# NEVANLINNA-PICK INTERPOLATION ON DISTINGUISHED VARIETIES IN THE BIDISK

MICHAEL T. JURY[1], GREG KNESE[2], AND SCOTT MCCULLOUGH[3]

Abstract. This article treats Nevanlinna-Pick interpolation in the setting of a special class of algebraic curves called distinguished varieties. An interpolation theorem, along with additional operator theoretic results, is given using a family of reproducing kernels naturally associated to the variety. The examples of the Neil parabola and doubly connected domains are discussed.

## 1. Introduction

Versions of the Nevanlinna-Pick interpolation theorem stated in terms of the positivity of a family of Pick matrices have a long tradition beginning with Abrahamse's interpolation Theorem on multiply connected domains [Ab]. The list [AD, Ag, AM99, AM00, AM02, AM03, B, BB, BBtH, BCV, BTV, CLW, DH, DPRS, FF, H, MS, Pa, P, S, R1, R2] is just a sample of now classic papers and newer results in this direction related to the present paper. Here we consider Pick interpolation on a distinguished variety. For general facts about distinguished varieties we have borrowed heavily from [AMS, AM05, Kn1, Kn2].

The classical Nevanlinna-Pick interpolation theorem says that, given points $z_1, \dots z_n$ in the unit disk $\mathbb{D} \subset \mathbb{C}$ and points $\lambda_1, \dots \lambda_n$ in $\mathbb{D}$, there exists a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with $f(z_i) = \lambda_i$ for each $i$ if and only if the $n \times n$ *Pick matrix*

$$\left( \frac{1 - \lambda_i \overline{\lambda_j}}{1 - z_i \overline{z_j}} \right)_{i,j=1}^{n} \tag{1}$$

is positive semidefinite. From the modern point of view (that is, the point of view of "function-theoretic operator theory"), one interprets this condition as checking the positivity of $(1 - \lambda_i \overline{\lambda_j})$ against the *Szegő*

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*kernel* $k(z_i, z_j) := (1 - z_i\overline{z_j})^{-1}$ on the interpolation nodes $z_1, \dots z_n$. The Szegő kernel $k(z, w)$ is the reproducing kernel for the Hardy space $H^2(\mathbb{D})$, which is a Hilbert space of holomorphic functions on $\mathbb{D}$. Thus, this point of view repackages the constraint $f : \mathbb{D} \to \mathbb{D}$ (that is, $|f|$ is bounded by 1 in $\mathbb{D}$) as the condition that $f$ multiply $H^2(\mathbb{D})$ into itself contractively. See [AM02] for an extended exposition of this point of view.

One strength of this perspective is that gives a natural framework in which to pose and solve interpolation problems on other domains (in $\mathbb{C}$ or $\mathbb{C}^n$). For example, Abrahamse [Ab] considered the analogous interpolation problem in a $g$-holed planar domain $R$. His theorem says, for a canonical family of reproducing kernels $k_t$ on $R$ naturally parametrized by the $g$-torus $\mathbb{T}^g$, that given $z_j$'s in $R$ and $\lambda_j$'s in $\mathbb{D}$, there exists a holomorphic interpolating function $f : R \to \mathbb{D}$ if and only if each of the family of Pick matrices

$$[(1 - \lambda_i\overline{\lambda_j})k_t(z_i, z_j)]_{i,j=1}^n \tag{2}$$

is positive semidefinite. More recently, Davidson, Paulsen, Raghupathi and Singh [DPRS] considered the original Pick problem on the disk, but with the additional constraint that $f'(0) = 0$. Again, positivity against a particular family of kernels is necessary and sufficient. A very general approach to interpolation via kernel families may be found in [JKM].

On the other hand, *distinguished varieties* have recently emerged as a new venue in which to investigate function-theoretic operator theory [AMS,AM05,Kn1,Kn2]. By one definition, a distinguished variety is an algebraic variety $\mathcal{Z} \subset \mathbb{C}^2$ with the property that if $(z, w) \in \mathcal{Z}$, then $|z|$ and $|w|$ are either both less than 1, both greater than 1, or both equal to 1. (We shall usually consider only the intersection $\mathcal{V} = \mathcal{Z} \cap \mathbb{D}^2$, and by abuse of language refer to this as a distinguished variety as well.) The simplest non-trivial example is the *Neil parabola* $\mathcal{N} = \{(z, w) : z^3 = w^2\}$. Just as von Neumann's inequality and the Sz.-Nagy-Foias dilation theorem establish a connection between contractive operators on Hilbert space and function theory in the unit disk $\mathbb{D}$, the function theory on a distinguished variety is linked to the study of pairs of commuting contractions $S, T$ on Hilbert space which obey a polynomial relation $p(S, T) = 0$.

