Dilations and constrained algebras

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T - a bounded operator on Hilbert space H

$K \subset \mathbb{C}$ - compact

Definition (von Neumann, 1949)

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Say K is a **complete spectral set** for T if

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for all $n \times n$ matrices $[r_i j]$ of rational functions, and all n.

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Theorem (Arveson, 1972)

K is a complete spectral set for T if and only if T has a **normal ∂K-dilation**. *That is, there exist*

- *A Hilbert space $\mathcal{L} \supset \mathcal{H}$,*
- *a normal operator N on $\mathcal{L}$ with spectrum in ∂K*

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Say **rational dilation holds on** K if whenever K is spectral for T, then K is complete spectral for T.

Examples:

- rational dilation holds for K=disk (Sz.-Nagy 1952)
- rational dilation holds for K=annulus (Agler 1985)
- rational dilation **fails** for K=two-holed domain (Dritschel-McCullough 2005, Agler-Harland-Raphael 2008)

$R(K)$=uniform closure of rational functions in $C(K)$

all the above examples: $R(K)$ is a **hypo-Dirichlet algebra** on ∂K:

$$\overline{\text{Re } R(K)} \subset C_{\mathbb{R}}(\partial K)$$

has finite codimension.

When the codimension is 0, rational dilation always holds (Berger-Foais-Lebow)

The variety $z^2 = w^2$ in $\mathbb{D}^2$:

$$\begin{aligned}\mathcal{Y} = \{z^2 = w^2\} &= \{z = w\} \cup \{z = -w\} \\ &:= \mathcal{Y}_+ \cup \mathcal{Y}_-\end{aligned}$$

$\partial\mathcal{Y} = \partial\mathcal{Y}_+ \cup \partial\mathcal{Y}_-$ (a union of two circles)

$R(\mathcal{Y})$ is a hypo-Dirichlet algebra on $\partial\mathcal{Y}$ (codimension 1)

The variety

$$z^2 = \frac{w^2 - r^2}{1 - r^2 w^2}$$

in $\mathbb{D}^2$ is an annulus.

$\mathcal{Y}$ is the limiting case $r \to 0$

Rational dilation holds on the annulus, does it hold on $\mathcal{Y}$?

The Neil parabola:

$$\mathcal{N} = \{z^2 = w^3\} \subset \mathbb{D}^2$$

$R(\mathcal{N}) \cong \{f \in R(\mathbb{D}) : f'(0) = 0\}$

$R(\mathcal{N})$ is hypo-Dirichlet on $\partial\mathcal{N}$ (codimension 2)

Does rational dilation hold on $\mathcal{N}$?

Theorem (1)

Rational dilation holds on the variety $z^2 = w^2$.

Theorem (2)

Rational dilation fails on the variety $z^2 = w^3$.

Rational dilation on the annulus:

$$\mathbb{A}_q = \{0 < q < |z| < 1\}$$

Theorem (Agler 1985)

Rational dilation holds on $\mathbb{A}_q$.

Goal: give a proof of Agler's theorem that can be extended to cover the $z^2 = w^2$ variety.

$\mathbb{A}_q = \{0 < q < |z| < 1\}$

inner boundary $\partial_0 = \{|z| = q\}$, outer boundary $\partial_1 = \{|z| = 1\}$

Let $u \geq 0$, harmonic in $\mathbb{A}$. Then

$$u(z) = \int_{\partial \mathbb{A}} P(z, \zeta)\, d\mu(\zeta).$$

$u = \mathrm{Re} f \dots$ if and only if $\mu(\partial_0) = \mu(\partial_1)$

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Still true for $n \times n$ matrix-valued u; then μ is a positive matrix-valued measure (POVM)

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Let $\sigma(T) \subset \mathbb{A}_q$ and suppose $\mathbb{A}_q$ is spectral for T.

