\cdot,\mu)\rangle_{H^2},\end{aligned}$$

V is an isometry from a dense subset of H^2 onto a dense subset of $H(k_f)$, so it extends to a unitary operator (which we also denote by V) from H^2 onto $H(k_f)$. It is now easy to verify that $T_\psi = V^* M_\phi V$.

□

The proof shows that every function in $H(k_f)$ has the form $g \circ f$, with $g \in H^2$, and $\|g \circ f\|_{H(k_f)} = \|g\|_{H^2}$ (this is evident for kernel functions and follows in general by taking limits). The Proposition also shows that every analytic Toeplitz operator gives rise to a multiplier of $H(k_f)$:

Corollary 1. Let V be as in the Proposition. For any $\psi \in H^\infty$, $\psi \circ f$ is a multiplier of $H(k_f)$ and $VT_\psi V^* = M_{\psi\circ f}$.

Proof. This is an immediate consequence of the preceding remark; alternatively we could observe that $(\psi \circ f) \circ f^{-1} = \psi|_G$ and apply Propositions 1 and 2. □

From now on we assume that f is such that M_z is bounded on $H(k_f)$; by Proposition 1 this means that there exists an H^∞ function ψ on $\mathbb{D}$ such that $\psi|_G = f^{-1}$. The next proposition deals with the set of bounded operators that commute with M_z (here $\{T\}'$ denotes the set of bounded operators that commute with the operator T):

Proposition 3. *Suppose M_z is bounded on $H(k_f)$ with $M_z \cong T_\psi$. If ψ is univalent as an analytic function on $\mathbb{D}$, then $\{M_z\}' = Mult(H(k_f))$.*

Proof. Since every multiplier commutes with M_z, it is enough to show that $\{M_z\}' \subseteq Mult(H(k_f))$; by Proposition 2 and its corollary, this is equivalent to $\{T_\psi\}' \subseteq Mult(H^2)$. So suppose A is a bounded operator on H^2 that commutes with T_ψ. Then A^* commutes with T_ψ^*, and hence with $T_\psi^* - \overline{\psi(\lambda)}I$ for every $\lambda \in \mathbb{D}$. Thus $ker(T_\psi^* - \overline{\psi(\lambda)}I)$ is invariant for A^*. Since ψ was assumed univalent, $ker(T_\psi^* - \overline{\psi(\lambda)}I)$ is the one–dimensional space spanned by $s(\cdot, \lambda)$. Thus $A^*s(\cdot, \lambda) = \overline{a(\lambda)}s(\cdot, \lambda)$ for some complex number $\overline{a(\lambda)}$. Now for every $g \in H^2$ and every $\lambda \in \mathbb{D}$, we have

$$\langle Ag, s(\cdot, \lambda)\rangle = \langle g, A^*s(\cdot, \lambda)\rangle = a(\lambda)g(\lambda).$$

So $ag = Ag$ is in H^2 and $A = T_a$.

□

The fact that $\{T_\psi\}' = Mult(H^2)$ when ψ is univalent is actually a special case of a much more general theorem about the commutant of an analytic Toeplitz operator—see [17]. The proof also shows that in this situation, M_z^* is in the Cowen-Douglas class (see [6]).

Our first result will invoke the following theorem of D. Sarason [13] :

Theorem 2. *Let $\psi \in H^\infty$. Then T_ψ has the same invariant subspaces as T_z if and only if ψ is a weak-star generator of H^∞.*

We can now describe those kernels k_f for which every M_z–invariant subspace is hyperinvariant:

Theorem 3. *With notations as above, the following are equivalent:*

1. *Every M_z-invariant subspace of $H(k_f)$ is hyperinvariant.*
2. *Every M_z-invariant subspace of $H(k_f)$ is M_f-invariant.*
3. *Every T_ψ-invariant subspace of H^2 is T_z-invariant.*
4. *ψ is a weak-star generator of H^∞.*

Proof. (1) $\Rightarrow$ (2) is trivial. By Proposition 2, $M_z \cong T_\psi$ and $M_f \cong T_z$; the equivalence of (2) and (3) follows. (3) $\Leftrightarrow$ (4) is just an application of Theorem 2 above, together with the fact that T_z-invariant subspaces are invariant for every analytic Toeplitz operator, the consequence of Beurling's theorem discussed earlier.

It remains to show (2) $\Rightarrow$ (1). Since (2) $\Leftrightarrow$ (4), the function ψ is a weak-star generator of H^∞ and hence is univalent in $\mathbb{D}$ (see [14]), so by Proposition 3, the only bounded operators on $H(k_f)$ that commute with M_z are the multipliers. By Theorem 1, each M_f-invariant subspace is invariant for every multiplier, and (1) follows. □

Turning now to cyclic vectors, we show that the kernels k_f are total Pick kernels, and describe the cyclicity of M_z on BLH subspaces in terms of the cyclicity of analytic Toeplitz operators.