The purpose of this paper is to prove a Pick interpolation theorem for bounded analytic functions on distinguished varieties. The main theorem identifies a canonical collection of kernels over the variety. Each kernel corresponds to commuting pair of isometries with finite rank defect and Taylor spectrum in the closure of the variety. It turns out, in fact, that two of the examples mentioned above (constrained interpolation in the disk, and interpolation on multiply connected domains) can be recast as interpolation problems on distinguished varieties. In addition to the interpolation theorem, this article contains information about such pairs of isometries, including a geometric picture of the (minimal) unitary extension with spectrum in the intersection of the boundary of the bidisk and the closure of the distinguished variety. We also prove a polynomial approximation theorem for bounded analytic functions on varieties. The remainder of this introductory section provides some background material and states the main results more fully.

1.1. **Distinguished Varieties.** A subset $\mathcal{V}$ of $\mathbb{D}^2$ is a *distinguished variety* if there exists a square free polynomial $p \in \mathbb{C}[z, w]$ such that

$$\mathcal{V} = \mathcal{Z}_p \cap \mathbb{D}^2$$

and

$$\mathcal{Z}_p \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2.$$

Here $\mathcal{Z}_p$ is the zero set of $p$;

$$\mathbb{D}^2 = \{(z, w) \in \mathbb{C} : |z|,\ |w| < 1\}$$

is the bidisk; $\mathbb{T}^2 = \{(z, w) \in \mathbb{C}^2 : |z| = 1 = |w|\}$ is the distinguished boundary of the bidisk; and

$$\mathbb{E}^2 = \{(z, w) \in \mathbb{C}^2 : |z|,\ |w| > 1\}$$

denotes the *exterior bidisk*. An alternate, but equivalent, definition of distinguished variety is an algebraic set in the bidisk that exits through the distinguished boundary $\mathbb{T}^2 = (\partial\mathbb{D})^2$ (see [AM05], [Kn1]).

The polynomial $p$ can be chosen to have the symmetry

$$p(z, w) = z^n w^m \overline{p(\frac{1}{\overline{z}}, \frac{1}{\overline{w}})} \tag{3}$$

and hence $\mathcal{Z}_p$ is invariant under the map $(z, w) \to (\frac{1}{\overline{z}}, \frac{1}{\overline{w}})$ [Kn1].

Write $(n, m)$ for the bidegree of $p$; i.e. $p$ has degree $n$ in $z$ and $m$ in $w$. By a fundamental result of Agler and McCarthy [AM05], $\mathcal{V}$ admits a determinantal representation

$$\mathcal{V} = \{(z, w) \in \mathbb{D}^2 : \det(wI_m - \Phi(z)) = 0\} \tag{4}$$

where $\Phi$ is an $m \times m$ rational matrix function which is analytic on the closed disk $\overline{\mathbb{D}}$ and unitary on $\partial\mathbb{D}$.

Given such a $\Phi$ there exist (row) vector-valued polynomials

$$Q(z, w) = (q_1(z, w) \ldots q_m(z, w)), \quad P(z, w) = (p_1(z, w) \ldots p_n(z, w)),$$

such that

$$\Phi(z)^* Q(z,w)^* = \overline{w} Q(z,w)^* \tag{5}$$

and

$$(1 - w\overline{\eta}) Q(z,w) Q(\zeta,\eta)^* = (1 - z\overline{\zeta}) P(z,w) P(\zeta,\eta)^* \tag{6}$$

for all $(z,w)$ and $(\zeta,\eta)$ in $\mathcal{V}$. In fact, such polynomials can be chosen such that $Q$ has degree at most $m-1$ in $w$ and $P$ has degree at most $n-1$ in $z$ [Kn1]. Moreover, every pair $P, Q$ of polynomial vector functions satisfying (6) on $\mathcal{V}$ arises in this way; i.e., there is a rational $\Phi$ such that (4) and (5) hold. This last assertion is a consequence of Lemma 4.3 below.

