C^*-ALGEBRAS GENERATED BY GROUPS OF COMPOSITION OPERATORS

MICHAEL T. JURY

Abstract. We compute the C*-algebra generated by a group of composition operators acting on certain reproducing kernel Hilbert spaces over the disk, where the symbols belong to a non-elementary Fuchsian group. We show that such a C*-algebra contains the compact operators, and its quotient is isomorphic to the crossed product C*-algebra for the action of the group on the boundary circle. In addition we show that the C*-algebras obtained from composition operators acting on a natural family of Hilbert spaces on the disk are in fact isomorphic, and also determine the same *Ext*-class, which can be related to known extensions of the crossed product.

1. Introduction

The purpose of this paper is to begin a line of investigation suggested by recent work of Moorhouse et al. [**12, 8, 7**]: to describe, in as much detail as possible, the C^*-algebra generated by a set of composition operators acting on a Hilbert function space. In this paper we consider a class of examples which, while likely the simplest from the point of view of composition operators, nonetheless produces C^*-algebras which are of great interest both intrinsically and for applications. In fact, the C^*-algebras we obtain are objects of current interest among operator algebraists, which appear in the study of hyperbolic dynamics [**5**], noncommutative geometry [**4**], and even number theory [**11**].

Let f belong to the Hardy space $H^2(\mathbb{D})$. For an analytic function $\gamma : \mathbb{D} \to \mathbb{D}$, the *composition operator with symbol* γ is the linear operator defined by

$$(C_\gamma f)(z) = f(\gamma(z))$$

In this paper, we will be concerned with the C^*-algebra

$$\mathcal{C}_\Gamma = C^*(\{C_\gamma : \gamma \in \Gamma\})$$

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where Γ is a discrete group of (analytic) automorphisms of $\mathbb{D}$ (i.e. a *Fuchsian group*). For reasons to be described shortly, we will further restrict ourselves to *non-elementary* Fuchsian groups (i.e. groups Γ for which the Γ-orbit of 0 in $\mathbb{D}$ accumulates at at least three points of the unit circle $\mathbb{T}$.) Our main theorem shows that $\mathcal{C}_\Gamma$ contains the compact operators, and computes the quotient $\mathcal{C}_\Gamma/\mathcal{K}$:

Theorem 1.1. *Let Γ be a non-elementary Fuchsian group, and let $\mathcal{C}_\Gamma$ denote the C^*-algebra generated by the set of composition operators on H^2 with symbols in Γ. Then there is an exact sequence*

$$0 \longrightarrow \mathcal{K} \xrightarrow{\iota} \mathcal{C}_\Gamma \xrightarrow{\pi} C(\mathbb{T}) \times \Gamma \longrightarrow 0 \tag{1.1}$$

Here $C(\mathbb{T}) \times \Gamma$ is the crossed product C^*-algebra obtained from the action α of Γ on $C(\mathbb{T})$ given by

$$\alpha_\gamma(f)(z) = f(\gamma^{-1}(z))$$

(Since the action of Γ on $\mathbb{T}$ is amenable, the full and reduced crossed products coincide; we will discuss this further shortly.) We will recall the relevant definitions and facts we require in the next section.

There is a similar result for the C^*-algebras

$$\mathcal{C}_\Gamma^n = C^*\left(\{C_\gamma \in \mathcal{B}(A_n^2) | \gamma \in \Gamma\}\right),$$

acting on the family of reproducing kernel Hilbert spaces A_n^2 (defined below), namely there is an extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{C}_\Gamma^n \longrightarrow C(\mathbb{T}) \times \Gamma \longrightarrow 0 \tag{1.2}$$

and for each n this extension is strongly equivalent to the extension (1.1). It follows that each of these extensions represents the same element of the *Ext* group $Ext(C(\mathbb{T}) \times \Gamma, \mathcal{K})$, and that $\mathcal{C}_\Gamma$ and $\mathcal{C}_\Gamma^n$ are isomorphic as C^*-algebras.

Finally, we will compare the extension determined by $\mathcal{C}_\Gamma$ to two other recent constructions of extensions of $C(\mathbb{T}) \times \Gamma$. We show that the *Ext*-class of $\mathcal{C}_\Gamma$ coincides with the class of the Γ-equivariant Toeplitz extension of $C(\mathbb{T})$ constructed by J. Lott [**10**], and differs from the extension of crossed products by co-compact groups constructed by H. Emerson [**5**]. Finally we show that this extension in fact gives rise to a Γ-equivariant KK_1-cycle for $C(\mathbb{T})$ which also accords with the construction in [**10**].

2. Preliminaries

We will consider C*-algebras generated by composition operators which act on a family of reproducing kernel Hilbert spaces on the unit disk. Specifically we will consider the spaces of analytic functions A^2_n, where A^2_n is the space with reproducing kernel

$$k^n(z, w) = (1 - z\overline{w})^n$$

When $n = 1$ this space is the Hardy space H^2, and its norm is given by

$$\|f\|^2 = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta$$

For $n \geq 2$, the norm on A^2_n is given by

$$\|f\|^2 = \frac{n-1}{\pi} \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^{n-2} \, dA(z)$$

An analytic function $\gamma : \mathbb{D} \to \mathbb{D}$ defines a *composition operator* C_γ on A^2_n by

$$(C_\gamma f)(z) = f(\gamma(z))$$

In this paper, we will only consider cases where γ is a Möbius transformation; in these cases C_γ is easily seen to be bounded, by changing variables in the integrals defining the norms. An elementary calculation shows that if $\gamma : \mathbb{D} \to \mathbb{D}$ is analytic, then

$$C^*_\gamma k_w(z) = k_{\gamma(w)}(z)$$

where k is any of the reproducing kernels k^n.

We recall here the definitions of the full and reduced crossed product C^*-algebras; we refer to **[13]** for details. Let a group Γ act by homeomorphisms on a compact Hausdorff space X. This induces an action of Γ on the commutative C^*-algebra $C(X)$ via

$$(\gamma \cdot f)(x) = f(\gamma^{-1} \cdot x)$$

The *algebraic crossed product* $C(X) \times_{alg} \Gamma$ consists of formal finite sums $f = \sum_{\gamma\in\Gamma} f_\gamma[\gamma]$, where $f_\gamma \in C(X)$ and the $[\gamma]$ are formal symbols. Multiplication is defined in $C(X) \times_{alg} \Gamma$ by

$$\left(\sum_{\gamma\in\Gamma} f_\gamma[\gamma]\right)\left(\sum_{\gamma\in\Gamma} f'_\gamma[\gamma']\right) = \sum_{\delta\in\Gamma}\sum_{\gamma\gamma'=\delta} f_\gamma(\gamma \cdot f'_{\gamma'})[\delta]$$

For $f = \sum_\gamma f_\gamma[\gamma]$, define $f^* \in C(X) \times_{alg} \Gamma$ by

$$f^* = \sum_{\gamma\in\Gamma} (\gamma \cdot \overline{f_{\gamma^{-1}}})[\gamma]$$

With this multiplication and involution, $C(X) \times_{alg} \Gamma$ becomes a $*$-algebra, and we may construct a C^*-algebra by closing the algebraic crossed product with respect to a C^*-norm.