We want to prove

$$\|G(T)\| \leq 1$$

for all rational matrix-valued $G(z)$ with

$$\|G\|_{\mathbb{A}} = \sup_{\mathbb{A}} \|G(z)\| \leq 1,$$

First reduction:

$$F(z) = (I_n + G(z))(I_n - G(z))^{-1}$$

Then $\mathrm{Re}\ F(z) \succeq 0$ and

$$\|G(T)\| \leq 1 \quad \text{if and only if } \mathrm{Re}\ F(T) \succeq 0$$

Second reduction: we want Re $F(T) \succeq 0$.

$$\mathrm{Re} F_\mu(z) = \int_{\partial\mathbb{A}} P(z,\zeta)\, d\mu(\zeta)$$

with $\mu(\partial_0) = \mu(\partial_1) = I_n$

F_μ is a Choquet integral

$$F_\mu(z) = \int_{\mathcal{E}} F_\lambda(z)\, dE_\mu(\lambda)$$

So we don't need check all F_μ, just the extreme points F_λ...

Second reduction (continued)

F_μ is a Choquet integral

$$F_\mu(z) = \int_{\mathcal{E}_n} F_\lambda(z)\, dE_\mu(\lambda)$$

where $\mathcal{E}_n$ is the set of extreme points of the compact, convex set

$$\Gamma_n := \{n \times n \text{ matrix-valued } \mu, \mu(\partial_0) = \mu(\partial_1) = I_n\}$$

When $n = 1$, exteme points are pairs of point masses (one on ∂_0, one on ∂_1).

$$\Gamma_n := \{n \times n \text{ matrix-valued } \mu, \mu(\partial_0) = \mu(\partial_1) = I_n\}$$

Theorem (Ball,Guerra-Huamán 2013)

If λ is an extreme point of Γ_n then λ is finitely supported:

$$\lambda = \sum_{j=1}^{m} A_j \delta_{\alpha_j} + \sum_{j=1}^{m} B_j \delta_{\beta_j}, \quad \alpha_j \in \partial_0, \quad \beta_j \in \partial_1$$

with A_j, B_j positive, rank one, and $\sum A_j = \sum B_j = I$

Not all of these are extreme–but it will suffice to check all of these.

Third reduction (Naimark dilation trick):

Lemma

If $A_1, \ldots A_m$, $B_1, \ldots B_m$ are positive, rank one, $n \times n$ matrices and $\sum A_j = \sum B_j = I_{n\times n}$ then there exist

- *rank one projections $P_1, \ldots P_m$, $Q_1, \ldots Q_m$ with $\sum P_j = \sum Q_j = I_{m\times m}$, and*
- *an isometry $V : \mathbb{C}^n \to \mathbb{C}^m$*

such that

$$A_j = V^* P_j V, \quad B_j = V^* Q_j V$$

We may thus assume λ is a **spectral measure** on each boundary component

Final reduction (rotation trick):

$$\lambda = \sum_{j=1}^{n} P_j \delta_{\alpha_j} + \sum_{j=1}^{n} Q_j \delta_{\beta_j}$$

Assume $1 \notin \mathrm{supp}(\lambda)$.

By transforming $F = F_\lambda$ appropriately we may sweep all the mass on ∂_1 onto a single point:

Lemma

Let $G = (F - I)(F + I)^{-1}$ and $U = G(1)$ (unitary). Then for

$$F_{\widetilde{\lambda}} := (I + U^* G)(I - U^* G)^{-1}$$

we have $\widetilde{\lambda}|_{\partial_1} = I_n \delta_1$.