A pair $P$ and $Q$ satisfying (6) determines the positive definite kernel $K : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$,

$$K(z,w) = \frac{Q(z,w)Q(\zeta,\eta)^*}{1 - z\overline{\zeta}} = \frac{P(z,w)P(\zeta,\eta)^*}{1 - w\overline{\eta}}$$

on $\mathcal{V} \times \mathcal{V}$.

It is natural to generalize the construction of the kernel $K$ , allowing for matrix-valued $P$ and $Q$. Let $M_{\alpha,\beta}$ denote the set of $\alpha \times \beta$ matrices with entries from $\mathbb{C}$.

**Definition 1.1.** A *rank* $\alpha$ *admissible pair*, synonymously a $\alpha$-*admissible pair*, is a pair $(P, Q)$ of matrix polynomials $P, Q$ in two variables such that,

(i) $Q(z,w)$ is $M_{\alpha,m\alpha}$-valued and $P(z,w)$ is $M_{\alpha,n\alpha}$-valued;
(ii) both $Q(z,w)$ and $P(z,w)$ have rank $\alpha$ (that is, full rank) at some point of each irreducible component of $\mathcal{Z}_p$; and
(iii) for $(z,w), (\zeta,\eta) \in \mathcal{Z}_p$,

$$\frac{Q(z,w)Q(\zeta,\eta)^*}{(1 - z\overline{\zeta})} = \frac{P(z,w)P(\zeta,\eta)^*}{(1 - w\overline{\eta})}. \tag{7}$$

A pair $(P, Q)$ is *admissible* if it is $\alpha$-admissible for some $\alpha$.

**Remark 1.2.** Though both sides of (7) define meromorphic functions on $\mathbb{C}^2 \times \mathbb{C}^2$, we emphasize that the equality is assumed to hold only on $\mathcal{Z}_p \times \mathcal{Z}_p$.

Let $\tilde{\mathcal{V}}$ denote the intersection of $\mathcal{Z}_p$ with the exterior bidisk $\mathbb{E}^2$. In view of equation (3), $\tilde{\mathcal{V}} = \{(\frac{1}{z}, \frac{1}{w}) : (z,w) \in \mathcal{V}\}$.

**Definition 1.3.** A rank $\alpha$ admissible pair $(P, Q)$ determines (positive semidefinite) kernels $K : \mathcal{V} \times \mathcal{V} \to M_\alpha$ and $\tilde{K} : \tilde{\mathcal{V}} \times \tilde{\mathcal{V}} \to M_\alpha$,

$$\begin{aligned} K((z,w),(\zeta,\eta)) =& \frac{Q(z,w)Q(\zeta,\eta)^*}{1 - z\overline{\zeta}} = \frac{P(z,w)P(\zeta,\eta)^*}{1 - w\overline{\eta}} \\ \tilde{K}((z,w),(\zeta,\eta)) =& \frac{Q(z,w)Q(\zeta,\eta)^*}{z\overline{\zeta} - 1} = \frac{P(z,w)P(\zeta,\eta)^*}{w\overline{\eta} - 1}, \end{aligned}$$

which we call an *admissible pair* of kernels.

**Remark 1.4.** The kernel $\tilde{K}$ can also be written as

$$\tilde{K}((z,w),(\zeta,\eta)) = \frac{1}{z\overline{\zeta}} \frac{Q(z,w)Q(\zeta,\eta)^*}{1 - \frac{1}{z}\frac{1}{\overline{\zeta}}} = \frac{1}{w\overline{\eta}} \frac{P(z,w)P(\zeta,\eta)^*}{1 - \frac{1}{w}\frac{1}{\overline{\eta}}}.$$

The corresponding reproducing Hilbert spaces are denoted $H^2(K)$ and $H^2(\tilde{K})$. The operators $S = M_z$ and $T = M_w$ of multiplication by $z$ and $w$ respectively are contractions on $H^2(K)$. Likewise the operators $\tilde{S}$ and $\tilde{T}$ of multiplication by $\frac{1}{z}$ and $\frac{1}{w}$ respectively are contractions on $H^2(\tilde{K})$. We will see later that they are in fact isometric.