To obtain a C^*-norm, one constructs $*$-representations of $C(X)\times_{alg}\Gamma$ on Hilbert space. To do this, we first fix a faithful representation π of $C(X)$ on a Hilbert space $\mathcal{H}$. We then construct a representation σ of the algebraic crossed product on $\mathcal{H}\otimes\ell^2(\Gamma)=\ell^2(\Gamma,\mathcal{H})$ as follows: define a representation $\tilde{\pi}$ of $C(X)$ by its action on vectors $\xi\in\ell^2(\Gamma,\mathcal{H})$

$$(\tilde{\pi}(f))(\xi)(\gamma)=\pi(f\circ\gamma)\xi(\gamma)$$

Represent Γ on $\ell^2(\Gamma,\mathcal{H})$ by left translation:

$$(U(\gamma))(\xi)(\eta)=\xi(\gamma^{-1}\eta)$$

The representation σ is then given by

$$\sigma\left(\sum f_\gamma[\gamma]\right)=\sum\tilde{\pi}(f_\gamma)U(\gamma)$$

The closure of $C(X)\times_{alg}\Gamma$ with respect the norm induced by this representation is called the *reduced crossed product* of Γ and $C(X)$, and is denoted $C(X)\times_r\Gamma$. The *full crossed product*, denote $C(X)\times\Gamma$, is obtained by taking the closure of the algebraic crossed product with respect to the maximal C^*-norm

$$\|f\|=\sup_\pi\|\pi(f)\|$$

where the supremum is taken over *all* $*$-representations π of $C(X)\times_{alg}\Gamma$ on Hilbert space. When Γ is discrete, $C(X)\times\Gamma$ contains a canonical subalgebra isomorphic to $C(X)$, and there is a natural surjective $*$-homomorphism $\lambda: C(X)\times\Gamma\to C(X)\times_r\Gamma$.

The full crossed product is important because of its universality with respect to covariant representations. A *covariant representation* of the pair (Γ,X) consists of a faithful representation π of $C(X)$ on Hilbert space, together with a unitary representation u of Γ on the same space satisfying the covariance condition

$$u(\gamma)\pi(f)u(\gamma)^*=\pi(f\circ\gamma^{-1})$$

for all $f\in C(X)$ and all $\gamma\in\Gamma$. If $\mathcal{A}$ denotes the C*-algebra generated by the images of π and u, then there is a surjective *-homomorphism from $C(X)\times\Gamma$ to $\mathcal{A}$; equivalently, any C*-algebra generated by a covariant representation is isomorphic to a quotient of the full crossed product.

We now collect properties of group actions on topological spaces which we will require in what follows. Let a group Γ act by homeomorphisms on a locally compact Hausdorff space X. The action of Γ on X is called *minimal* if the set $\{\gamma\cdot x|\gamma\in\Gamma\}$ is dense in X for each $x\in X$, and called *topologically free* if, for each $\gamma\in\Gamma$, the set of points fixed by γ has empty interior.

Suppose now Γ is discrete and X is compact. Let $\mathrm{Prob}(\Gamma)$ denote the set of finitely supported probability measures on Γ. We say Γ *acts amenably on* X if there exists a sequence of weak-* continuous maps $b_x^n : X \to \mathrm{Prob}(\Gamma)$ such that for every $\gamma \in \Gamma$,

$$\lim_{n\to\infty} \sup_{x\in X} \|\gamma \cdot b_x^n - b_{\gamma\cdot x}^n\|_1 = 0$$

where Γ acts on the functions b_x^n via $(\gamma \cdot b_x^n)(z) = b_x^n(\gamma^{-1}\cdot z)$, and $\|\cdot\|_1$ denotes the l^1-norm on Γ.

Theorem 2.1. **[2]** *Let Γ be a discrete group acting on a compact Hausdorff space X, and suppose that the action is topologically free. If $\mathcal{J}$ is an ideal in $C(X) \times \Gamma$ such that $C(X) \cap \mathcal{J} = 0$, then $\mathcal{J} \subseteq \mathcal{J}_\lambda$, where $\mathcal{J}_\lambda$ is the kernel of the surjection of the full crossed product onto the reduced crossed product.*

Theorem 2.2. **[15, 16]** *If the Γ acts amenably on X, then the full and reduced crossed product C^*-algebras coincide, and this crossed product is nuclear.*

In this paper we are concerned with the case $X = \mathbb{T}$ and $G = \Gamma$, a non-elementary Fuchsian group. The action of Γ on $\mathbb{T}$ is always amenable, so by Theorem 2.2 the full and reduced crossed products coincide and $C(\mathbb{T}) \times \Gamma$ is nuclear.

We also require some of the basic terminology concerning Fuchsian groups. Fix a base point $z_0 \in \mathbb{D}$. The *limit set* of Γ is the set accumulation points of the orbit $\{\gamma(z_0) : \gamma \in \Gamma\}$ on the boundary circle; it is closed and does not depend on the choice of z_0. The limit set can be one of three types: it is either finite (in which case it consists of at most two points), a totally disconnected perfect set (hence uncountable), or all of the circle. In the latter case we say Γ is of the *first kind*, and of the *second kind* otherwise. If the limit set is finite, Γ is called *elementary*, and *non-elementary* otherwise.

Though we do not require it in this paper, it is worth recording that if Γ is of the first kind, then $C(\mathbb{T}) \times \Gamma$ is simple, nuclear, purely infinite, and belongs to the bootstrap category $\mathcal{N}$ (and is clearly separable and unital). Hence, it satisfies the hypotheses of the Kirchberg-Phillips classification theorem, and is classified up to isomorphism by its (unital) K-theory **[9, 1]**.

Finally, we recall briefly the basic facts about extensions of C*-algebras and the *Ext* functor. An exact sequence of C*-algebras

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0 \tag{2.1}$$

is called an *extension* of A by B. (In this paper we consider only extensions for which $B = \mathcal{K}$, the C*-algebra of compact operators on a separable Hilbert space.) Associated to an extension of A by $\mathcal{K}$ is a $*$-homomorphism τ from A to the Calkin algebra $\mathcal{Q}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$; this τ is called the *Busby map* associated to the extension. Conversely, given a map $\tau : A \to \mathcal{Q}(\mathcal{H})$ there is, via the pull-back construction, a unique extension having Busby map τ; we will thus speak of an extension and its Busby map interchangeably. Two extensions of A by $\mathcal{K}$ with Busby maps $\tau_1 : A \to \mathcal{Q}(\mathcal{H}_1)$ and $\tau_2 : A \to \mathcal{Q}(\mathcal{H}_2)$ are *strongly unitarily equivalent* if there is a unitary $u : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\pi(u)\tau_1(a)\pi(u)^* = \tau_2(a) \tag{2.2}$$

for all $a \in A$ (here π denotes the quotient map from $\mathcal{B}(\mathcal{H}_1)$ to $\mathcal{Q}(\mathcal{H}_1)$). We say τ_1 and τ_2 are *unitarily equivalent*, written $\tau_1 \sim_u \tau_2$, if (2.2) holds with u replaced by some v such that $\pi(v)$ is unitary. An extension τ is *trivial* if it lifts to a $*$-homomorphism, that is there exists a $*$-homomorphism $\rho : A \to \mathcal{B}(\mathcal{H})$ such that $\tau(a) = \pi(\rho(a))$ for all $a \in A$. Two extensions τ_1, τ_2 are *stably equivalent* if there exist trivial extensions σ_1, σ_2 such that $\tau_1 \oplus \sigma_1$ and $\tau_2 \oplus \sigma_2$ are strongly unitarily equivalent; stable equivalence is an equivalence relation. If A is separable and nuclear (which will always be the case in this paper) then the stable equivalence classes of extensions of A by $\mathcal{K}$ form an abelian group (where addition is given by direct sum of Busby maps) called $Ext(A, \mathcal{K})$, abbreviated $Ext(A)$. Finally, each element of $Ext(A)$ determines an *index homomorphism* $\partial : K_1(A) \to \mathbb{Z}$ obtained by lifting a unitary in $M_n(A)$ representing the K_1 class to a Fredholm operator in $M_n(E)$ and taking its Fredholm index.