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- ~~Re $F_\lambda(T) \succeq 0$ for all finitely supported spectral λ~~
- Re $F_\lambda(T) \succeq 0$ for all finitely supported spectral λ with only $I_n\delta_1$ on outer boundary

Consider F_λ with

$$\lambda = \sum_{j=1}^{n} P_j \delta_{\alpha_j} + I\delta_1.$$

$$\lambda = \sum_{j=1}^{n} P_j \delta_{\zeta_j} + \sum_{j=1}^{n} P_j \delta_1$$

(recall $\sum P_j = I$).

$$\lambda = \sum_{j=1}^{n} P_j \cdot (\delta_{\alpha_j} + \delta_1)$$

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Put

$$\rho_j = \delta_{\alpha_j} + \delta_1$$

Poisson integral of ρ_j is real part of some holomorphic f_j. Since P_j are orthogonal,

$$F_\lambda = \text{diag } (f_1, \dots f_n)$$

But each Re $f_j(T) \succeq 0$ by the spectral set hypothesis, so Re $F(T) \succeq 0$. $\square$

The variety $z^2 = w^2$ in $\mathbb{D}^2$:

$$\begin{aligned} \mathcal{Y} = \{z^2 = w^2\} &= \{z = w\} \cup \{z = -w\} \\ &:= \mathcal{Y}_+ \cup \mathcal{Y}_- \end{aligned}$$

$\psi_\pm(t) = (t, \pm t)$ parametrize $\mathcal{Y}_\pm$ by $\mathbb{D}$

Holomorphic functions on $\mathcal{Y}$:

$F : \mathcal{Y} \to \mathbb{C}$

$$F_{\pm} = F|_{\mathcal{Y}_{\pm}}$$

are holomorphic on $\mathcal{Y}_{\pm}$ and $F_{+}(0) = F_{-}(0)$. Conversely:

Lemma

If $F_{\pm}$ are holomorphic in $\mathbb{D}$ and $F_{+}(0) = F_{-}(0) = 0$, then

$$\widetilde{F}(z, w) := (1 - (z - w))F_{+}(\frac{z + w}{2}) + (1 - (z + w))F_{-}(\frac{z - w}{2})$$

is holomrphic in $\mathbb{D}^2$ and restricts to $F_{\pm}$ on $\mathcal{Y}_{\pm}$.

To solve the Dirichlet problem on $\mathcal{Y}$, solve on each disk $\mathcal{Y}_\pm$ and require the solutions to match at 0. Thus:

Lemma

μ on $\partial\mathcal{Y} = \partial_+ \cup \partial_-$ is the boundary measure of Re F if and only if $\mu(\partial_+) = \mu(\partial_-)$.

Same for matrix valued F, μ.

Proof for the annulus works:

Theorem (1)

Rational dilation holds on the variety $z^2 = w^2$.

Note that by the spectral set condition,

$$\|S^2 - T^2\| \leq \|z^2 - w^2\|_{\mathcal{Y}} = 0.$$

The dilation of S, T will be a pair of commuting normal operators U, V with spectrum in $\partial\mathcal{Y}$: thus

$$U, V \text{ unitary, and } U^2 = V^2$$

Which S, T have $\mathcal{Y}$ as a spectral set?

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Theorem

Let S, T be commuting operators satisfying $S^2 = T^2$. Then S, T dilate to commuting unitaries with $U^2 = V^2$ if and only if

$$\|\gamma S + (1-\gamma)T\| \leq 1$$

for all $|\gamma - \frac{1}{2}| = \frac{1}{2}$.

A homomorphism

$$\pi : R(K) \to B(H)$$

is **contractive** if

$$\|\pi(f)\|_{B(H)} \leq \|f\|_\infty$$

for all $f \in R(K)$,
and **completely contractive** if

$$\|[\pi(F_{ij})]\|_{B(H)} \leq \|[F_{ij}\|_\infty$$

for all matrices $F \in M_n(R(K))$.

Theorem (2)

Rational dilation fails on the Neil parabola $\mathcal{N}$.

In fact: there is a representation

$$\pi : R(\mathcal{N}) \to M_{12\times 12}(\mathbb{C})$$

contractive but not 2-contractive.