These pairs of operators $(S, T)$ play the role of *bundle shifts* [AD, S] over $\mathcal{V}$, terminology which is explained by Theorems 1.9 and Theorem 1.10 below. In addition to these theorems, a main result of this paper is a version of Pick interpolation for $\mathcal{V}$ - Theorem 1.6. In the next subsection we describe these results more fully.

## 1.2. Main Results.

The main result of this paper is Theorem 1.6, the Pick interpolation theorem on distinguished varieties. It is based on three subsidiary results, each of some interest in its own right.

**Definition 1.5.** *Let $\mathcal{V}$ be a variety. A function $f : \mathcal{V} \to \mathbb{C}$ is* holomorphic *at a point $(z, w) \in \mathcal{V}$ if there exists an open set $U \subset \mathbb{C}^2$ containing $(z, w)$ and a holomorphic function $F : U \to \mathbb{C}$ such that $F$ agrees with $f$ on $\mathcal{V} \cap U$.*

If $(z, w)$ is a smooth point of $\mathcal{V}$, then the (holomorphic) implicit function theorem tells us there is a local coordinate in a neighborhood $O$ of $(z, w)$ in $\mathcal{V}$ making this neighborhood into a Riemann surface; a function is holomorphic at $(z, w)$ by the above definition if and only if it is holomorphic as a function of the local coordinate. The importance of the definition, therefore, is that it allows us to make sense of holomorphicity near singular points of $\mathcal{V}$. See [T02, Chapter 3].

We let $H^\infty(\mathcal{V})$ denote the set of functions that are bounded on $\mathcal{V}$ and holomorphic at every point of $\mathcal{V}$. It is a Banach algebra under the supremum norm

$$\|f\|_\infty := \sup_{(z,w)\in\mathcal{V}} |f(z,w)|. \tag{8}$$

(That $H^\infty(\mathcal{V})$ is complete follows from the non-trivial fact that a locally uniform limit of functions holomorphic on $\mathcal{V}$ is holomorphic on $\mathcal{V}$, see [T02] Theorem 11.2.5.)

**Theorem 1.6.** *Let $(z_1,w_1),\dots,(z_n,w_n)\in\mathcal{V}$ and $\lambda_1,\dots,\lambda_n\in\mathbb{D}$ be given. There exists an $f\in H^\infty(\mathcal{V})$ such that $\|f\|_\infty\le 1$ and*

$$f(z_j,w_j)=\lambda_j \text{ for each } j=1,\dots n$$

*if and only if*

$$(1-\lambda_\ell\overline{\lambda_j})K((z_j,w_j),(\zeta_\ell,\eta_\ell))$$

*is positive semi-definite for every admissible kernel $K$.*

The proof of this theorem splits into two parts. We first introduce an auxiliary algebra $H^\infty_{\mathcal{K}}(\mathcal{V})$ of bounded analytic functions on $\mathcal{V}$ (but equipped with a norm $\|\cdot\|_{\mathcal{V}}$ that is defined differently than the supremum norm) and prove that the condition of Theorem 1.6 is necessary and sufficient for interpolation in this algebra. The second part of the proof consists in showing that in fact $H^\infty_{\mathcal{K}}(\mathcal{V})=H^\infty(\mathcal{V})$ isometrically (*a priori* it is obvious only that $H^\infty_{\mathcal{K}}(\mathcal{V})\subseteq H^\infty(\mathcal{V})$ contractively). This second part in turn splits in two: we first prove a unitary dilation theorem for algebraic pairs of isometries; as a corollary we find that $H^\infty_{\mathcal{K}}$ contains all polynomials $q$, and $\|q\|_{\mathcal{V}}=\|q\|_\infty$. Finally we prove a polynomial approximation result for $H^\infty(\mathcal{V})$ (Theorem 1.12) which allows us to extend this isometry to all of $H^\infty(\mathcal{V})$.

1.2.1. *Interpolation in $H^\infty_{\mathcal{K}}(\mathcal{V})$.* We now define the auxiliary algebra $H^\infty_{\mathcal{K}}(\mathcal{V})$ in which we will interpolate. Let $W$ be a set, $n$ a positive integer, and let $M_n$ denote the $n\times n$ matrices with entries from $\mathbb{C}$. A function $f: W\times W\to M_n$ is positive semi-definite if, for each finite subset $F\subset W$, the $n|F|\times n|F|$ matrix

$$\big(f(u,v)\big)_{u,v\in F}$$

is positive semi-definite; we write $f(u,v)\succeq 0$.