3. The Extensions $\mathcal{C}_\Gamma$ and $\mathcal{C}_\Gamma^n$

3.1. **The Hardy space.**

This section is devoted to the proof of our main theorem, in the case of the Hardy space:

Theorem 3.1. *Let Γ be a non-elementary Fuchsian group, and let $\mathcal{C}_\Gamma$ denote the C^*-algebra generated by the set of composition operators on H^2 with symbols in Γ. Then there is an exact sequence*

$$0 \longrightarrow \mathcal{K} \xrightarrow{\iota} \mathcal{C}_\Gamma \xrightarrow{\pi} C(\mathbb{T}) \times \Gamma \longrightarrow 0 \tag{3.1}$$

While the proof in the general case of A_n^2 follows similar lines, we prove the H^2 case first since it is technically simpler and illustrates the main ideas. The proof splits into three parts: first, we prove that $\mathcal{C}_\Gamma$ contains the unilateral shift S and hence the compacts (Proposition 3.4). We then prove that the quotient $\mathcal{C}_\Gamma/\mathcal{K}$ is generated by a covariant representation of the topological dynamical system $(C(\mathbb{T}), \Gamma)$.

Finally we prove that the C^*-algebra generated by this representation is all of the crossed product $C(\mathbb{T}) \times \Gamma$.

We first require two computational lemmas.

Lemma 3.2. *Let γ be an automorphism of $\mathbb{D}$ with $a = \gamma^{-1}(0)$ and let*

$$f(z) = \frac{1 - \overline{a}z}{(1 - |a|^2)^{1/2}}$$

Then

$$C_\gamma C_\gamma^* = M_f M_f^*$$

where M_f denotes the operator of multiplication by f.

Proof. It suffices to show

$$\langle C_\gamma C_\gamma^* k_w, k_z \rangle = \langle M_f M_f^* k_w, k_z \rangle \tag{3.2}$$

for all z, w in $\mathbb{D}$. Since $C_\gamma^* k_w = k_{\gamma(w)}$ and $M_f^* k_w = \overline{f(w)} k_w$, (3.2) reduces to the well-known identity

$$\frac{1}{1 - \gamma(z)\overline{\gamma(w)}} = \frac{(1 - \overline{a}z)(1 - a\overline{w})}{(1 - |a|^2)(1 - z\overline{w})}. \tag{3.3}$$

Lemma 3.3. *Let $\gamma(z)$ be an automorphism of $\mathbb{D}$ and let $S = M_z$. Then*

$$M_\gamma C_\gamma = C_\gamma S$$

Proof. For all $g \in H^2$,

$$\begin{aligned} (C_\gamma S)(g)(z) &= \gamma(z) g(\gamma(z)) \\ &= (M_\gamma C_\gamma)(g)(z) \end{aligned}$$

Proposition 3.4. *Let Γ be a non-elementary Fuchsian group. Then $\mathcal{C}_\Gamma$ contains the unilateral shift S.*

Proof. Let Λ denote the limit set of Γ. Since Γ is non-elementary, there exist three distinct points λ_1, λ_2, $\lambda_3 \in \Lambda$. For each λ_i there exists a sequence $(\gamma_{n,i})_{n=1}^\infty \subset \Gamma$ such that $\gamma_{n,i}(0) \to \lambda_i$. Let $a_i^n = \gamma_{n,i}(0)$. By Lemma 3.2,

$$(1 - |a_i^n|^2) C_{\gamma_{n,i}^{-1}} C_{\gamma_{n,i}^{-1}}^* = (1 - \overline{a_i^n} S)(1 - \overline{a_i^n} S)^*$$

As $n \to \infty$, $a_i^n \to \lambda_i$ and the right-hand side converges to

$$1 - \overline{\lambda} S - \lambda S^* + S S^*$$

in $\mathcal{C}_\Gamma$. Taking differences of these operators for different values of i, we see that $\Re[\mu_1 S]$, $\Re[\mu_2 S] \in \mathcal{C}_\Gamma$, with $\mu_1 = \overline{\lambda_1 - \lambda_2}$ and $\mu_2 = \overline{\lambda_2 - \lambda_3}$. Note that since λ_1, λ_2, λ_3 are distinct points on the circle, the complex numbers μ_1 and μ_2 are linearly independent over $\mathbb{R}$. We now show there exist scalars a_1, a_2 such that

$$a_1 \Re[\mu_1 S] + a_2 \Re[\mu_2 S] = S$$

which proves the lemma. Such scalars must solve the linear system

$$\begin{pmatrix} \mu_1 & \mu_2 \\ \overline{\mu_1} & \overline{\mu_2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.4}$$

Writing $\mu_1 = \alpha_1 + i\beta_1$, $\mu_2 = \alpha_2 + i\beta_2$, a short calculation shows that

$$\det \begin{pmatrix} \mu_1 & \mu_2 \\ \overline{\mu_1} & \overline{\mu_2} \end{pmatrix} = -2i \det \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}$$

The latter determinant is nonzero since μ_1, μ_2 are linearly independent over $\mathbb{R}$, and the system is solvable. □

The above argument does not depend on the discreteness of Γ; indeed it is a refinement of an argument due to J. Moorhouse **[12]** that the C^*-algebra generated by *all* Möbius transformations contains S. The argument is valid for any group which has an orbit with three accumulation points on $\mathbb{T}$; e.g. the conclusion holds for any dense subgroup of $Aut(\mathbb{D}) \cong PSU(1,1)$.

Proof of Theorem 3.1 For an automorphism γ of $\mathbb{D}$ set

$$U_\gamma = (C_\gamma C_\gamma^*)^{-1/2} C_\gamma,$$

the unitary appearing in the polar decomposition of C_γ. By Lemma 3.2, $C_\gamma C_\gamma^* = T_f T_f^*$, so we may write

$$U_\gamma = T_{|f|^{-1}} C_\gamma + K$$

for some compact K. (Here we have used the fact the Toeplitz operators with continuous symbols commute modulo the compact operators.) Now by Lemma 3.3, if p is any analytic polynomial,

$$\begin{aligned} U_\gamma T_p &= T_{|f|^{-1}} C_\gamma T_p + K' \\ &= T_{|f|^{-1}} T_{p\circ\gamma} C_\gamma + K' \\ &= T_{p\circ\gamma} T_{|f|^{-1}} C_\gamma + K'' \\ &= T_{p\circ\gamma} U_\gamma + K'' \end{aligned}$$

again using the fact that $T_{|f|^{-1}}$ and T_p commute modulo $\mathcal{K}$. Taking adjoints and sums shows that

$$U_\gamma T_q U_\gamma^* = T_{q\circ\gamma} + K \tag{3.5}$$

for any trigonometric polynomial q. We next show that, for Möbius transformations γ, η,

$$U_\gamma U_\eta = U_{\eta\circ\gamma} + K$$

for some compact K. To see this, write

$$\gamma(z) = \frac{az+b}{cz+d}, \qquad \eta(z) = \frac{ez+f}{gz+h}$$

Then

$$U_\gamma = T_{|cz+d|^{-1}} C_\gamma + K_1, \qquad U_\eta = T_{|gz+h|^{-1}} C_\eta + K_2.$$