Proposition 4. *The kernels k_f are total Pick kernels.*

Proof. We observe that $\mathcal{M}$ is a BLH subspace of $H(k_f)$ if and only if $V^*\mathcal{M}$ is a BLH subspace (i.e. shift invariant subspace) of H^2, where V is the unitary map of Proposition 2. By Beurling's theorem, $V^*\mathcal{M} = \phi H^2$, where ϕ is an inner function, so

$$\mathcal{M} = VV^*\mathcal{M} = V\phi H^2 = (\phi \circ f)H(k_f).$$

A straightforward calculation then shows that

$$P_{\mathcal{M}} k_f(z,w) = \phi(f(z))\overline{\phi(f(w))}k_f(z,w).$$

Recall that if ϕ_1 and ϕ_2 are inner functions and $\phi_1 H^2 \subseteq \phi_2 H^2$, then ϕ_1/ϕ_2 is inner. Let $\mathcal{M}$ and $\mathcal{N}$ be BLH subspaces of $H(k_f)$ with $\mathcal{M} \subseteq \mathcal{N}$, so there exist inner functions ϕ_1 and ϕ_2 with ϕ_2 dividing ϕ_1 so that $\mathcal{M} = (\phi_1 \circ f)H(k_f)$ and $\mathcal{N} = (\phi_2 \circ f)H(k_f)$. Then

$$\begin{aligned}\frac{P_{\mathcal{M}}k_f(z,w)}{P_{\mathcal{N}}k_f(z,w)} &= \frac{\phi_1(f(z))\overline{\phi_1(f(w))}k_f(z,w)}{\phi_2(f(z))\overline{\phi_2(f(w))}k_f(z,w)} \\ &= \frac{\phi_1}{\phi_2}(f(z))\overline{\frac{\phi_1}{\phi_2}(f(w))}\end{aligned}$$

The last expression is a positive semidefinite function on $\mathbb{D} \times \mathbb{D}$, so k_f is a total Pick kernel.

□

Theorem 4. *Let k_f be a complete Pick kernel, and suppose M_z is bounded, so that $M_z \cong T_\psi$. Then the following are equivalent:*

(1) *M_z is cyclic on some BLH subspace $\mathcal{M} \subseteq H(k_f)$.*
(2) *M_z is cyclic on every BLH subspace $\mathcal{M} \subseteq H(k_f)$.*
(3) *T_ψ is cyclic on H^2.*

Proof. In the proof of Proposition 4 we showed that the BLH subspaces of $H(k_f)$ have the form $(\phi \circ f)H(k_f)$ where ϕ is an inner function. From this it follows that a function g in a BLH subspace $\mathcal{M} \subseteq H(k_f)$ is cyclic for $M_z|_{\mathcal{M}}$ if and only if $g/(\phi \circ f)$ is cyclic for M_z on $H(k_f)$, from this follows the equivalence of assertions (1) and (2). Since $M_z \cong T_\psi$, M_z is cyclic on the BLH subspaces if and only if T_ψ is cyclic on H^2.

□

3. Examples and Remarks

Using Theorem 3 we can construct an analytic Pick kernel for which M_z is bounded but for which there exist M_z-invariant subspaces which are not hyperinvariant. To do this, we need only exhibit a univalent analytic function $f : \mathbb{D} \to G$, (with $G \subseteq \mathbb{D}$) so that f^{-1} extends to a function $\psi \in H^\infty$ which is *not* a weak-star generator of H^∞.

Consider a bounded, simply connected domain $\Omega \subseteq \mathbb{C}$, and let ψ be a Riemann map from $\mathbb{D}$ onto Ω. Sarason shows in [14] that ψ fails to be a weak-star generator of H^∞ if and only if Ω has the following property:

(∗) *There exists a domain B containing Ω properly such that* $\sup_{z\in B}|f(z)| = \sup_{z\in\Omega}|f(z)|$ *for all f bounded and analytic in B.*

For example, let B be an open disk, and fix a point $z_0 \in B$ (not the center). Let $\alpha = dist(z_0, \partial B)$ and let $\overline{D}$ be the closed disk of radius α centered at z_0. Then $B \backslash \overline{D}$ has property (∗), by the maximum modulus principle.

We now let Ω be a domain satisfying (∗) with $\mathbb{D} \subseteq \Omega$ and let $\psi : \mathbb{D} \to \Omega$ be a Riemann map. Taking f to be the restriction of ψ^{-1} to $\mathbb{D}$ does the job.

Regarding cyclic vectors, we can use Theorem 4 to exhibit a total Pick kernel for which multiplication by z is bounded above and below, but such that M_z is not cyclic on any of the BLH subspaces. By Theorem 4 it will suffice to exhibit an analytic Toeplitz operator which is bounded below but is not cyclic. For this a Riemann map ϕ of $\mathbb{D}$ onto a slit disk which contains $\mathbb{D}$ suffices (see [16]); we then let $f = \phi^{-1}|_{\mathbb{D}}$ as in the previous example. The cyclicity of analytic Toeplitz operators is a very subtle problem, which has not been solved completely; it is for this reason

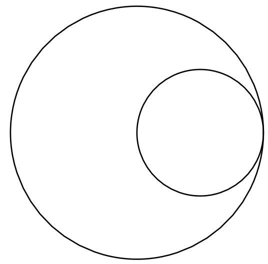

FIGURE 1. The domain $B \backslash \overline{D}$

that we regard the "characterization" of Theorem 4 as incomplete. We remark, however, that T_ψ is cyclic when ψ is a weak-star generator of H^∞, so combining Theorem 3 and the above remarks tells us that if every M_z–invariant subspace of $H(k_f)$ is hyperinvariant (and hence a BLH subspace), then M_z is cyclic on each of these spaces.

In general, we expect the kernels k_f to be a good source of counterexamples for questions regarding multiplication by z on spaces possessing a complete Pick kernel: as seen in these two examples, one can choose f so that M_z is unitarily equivalent to an analytic Toeplitz operator with "bad" properties.

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