Identify $R(\mathcal{N})$ with subalgebra of $\mathcal{A}(\mathbb{D})$:

$$R(\mathcal{N}) = \{f \in \mathcal{A}(\mathbb{D}) \ : \ f'(0) = 0\}.$$

Convexity approach again:

Let

$$\mathcal{P} = \left\{ \frac{1+f}{1-f} : f \in H^\infty(\mathcal{N}), f(0) = 1 \right\}$$

The set $\mathcal{P}$ is compact and convex, let

$$\mathcal{E} = \text{extreme points of } \mathcal{P}$$

For each $f \in \mathcal{P}$ we have a Choquet integral

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The functions

$$\{\phi : \frac{1+\phi}{1-\phi} \in \mathcal{E}\}$$

are called test functions for $H^\infty(\mathcal{N})$.

Representing the unit ball of $H^\infty(\mathcal{N})$:

Rearringing the Choquet integral we have

$$1 - f(z)f(w)^* = \int_{\mathcal{E}} 1 - \phi_t(z)\phi_t(w)^* \, d\mu_{zw}(t)$$

where μ is a positive measure valued kernel on $\mathbb{D} \times \mathbb{D}$. To proceed we need this more explicitly...

$\mathbb{D}^* =$ one-point compactification of $\mathbb{D}$

$\psi_\lambda(z) = z^2 \frac{\lambda - z}{1-\overline{\lambda}z}, \quad \psi_*(z) = z^2$ (test functions)

Theorem (Pickering)

If f lies in the unit ball of $R(\mathcal{N})$ and is smooth across the boundary then

$$1 - f(z)f(w)^* = \int_{\mathbb{D}^*} (1 - \psi_t(z)\psi_t(w)^*)\, d\mu_{z,w}(t)$$

where μ is a positive $M(\mathbb{D}^*)$-valued kernel.

The test functions can be pushed back down to $\mathcal{N}$, we get

$$\phi_\lambda(z, w) = z\frac{\lambda z - w}{z - \lambda^* w}, \phi_*(z) = z$$

Pickering also shows no (closed) subcollection of test functions suffices.

Corollary: a pair of commuting, contractive, invertible matrices X, Y with $X^3 = Y^2$ give a contractive representation of $\mathcal{A}(\mathcal{N})$ if and only if

$$X(\lambda X - Y)(X - \lambda^* Y)^{-1}$$

is contractive for all $\lambda \in \mathbb{D}$.

Loosely, moving to matrix valued F, if F every $n \times n$ matrix function also has a representation

$$1 - F(z)F(w)^* = \int_{\mathbb{D}^*} (1 - \psi_t(z)\psi_t(w)^*) \, d\mu_{z,w}(t)$$

then one can pass (nice) representaitons inside the integral to conclude contractive implies completely contractive.

Conversely, let $\mathfrak{F}$ be a finite set and form the closed, convex cone

$$C_{\mathfrak{F}} = \left\{ H(z,w) = \int_{\mathbb{D}^*} (1 - \psi_t(z)\psi_t(w)^*)\, d\mu_{z,w}(t) \right\}$$

where $\mu_{z,w}$ are matrix-valued measures.
If for some F we have

$$I - F(z)F(w)^* \notin C_{\mathfrak{F}}$$

then we can separate $I - F(z)F(w)^*$ from $C_{\mathfrak{F}}$ with a positive functional, apply GNS to get a representation that is contractive but NOT completely contractive.

Key step: if F is a matrix inner function in $M_2 \otimes \mathcal{A}(\mathcal{N})$, an integral representation for F imposes constraints on its zeroes...

$F(z) = z^2\Phi(z)$, Φ rational, inner, degree 2,

Theorem

If F is representable as

$$I - F(z)F(w)^* = \int_{\mathbb{D}^*} (1 - \psi_t(z)\psi_t(w)^*)\, d\mu_{z,w}(t)$$

for z, w in a large finite set $\mathfrak{F}$, then either

$$\Phi \simeq \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 0 & \phi_1\phi_2 \end{pmatrix}$$

One can write down Φ for which this fails, so rational dilation fails on $\mathcal{N}$. $\square$