Note that

$$U_{\eta\circ\gamma} = T_{|g(az+b)+h(cz+d)|^{-1}} C_{\eta\circ\gamma} + K_3.$$

Then

$$\begin{aligned} U_\gamma U_\eta &= T_{|cz+d|^{-1}} C_\gamma T_{|gz+h|^{-1}} C_\eta + K \\ &= T_{|cz+d|^{-1}} T_{|g\gamma(z)+h|^{-1}} C_\gamma C_\eta + K' \\ &= T_{|g(az+b)+h(cz+d)|^{-1}} C_{\eta\circ\gamma} + K' \\ &= U_{\eta\circ\gamma} + K'' \end{aligned}$$

Observe now that $\mathcal{C}_\Gamma$ is equal to the C*-algebra generated by S and the unitaries U_γ. We have already shown that $\mathcal{C}_\Gamma$ contains S and each U_γ, for the converse we recall that $C_\gamma = (C_\gamma C_\gamma^*)^{1/2} U_\gamma$, and since $C_\gamma C_\gamma^*$ lies in $C^*(S)$ by Lemma 3.2, we see that C_γ lies in the C*-algebra generated by S and U_γ. Letting π denote the quotient map $\pi : \mathcal{C}_\Gamma \to \mathcal{C}_\Gamma/\mathcal{K}$, it follows hat $\mathcal{C}_\Gamma/\mathcal{K}$ is generated as a C^*-algebra by a copy of $C(\mathbb{T})$ and the unitaries $\pi(U_\gamma)$, and the map $\gamma \to \pi(U_{\gamma^{-1}})$ defines a unitary representation of Γ. Let $\alpha : \Gamma \to \mathrm{Aut}(C(\mathbb{T}))$ be given by

$$\alpha_\gamma(f)(z) = f(\gamma^{-1}(z))$$

Then by (3.5),

$$\pi(U_{\gamma^{-1}})\pi(T_f)\pi(U_{\gamma^{-1}}^*) = \pi(T_{f\circ\gamma^{-1}}) = \pi(T_{\alpha_\gamma(f)}) \tag{3.6}$$

for all trigonometric polynomials f, and hence for all $f \in C(\mathbb{T})$ by continuity. Thus, $\mathcal{C}_\Gamma/\mathcal{K}$ is generated by $C(\mathbb{T})$ and a unitary representation of Γ satisfying the relation (3.6). Therefore there is a surjective $*$-homomorphism $\rho : C(\mathbb{T}) \times \Gamma \to \mathcal{C}_\Gamma/\mathcal{K}$ satisfying

$$\rho(f) = \pi(T_f), \qquad \rho(u_\gamma) = \pi(U_{\gamma^{-1}})$$

for all $f \in C(\mathbb{T})$ and all $\gamma \in \Gamma$. Let $\mathcal{J} = \ker \rho$. The theorem will be proved once we show $\mathcal{J} = 0$. To do this, we use Theorems 2.1 and 2.2. Indeed, it suffices to show that $C(\mathbb{T}) \cap \mathcal{J} = 0$: then $\mathcal{J} \subset \mathcal{J}_\lambda$ by Theorem 2.1, and $\mathcal{J}_\lambda = 0$ by Theorem 2.2.

To see that $C(\mathbb{T}) \cap \mathcal{J} = 0$, choose $f \in C(\mathbb{T}) \cap \mathcal{J}$. Then $\pi(T_f) = \rho(f) = 0$, which means that T_f is compact. But then $f = 0$, since nonzero Toeplitz operators are non-compact. □

3.2. **Other Hilbert spaces.** In this section we prove the analogue of Theorem 3.1 for composition operators acting on the reproducing kernel Hilbert spaces with kernel given by

$$k(z,w) = \frac{1}{(1-z\overline{w})^n}$$

for integers $n \geq 2$.

We let A^2_n denote the Hilbert function space on $\mathbb{D}$ with kernel $k(z,w) = (1-z\overline{w})^{-n}$. For $n \geq 2$, this space consists of those analytic functions in $\mathbb{D}$ for which

$$\frac{n-1}{\pi}\int_{\mathbb{D}} |f(w)|^2(1-|w|^2)^{n-2}dA(w)$$

is finite, and the square root of this quantity gives the norm on A^2_n.

We fix $n \geq 2$ for the remainder of this section, and let T denote the operator of multiplication by z on A^2_n. The operator T is a contractive weighted shift, and essentially normal. Moreover, if f is any bounded analytic function on $\mathbb{D}$, then multiplication by f is bounded on A^2_n and is denoted M_f. If γ is a Möbius transformation, then C_γ is bounded on A^2_n. We let $\mathcal{C}^n_\Gamma$ denote the C*-algebra generated by the operators $\{C_\gamma : \gamma \in \Gamma\}$ acting on A^2_n.

We collect the following computations together in a single Lemma, the analogue for A^2_n of the lemmas at the beginning of the previous section.

Lemma 3.5. *For $\gamma \in \Gamma$, with $a = \gamma^{-1}(0)$ and n fixed, let*

$$f(z) = \frac{(1-\overline{a}z)^n}{(1-|a|^2)^{n/2}}$$

Then $C_\gamma C_\gamma^* = M_f M_f^*$, and $M_\gamma C_\gamma = C_\gamma T$.

Proof. As in the Hardy space, for the first equation it suffices to prove that

$$\langle C_\gamma C_\gamma^* k_w, k_z\rangle = \langle T_f T_f^* k_w, k_z\rangle \tag{3.7}$$

for all z, w in $\mathbb{D}$. This is the same as the identity

$$\left(\frac{1}{1-\gamma(z)\overline{\gamma(w)}}\right)^n = \frac{(1-\overline{a}z)^n(1-a\overline{w})^n}{(1-|a|^2)^n(1-z\overline{w})^n}. \tag{3.8}$$

which is just (3.3) raised to the power n. As before, the relation $M_\gamma C_\gamma = C_\gamma T$ follows immediately from the definitions. □

The proof of Theorem 3.8 below follows the same lines as that for the Hardy space, except that it requires more work to show that $\mathcal{C}^n_\Gamma$ contains T. We first prove that $\mathcal{C}^n_\Gamma$ is irreducible.

Proposition 3.6. *The C^*-algebra $\mathcal{C}_\Gamma^n$ is irreducible.*

Proof. We claim that the operator $T^n = M_{z^n}$ lies in $\mathcal{C}_\Gamma^n$; we first show how irreducibility follows from this and then prove the claim.

By [**14**, Theorem 14], the only reducing subspaces of T^n are direct sums of subspaces of the form

$$X_k = \overline{\text{span}}\{z^{k+mn} | m = 0, 1, 2, \dots\}$$

for $0 \le k \le n-1$. Thus, since $T^n \in \mathcal{C}_\Gamma^n$, these are the only possible reducing subspaces for $\mathcal{C}_\Gamma^n$. Suppose, then, that X is a nontrivial direct sum of distinct subspaces X_k (i.e. $X \neq A_n^2$). Observe that, for each k, either X contains X_k as a summand, or X is orthogonal to X_k. In the latter case (which we assume holds for some k), the k^{th} Taylor coefficient of every function in X vanishes.

It is now easy to see that X cannot be reducing (or even invariant) for $\mathcal{C}_\Gamma^n$. Indeed, if X does not contain X_0 as a summand, then every function $f \in X$ vanishes at the origin, but if $\gamma \in \Gamma$ does not fix the origin then $C_\gamma f \notin X$. On the other hand, if X contains the scalars, then consider the operator $F(\lambda) = (1 - \overline{\lambda} T)^n (1 - \lambda T^*)^n$, for $\lambda \in \Lambda$. Since T^* annihilates the scalars, we have $F(\lambda)1 = (1 - \overline{\lambda} z)^n = p(z)$. But since $|\lambda| = 1$, the k^{th} Taylor coefficient of p does not vanish, so $p \notin X$. Finally, as $F(\lambda)$ belongs to $\mathcal{C}_\Gamma^n$, it follows that X is not invariant for $\mathcal{C}_\Gamma^n$ and hence $\mathcal{C}_\Gamma^n$ is irreducible.

To prove that $T^n \in \mathcal{C}_\Gamma^n$, we argue along the lines of Proposition 3.4, though the situation is somewhat more complicated. Using the same reproducing kernel argument as in that Proposition, we see that the operators

(3.9)

$$F(\lambda) = (1 - \overline{\lambda} T)^n (1 - \lambda T^*)^n$$

(3.10)

$$= \sum_{j=0}^n \sum_{k=0}^n (-1)^{(j+k)} \binom{n}{j} \binom{n}{k} \overline{\lambda}^k \lambda^j T^k T^{*j}$$

(3.11)

$$= \sum_{d=0}^n \binom{n}{d}^2 T^d T^{*d} + 2\Re \sum_{m=1}^n (-1)^m \overline{\lambda}^m \sum_{m \le k \le n} \binom{n}{k} \binom{n}{k-m} T^k T^{*k-m}$$

(3.12)

$$= \sum_{d=0}^n \binom{n}{d}^2 T^d T^{*d} + \sum_{m=1}^n \overline{\lambda}^m E_m + \lambda^m E_m^*$$

lie in $\mathcal{C}_\Gamma^n$ for all λ in the limit set Λ of Γ. Here we have adopted the notation

$$E_m = (-1)^m \sum_{m\leq k\leq n} \binom{n}{k}\binom{n}{k-m} T^k T^{*k-m}$$

Note in particular that $E_n = (-1)^n T^n$. Forming differences as λ ranges over Λ shows that $\mathcal{C}_\Gamma^n$ contains all operators of the form

$$G(\lambda,\mu) = F(\lambda) - F(\mu) = \sum_{m=1}^n (\overline{\lambda}^m - \overline{\mu}^m)E_m + (\lambda^m - \mu^m)E_m^*$$

for all $\lambda, \mu \in \Lambda$. We wish to obtain T^n as a linear combination of the $G(\lambda,\mu)$; since E_n is a scalar multiple of T^n it suffices to show that there exist $2n$ pairs $(\lambda_j,\mu_j) \in \Lambda\times\Lambda$ and $2n$ scalars α_j such that

$$E_n = \sum_{j=1}^{2n} \alpha_j G(\lambda_j,\mu_j)$$

Let L be the $2n\times 2n$ matrix whose j^{th} column is

$$\begin{pmatrix} \overline{\lambda_j} - \overline{\mu_j} \\ \lambda_j - \mu_j \\ \overline{\lambda_j}^2 - \overline{\mu_j}^2 \\ \lambda_j^2 - \mu_j^2 \\ \vdots \\ \overline{\lambda_j}^n - \overline{\mu_j}^n \\ \lambda_j^n - \mu_j^n \end{pmatrix}$$

We must therefore solve the linear system

$$L\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{2n-1} \\ \alpha_{2n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}$$

for which it suffices to show that the matrix L is nonsingular for some choice of the λ_j and μ_j in Λ. To do this, we fix $2n+1$ distinct points $z_0, z_1, \dots z_{2n}$ in Λ and set $\lambda_j = z_0$ for all j and $\mu_j = z_j$ for $j = 1, \dots 2n$.

The matrix L then becomes the matrix whose j^{th} column is

$$\begin{pmatrix} \overline{z_0} - \overline{z_j} \\ z_0 - z_j \\ \overline{z_0}^2 - \overline{z_j}^2 \\ z_0^2 - {z_j}^2 \\ \vdots \\ \overline{z_0}^n - \overline{z_j}^n \\ z_0^n - {z_j}^n \end{pmatrix}$$

To prove that L is nonsingular, we prove that its rows are linearly independent. To this end, let c_j and d_j, $j = 1, \dots n$ be scalars such that for each $k = 1, \dots 2n$,

$$\sum_{j=1}^{n} c_j (z_0^j - z_k^j) + \sum_{j=1}^{n} d_j (\overline{z_0}^j - \overline{z_k}^j) = 0 \tag{3.13}$$

To see that all of the c_j and d_j must be 0, consider the harmonic polynomial

$$P(z, \overline{z}) = \sum_{j=1}^{n} c_j (z_0^j - z^j) + \sum_{j=1}^{n} d_j (\overline{z_0}^j - \overline{z}^j)$$

By (3.13), P has $2n+1$ distinct zeroes on the unit circle, namely the points $z_0, \dots z_{2n}$. But this means that the rational function

$$Q(z) = \sum_{j=1}^{n} c_j (z_0^j - z^j) + \sum_{j=1}^{n} d_j (\overline{z_0}^j - \frac{1}{z^j})$$

also has $2n+1$ zeroes on the circle. But since the degree of Q is at most $2n$, it follows that Q must be identically zero, and hence all the c_j and d_j are 0.

Proposition 3.7. *$\mathcal{C}_\Gamma^n$ contains T.*

Proof. We have established that for each λ in the limit set, the operators

$$(1 - \overline{\lambda} T)^n (1 - \overline{\lambda} T)^{*n}$$

lie in $\mathcal{C}_\Gamma^n$. Now, since T is essentially normal, the difference

$$(1 - \overline{\lambda} T)^n (1 - \overline{\lambda} T)^{*n} - [(1 - \overline{\lambda} T)(1 - \overline{\lambda} T)^*]^n$$

is compact. Given two positive operators which differ by a compact, their (unique) positive n^{th} roots also differ by a compact. The positive n^{th} root of $F(\lambda)$ thus equal to

$$(1 - \overline{\lambda} T)(1 - \overline{\lambda} T)^* + K(\lambda)$$

for some compact operator $K(\lambda)$, and lies in $\mathcal{C}_\Gamma^n$. Forming linear combinations as in Proposition 3.4, we conclude that $T+K$ lies in $\mathcal{C}_\Gamma^n$ for some compact K. Now, as $\mathcal{C}_\Gamma^n$ is irreducible and contains a nonzero compact operator (namely, the self-commutator of $T+K$), it contains all the compacts, and hence T. □

We now prove the analogue of Theorem 3.1 for $\mathcal{C}_\Gamma^n$:

Theorem 3.8. *Let Γ be a non-elementary Fuchsian group, and let $\mathcal{C}_\Gamma^n$ denote the C^*-algebra generated by the set of composition operators on A^2 with symbols in Γ. Then there is an exact sequence*

$$0 \longrightarrow \mathcal{K} \xrightarrow{\ \iota\ } \mathcal{C}_\Gamma^n \xrightarrow{\ \pi\ } C(\mathbb{T}) \times \Gamma \longrightarrow 0 \tag{3.14}$$

Proof. The proof is similar to that of Theorem 3.1; we define the unitary operators U_γ in the same way (using the polar decomposition of C_γ), and check that the map that sends γ to $U_{\gamma^{-1}}$ is a unitary representation of Γ on A_n^2, modulo $\mathcal{K}$. The computation to check the covariance condition is essentially the same. As $\mathcal{C}_\Gamma^n$ is generated by the T and the unitaries U_γ, the quotient $\mathcal{C}_\Gamma^n/\mathcal{K}$ is generated by a copy of $C(\mathbb{T})$ (since $\mathbb{T}$ is the essential spectrum of T) and a representation of Γ which satisfy the covariance condition; the rest of the proof proceeds exactly as for H^2, up until the final step. At the final step in that proof, we have a Toeplitz operator T_f which is compact; this implies that $f=0$ in the H^2 case but not for the spaces A_n^2. However the symbol of a compact Toeplitz operator on A_n^2 must vanish at the boundary of $\mathbb{D}$, so we may still conclude that $f=0$ on $\mathbb{T}$ and the proof is complete.□

4. K-Homology of $C(\mathbb{T}) \times \Gamma$

4.1. *Ext* **classes of $\mathcal{C}_\Gamma$ and $\mathcal{C}_\Gamma^n$.** In this section we prove that the extensions of $C(\mathbb{T}) \times \Gamma$ determined by $\mathcal{C}_\Gamma$ and $\mathcal{C}_\Gamma^n$ determine the same element of the group $\mathrm{Ext}(C(\mathbb{T}) \times \Gamma)$, and we prove that $\mathcal{C}_\Gamma$ and $\mathcal{C}_\Gamma^n$ are isomorphic as C^*-algebras. (In general neither of these statements implies the other.) The reader is referred to [**3**, Chapter 15] for the basic definitions and theorems concerning extensions of C^*-algebras and the *Ext* group.

Theorem 4.1. *For each n, the extension of $C(\mathbb{T}) \times \Gamma$ determined by $\mathcal{C}_\Gamma^n$ is strongly equivalent to the extension determined by $\mathcal{C}_\Gamma$. Hence each of these extensions determines the same class in Ext$(C(\mathbb{T}) \times \Gamma)$ and the C^*-algebras $\mathcal{C}_\Gamma^n$ are mutually isomorphic.*

To prove the theorem we introduce an integral operator $V \in \mathcal{B}(A_{n+1}^2, A_n^2)$ and prove the following lemma, which describes the properties of V needed in the proof of the theorem.

Lemma 4.2. *Consider the integral operator defined on analytic polynomials $f(w)$ by*

$$(Vf)(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)}(1-|w|^2)^{-1/2}\,dA(w)$$

where dA is normalized Lebesgue area measure on $\mathbb{D}$. Then:

1. *For each $n \geq 0$, V extends to a bounded diagonal operator from A^2_{n+1} to A^2_n, and there exists a compact operator K_n such that $V + K_n$ is unitary.*
2. *If T_{n+1} and T_n denote multiplication by z on A^2_{n+1} and A^2_n respectively, then $VT_{n+1} - T_nV$ is compact.*
3. *If g is continuous on $\overline{\mathbb{D}}$ and analytic in $\mathbb{D}$, then the integral operator*

$$(V_{\overline{g}}f)(z) = \int_{\mathbb{D}} \frac{f(w)\overline{g(w)}}{1-\overline{w}z}(1-|w|^2)^{-1/2}dA(w)$$

is bounded from A^2_{n+1} to A^2_n, and is a compact perturbation of VM_g^*.

Proof. If $f(w) = w^k$, then the integral is absolutely convergent for each $z \in \mathbb{D}$, and we calculate

$$(Vf)(z) = \int_{\mathbb{D}} \frac{w^k}{1-\overline{w}z}(1-|w|^2)^{-1/2}dA(w) \tag{4.1}$$

$$= \sum_{j=0}^{\infty} \int_{\mathbb{D}} w^k \overline{w}^j z^j (1-|w|^2)^{-1/2}dA(w) \tag{4.2}$$

$$= \left(\int_{\mathbb{D}} |w|^{2k}(1-|w|^2)^{-1/2}dA(w) \right) z^k \tag{4.3}$$

$$= \alpha_k z^k \tag{4.4}$$

It is known that the sequence (α_k) defined by the above integral is asymptotically $(k+1)^{-1/2}$. Each space A^2_n has an orthonormal basis of the form $\beta_k z^k$, where the sequence β_k is asymptotically $(k+1)^{(n-1)/2}$. It follows that V intertwines orthonormal bases for A^2_{n+1} and A^2_n modulo a compact diagonal operator, which establishes the first statement. Moreover, the second statement also follows, by observing that the operators T_{n+1} and T_n are weighted shifts with weight sequences asymptotic to 1.

To prove the last statement, first suppose $g(w) = w$, and let $\beta_k z^k$ be an orthonormal basis for A^2_{n+1}. Then a direct computation shows that $V_{\overline{g}}$ is a weighted backward shift with weight sequence $\alpha_{j+1}\beta_j/\beta_{j+1}$, $j = 0, 1, \ldots$. Thus with respect to this basis, $V^*V_{\overline{z}}$ is a weighted shift with weight sequence $\alpha_{j+1}\beta_j/\alpha_j\beta_{j+1}$. Since $\lim_{j\to\infty}\alpha_{j+1}/\alpha_j = 1$, it follows that $V^*V_{\overline{z}}$ is a compact perturbation of M_z^*. A similar argument shows that $V^*V_{\overline{z}^m}$ is a compact perturbation of $M_{z^m}^*$. By linearity, the lemma then holds for polynomial g. Finally, if p_n is a sequence of polynomials such that M_{p_n} converges to M_g in the operator norm (which is equal to the supremum norm of the symbol), then M_{p_n} converges to M_g in the essential norm, and p_n converges to g uniformly, so that $V_{\overline{p_n}} \to V_{\overline{g}}$ in norm and the result follows.

Proof of Theorem 4.1

We will prove that the Busby maps of the extensions determined by $\mathcal{C}_\Gamma$ and $\mathcal{C}_\Gamma^n$ are strongly unitarily equivalent. By induction, it suffices to prove the equivalence between $\mathcal{C}_\Gamma^n$ and $\mathcal{C}_\Gamma^{n+1}$ for each n. We first establish some notation: for a function g in the disk algebra, we let M_g denote the multiplication operator with symbol g acting on A^2_{n+1}, and $\widetilde{M}_g$ the corresponding operator acting on A^2_n. Similarly, for the composition operators C_γ and the unitaries U_γ on A^2_{n+1}, a tilde denotes the corresponding operator on A^2_n. Now, since each of the algebras $\mathcal{C}_\Gamma^n, \mathcal{C}_\Gamma^{n+1}$ is generated by the operators M_z (resp. $\widetilde{M}_z$) and the unitaries U_γ (resp. $\widetilde{U}_\gamma$) on the respective Hilbert spaces, it suffices to produce a unitary U (or indeed a compact perturbation of a unitary) from A^2_{n+1} to A^2_n such that the operators $UM_z - \widetilde{M}_z U$ and $UU_\gamma - \widetilde{U}_\gamma U$ are compact for all $\gamma\in\Gamma$. In fact we will prove that the operator V of the previous lemma does the job.

To prove this, we will first calculate the operator $\widetilde{C}_\gamma V C_{\gamma^{-1}}$. Written as an integral operator,

$$(\widetilde{C}_\gamma V C_{\gamma^{-1}} f)(z) = \int_{\mathbb{D}} \frac{f(\gamma^{-1}(w))}{1-\overline{w}\gamma(z)}(1-|w|^2)^{-1/2}\,dA(w)$$

Applying the change of variables $w\to\gamma(w)$ to this integral, we obtain

$$(\widetilde{C}_\gamma V C_{\gamma^{-1}} f)(z) = \int_{\mathbb{D}} \frac{f(w)}{1-\overline{\gamma(w)}\gamma(z)}(1-|\gamma(w)|^2)^{-1/2}|\gamma'(w)|^2\,dA(w)$$

A little algebra shows that

$$1-|\gamma(w)|^2 = (1-|w|^2)|\gamma'(w)|$$

Furthermore, multiplying and dividing the identity (3.3) by $1-\overline{a}w$, we obtain

$$\frac{1}{1-\overline{\gamma(w)}\gamma(z)} = \frac{1-\overline{a}z}{(1-\overline{a}w)(1-\overline{w}z)}|\gamma'(w)|^{-1}$$

Thus the integral may be transformed into

$$(\widetilde{C}_\gamma V C_{\gamma^{-1}} f)(z) = \int_{\mathbb{D}} \frac{f(w)}{1-\overline{w}z}\frac{1-\overline{a}z}{1-\overline{a}w}|\gamma'(w)|^{1/2}(1-|w|^2)^{-1/2}\,dA(w)$$

Since γ' is analytic and non-vanishing in a neighborhood of the closed disk, there exists a function ψ in the disk algebra such that $|\psi|^2 = |\gamma'|^{1/2}$ (choose ψ to be a branch of $(\gamma')^{1/4}$). Thus we may rewrite this integral as

$$(\widetilde{C}_\gamma V C_{\gamma^{-1}} f)(z) = \int_{\mathbb{D}} \frac{f(w)}{1-\overline{w}z}\frac{1-\overline{a}z}{1-\overline{a}w}\psi(w)\overline{\psi(w)}(1-|w|^2)^{-1/2}\,dA(w) \tag{4.5}$$

$$= M_{1-\overline{a}z} V_{\overline{\psi}} M_{(1-\overline{a}z)^{-1}\psi(z)} \tag{4.6}$$

Now, applying the lemma to $V_{\overline{\psi}}$ and using the fact that V intertwines the multiplier algebras of A^2_{n+1} and A^2_n modulo the compacts, we have

$$\widetilde{C}_\gamma V C_{\gamma^{-1}} = M_{1-\overline{a}z} V_{\overline{\psi}} M_{(1-\overline{a}z)^{-1}\psi(z)} \tag{4.7}$$

$$= M_{1-\overline{a}z} V M^*_\psi M_\psi M_{(1-\overline{a}z)^{-1}} + K \tag{4.8}$$

$$= V M^*_\psi M_\psi + K \tag{4.9}$$

modulo the compacts. Multiplying on the right by $C_{\gamma^{-1}}$ and on the left by V^* we get

$$V^*\widetilde{C}_\gamma V = M^*_\psi M_\psi C_\gamma + K \tag{4.10}$$

Since $\psi^2(z) = (1-|a|^2)^{1/2}(1-\overline{a}z)^{-1}$, the calculations in the proof of 3.1 show that

$$(C_\gamma C^*_\gamma)^{-1/2} = M_{\psi^{n+1}} M^*_{\psi^{n+1}} \quad \text{and} \quad (\widetilde{C}_\gamma \widetilde{C}^*_\gamma)^{-1/2} = \widetilde{M}_{\psi^n} \widetilde{M}^*_{\psi^n}$$

Using the fact that V intertwines the C*-algebras generated by M_z and $\widetilde{M}_z$ modulo $\mathcal{K}$, and that these algebras are commutative modulo $\mathcal{K}$, we conclude after multiplying both sides of (4.10) on the left by $M_{\psi^n} M^*_{\psi^n}$ that

$$V^*\widetilde{U}_\gamma V = V^*\widetilde{M}_{\psi^n}\widetilde{M}^*_{\psi^n}\widetilde{C}_\gamma V + K = M_{\psi^{n+1}} M^*_{\psi^{n+1}} C_\gamma + K = U_\gamma + K$$

which completes the proof.

4.2. **Emerson's construction.** When Γ is co-compact, there is another extension of $C(\mathbb{T}) \times \Gamma$ which was constructed by Emerson [**5**], motivated by work of Kaminker and Putnam [**6**]; we will show that the *Ext*-class of $\mathcal{C}_\Gamma$ differs from the *Ext*-class of this extension.

The extension is constructed as follows: since Γ is co-compact, we may identify $\mathbb{T}$ with the Gromov boundary $\partial\Gamma$. Let f be a continuous function on $\partial\Gamma$, extend it arbitrarily to Γ by the Tietze extension theorem, denote this extended function by $\tilde{f}$. Let $e_x, x \in \Gamma$ be the standard orthonormal basis for $\ell^2(\Gamma)$, and let u_γ, $\gamma \in \Gamma$ denote the unitary operator of left translation on Γ. Define a map $\tau : C(\mathbb{T}) \times \Gamma \to \mathcal{Q}(\ell^2(\Gamma))$ by

$$\tau(f)e_x = \tilde{f}(x)e_x$$

and

$$\tau(\gamma)e_x = u_\gamma e_x = e_{\gamma x},$$

it can be shown that these expressions are well defined modulo the compact operators and determine a $*$-homomorphism from $C(\mathbb{T}) \times \Gamma$ to the Calkin algebra of $\ell^2(\Gamma)$, i.e. an extension of $C(\mathbb{T}) \times \Gamma$ by the compacts. Let π denote the quotient map $\pi : \mathcal{B}(\ell^2(\Gamma)) \to \mathcal{Q}(\ell^2(\Gamma))$, and consider the pull-back C^*-algebra E:

$$\begin{array}{ccc} E & \xrightarrow{\tilde{\tau}} & C(\mathbb{T}) \times \Gamma \\ \big\downarrow & & \big\downarrow{\scriptstyle \tau} \\ \mathcal{B}(\ell^2(\Gamma)) & \xrightarrow[\pi]{} & \mathcal{Q}(\ell^2(\Gamma)) \end{array} \tag{4.11}$$

We will show this extension is distinct from $\mathcal{C}_\Gamma$ by showing that it induces a different homomorphism from $K_1(C(\mathbb{T}) \times \Gamma)$ to $\mathbb{Z}$. Indeed, consider the class $[z]_1$ of the unitary $f(z) = z$ in $K_1(C(\mathbb{T}) \times \Gamma)$; the extension belonging to $\mathcal{C}_\Gamma$ sends this class to -1. On the other hand, for the extension τ we claim the function z lifts to a unitary in E; it follows that $\partial([z]_1) = 0$ in this case. To verify the claim, note that z lifts to a diagonal operator on $\ell^2(\Gamma)$, and if (x_n) is a subsequence in Γ tending to the boundary point $\lambda \in \mathbb{T}$, then $\tilde{f}(x_n) \to f(\lambda) = \lambda$. We may thus choose all the $\tilde{f}(x)$ nonzero, and by dividing $\tilde{f}(x)$ by its modulus, we may assume they are all unimodular. Thus z lifts to the unitary $diag(\tilde{f}(x)) \oplus z$ in E.

4.3. **Lott's construction.** In this section we relate the extension τ given by $\mathcal{C}_\Gamma$ to an extension recently constructed by J. Lott [**10**]. We first describe a construction of the extension σ_+ of [**10**]. Let $\mathcal{D}$ denote the Hilbert space of analytic functions on $\mathbb{D}$ with finite Dirichlet integral

$$D(f) = \int_{\mathbb{D}} |f'(z)|^2 dA(z)$$

equipped with the norm

$$\|f\|^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) = |f(0)|^2 + D(f)$$

If f is represented in $\mathbb{D}$ by the Taylor series $\sum_{n=0}^{\infty} a_n z^n$, then this norm is given by

$$\|f\|^2 = |a_0|^2 + \sum_{n=1}^{\infty} n|a_n|^2$$

The operator of multiplication by z on $\mathcal{D}$, denoted M_z, is a weighted shift with weight sequence asymptotic to 1, and hence is unitarily equivalent to a compact perturbation of the unilateral shift on H^2. It follows that there is a *-homomorphism $\rho : C(\mathbb{T}) \to \mathcal{Q}(\mathcal{D})$ with $\rho(z) = \pi(M_z)$. Now, by changing variables one checks that if γ is a Möbius transformation, $D(f \circ \gamma) = D(f)$. Let $\mathcal{D}_0$ denote the subspace of $\mathcal{D}$ consisting of those functions which vanish at the origin. It then follows from the definition of the norm in $\mathcal{D}$ that the operators

$$u_\gamma(f)(z) = f(\gamma(z)) - f(\gamma(0))$$

are unitary on $\mathcal{D}_0$, and form a unitary representation of Γ. We extend this representation to all of $\mathcal{D}$ by letting Γ act trivially on the scalars. Moreover, it is simple to verify (by noting that u_γ is a compact perturbation of the composition operator C_γ) that for all $\gamma \in \Gamma$,

$$u_\gamma M_z u_\gamma^* = M_{\gamma(z)}$$

modulo compact operators. Arguing as in the proof of Theorem 3.1, we conclude that the pair $(\rho(f), \pi(u_\gamma))$ determines an injective *-homomorphism from $C(\mathbb{T}) \times \Gamma$ to $\mathcal{Q}(\mathcal{D})$, which is unitarily equivalent to the Busby map σ_+ of **[10]**.

We may now state the main theorem of this subsection:

Theorem 4.3. *The Busby maps τ and σ_+ are unitarily equivalent.*

Proof. We first show that the Busby map $\tau : C(\mathbb{T}) \times \Gamma \to \mathcal{Q}(H^2)$ lifts to a completely positive map $\eta : C(\mathbb{T}) \times \Gamma \to \mathcal{B}(H^2)$. Define a unitary representation of Γ on $L^2(\mathbb{T})$ by

$$U(\gamma^{-1}) = M_{|\gamma'|^{1/2}} C_\gamma$$

Together with the usual representation of $C(\mathbb{T})$ as multiplication operators on L^2, we obtain a covariant representation of $(\Gamma, \mathbb{T})$ which in turn determines a representation $\rho : C(\mathbb{T}) \times \Gamma \to \mathcal{B}(L^2)$. Letting P denote the Riesz projection $P : L^2 \to H^2$, we next claim that the commutator $[\rho(a), P]$ is compact for all $a \in C(\mathbb{T}) \times \Gamma$, and so the pair (ρ, P) is an abstract Toeplitz extension of $C(\mathbb{T}) \times \Gamma$ by $\mathcal{K}$. To see this, it suffices to prove the compactness of the commutators

$$[M_f, P] \quad \text{and} \quad [U(\gamma), P].$$

Now, it is well known that $[M_f, P]$ is compact, and as $U(\gamma)$ has the form $M_g C_{\gamma^{-1}}$ it suffices to check that $[C_\gamma, P]$ is compact. It is easily checked that this latter commutator is rank one. Indeed, the range of P is invariant for C_γ, so $[C_\gamma, P] = PC_\gamma P^\perp$. If we expand $h \in L^2(\mathbb{T})$ in a Fourier series

$$h \sim \sum_{n\in\mathbb{Z}} \hat{h}(n)e^{in\theta}$$

then a short calculation shows that

$$PC_\gamma P^\perp h \sim (\sum_{n<0} \hat{h}(n)\overline{\gamma(0)}^{|n|}) \cdot 1$$

so $PC_\gamma P^\perp$ is rank one.

Thus, we have established that the completely positive map $\eta : C(\mathbb{T}) \times \Gamma \to \mathcal{B}(H^2)$ given by

$$\eta(a) = P\rho(a)P$$

is a homomorphism modulo compacts, and the calculations in the proof of Theorem 3.1 show that the map

$$\tau(a) = \pi(P\rho(a)P)$$

coincides with the Busby map associated to $\mathcal{C}_\Gamma$. Thus, η is a completely positive lifting of τ as claimed.

With the lifting η in hand, to prove the unitary equivalence of τ and σ_+ it will suffice to exhibit an operator $V : H^2 \to \mathcal{D}$ such that V is a compact perturbation of a unitary, and such that

$$\pi(V\eta(a)V^*) = \sigma_+(a)$$

for all $a \in C(\mathbb{T}) \times \Gamma$. Since the crossed product is generated by the function $f(z) = z$ and the formal symbols $[\gamma]$, it suffices to establish the above equality on these generators. In fact, we may use the operator V of Lemma 4.2. Since the map $z^n \to n^{-1/2}z^n$ is essentially unitary from H^2 to $\mathcal{D}$, the proofs of statements (1) and (2) of Lemma 4.2 are still valid. The conclusion of statement (3) holds provided the hypothesis on g is strengthened, by requiring that g by analytic in a neighborhood of $\overline{\mathbb{D}}$. Moreover, the arguments of Theorem 4.1 still apply, since the proof applies statement (3) of Lemma 4.2 only to the function ψ, which is indeed analytic across the boundary of $\mathbb{D}$. Thus, the arguments in the proof of Theorem 4.1 prove that V intertwines (modulo compacts) multiplication by z on H^2 and $\mathcal{H}$, and also intertwines (modulo compacts) C_γ acting on $\mathcal{H}$ with U_γ on H^2. Since u_γ is a compact perturbation of C_γ on $\mathcal{H}$, it follows that V intertwines U_γ and u_γ.

Now, the Busby map σ_+ takes the function z to the image of M_z in the Calkin algebra $\mathcal{Q}(\mathcal{H})$. Since $\eta(z) = M_z \in \mathcal{B}(H^2)$, the intertwining property of V (modulo compacts) may be written as

$$\pi(V\eta(z)V^*) = \sigma_+(z)$$

Similarly, since η applied to the formal symbol $[\gamma]$ (viewed as a generator of $C(\mathbb{T}) \times \Gamma$) is U_γ, the intertwining property for V with respect to U_γ and u_γ reads

$$\pi(V\eta([\gamma])V^*) = \sigma_+([\gamma])$$

Thus the equivalence of τ and σ_+ on generators is established, which proves the theorem. □

We conclude by observing that the covariant representation on L^2 described in the proof of the previous theorem gives rise to an equivariant KK_1-cycle for $C(\mathbb{T})$. Indeed, such a cycle consists of a triple (U, π, F) where (U, π) is a covariant representation on a Hilbert space $\mathcal{H}$ and F is a bounded operator on $\mathcal{H}$ such that the operators

$$F^2 - I,\ F - F^*,\ [U(\gamma), F],\ \text{and}\ [\pi(f), F]$$

are compact for all $\gamma \in \Gamma$ and $f \in C(\mathbb{T})$. The computations in the previous proof show that the triple $(U, \pi, 2P - I)$ satisfies all of these conditions, and essentially the same unitary equivalence argument shows that this cycle represents (up to a scalar multiple) the class of [**10**, Section 9.1].

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Department of Mathematics, University of Florida, Gainesville, Florida 32603

E-mail address: mjury@math.ufl.edu