**Definition 1.7.** Say a function $f:\mathcal{V}\to\mathbb{C}$ belongs to $H^\infty_{\mathcal{K}}(\mathcal{V})$ if there exists a real number $M>0$ such that

$$(M^2-f(z,w)f(\zeta,\eta))K((z,w),(\zeta,\eta))\succeq 0 \tag{9}$$

for all admissible kernels $K$. The norm $\|f\|_{\mathcal{V}}$ is defined to be the infimum of all $M$ such that (9) holds for all admissible $K$.

In other words, $\|f\|_{\mathcal{V}} \leq M$ if and only if for each admissible $K$, the operator $M_f$ of multiplication by $f$ is bounded on $H^2(K)$, with operator norm at most $M$; and thus

$$\|f\|_{\mathcal{V}} = \sup_K \|M_f\|_{B(H^2(K))}. \tag{10}$$

We are finally ready to state the interpolation theorem.

**Theorem 1.8.** *Let $(z_1, w_1), \dots, (z_n, w_n) \in \mathcal{V}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{D}$ be given. There exists an $f \in H^\infty_{\mathcal{K}}(\mathcal{V})$ such that $\|f\|_{\mathcal{V}} \leq 1$ and*

$$f(z_j, w_j) = \lambda_j \text{ for each } j = 1, \dots n$$

*if and only if*

$$(1 - \lambda_\ell \overline{\lambda_j}) K((z_j, w_j), (\zeta_\ell, \eta_\ell)) \succeq 0$$

*for every admissible kernel $K$.*

Theorem 1.8 is proved in Section 3.

The problem of extending a function defined on $\mathcal{V}$ to all of the bidisk is treated in [AM03].

1.2.2. *Commuting isometries with spectrum in $\overline{\mathcal{V}}$.*

**Theorem 1.9.** *Let $K$ be an admissible kernel and write*

$$S = M_z, \quad T = M_w$$

*for the coordinate multiplication operators on $H^2(K)$. Then:*

(i) *$S$ and $T$ are pure commuting isometries,*
(ii) *$p(S, T) = 0$, and*
(iii) *the Taylor spectrum of $(S, T)$ is contained in the closure of $\mathcal{V}$ in $\overline{\mathbb{D}^2}$.*

Theorem 1.9, with additional detail, is proved in Section 5. See [AKM] for more on pairs $(S, T)$ satisfying (i) and (ii) above for some polynomial $p$.

1.2.3. *Dilating commuting isometries with spectrum in $\overline{\mathcal{V}}$.*

**Theorem 1.10.** *Let $K$ be an admissible kernel. Then the isometries $(S, T)$ of the previous theorem admit a commuting unitary extension $(X, Y)$ such that $p(X, Y) = 0$ and the joint spectral measure for $(X, Y)$ lies in $\partial\mathcal{V}$.*

*In fact,*

$$X = \begin{pmatrix} S & \Sigma \\ 0 & \tilde{S}^* \end{pmatrix}$$

$$Y = \begin{pmatrix} T & \Gamma \\ 0 & \tilde{T}^* \end{pmatrix},$$

*for a canonical pair of operators* $\Sigma, \Gamma : H^2(\tilde{K}) \to H^2(K)$ *and where* $S, T, \tilde{S}, \tilde{T}$ *are defined immediately after Remark 1.4.*

This theorem is proved in Section 5.1.

If $f \in H^\infty_{\mathcal{K}}(\mathcal{V})$, it is always the case that $\|f\|_\infty \leq \|f\|_{\mathcal{V}}$. If $q$ is a polynomial, then the operator $M_q$ on $H^2(K)$ is equal to $q(S,T)$. The following corollary is then immediate from (10) and Theorem 1.10:

**Corollary 1.11.** *Every polynomial* $q(z,w)$ *belongs to* $H^\infty_{\mathcal{K}}(\mathcal{V})$, *and* $\|q\|_{\mathcal{V}} = \|q\|_\infty$.

To extend this result from polynomials to all of $H^\infty(\mathcal{V})$, we have the following approximation theorem and its corollary.

**Theorem 1.12.** *For each* $f \in H^\infty(\mathcal{V})$, *there exists a sequence of polynomials* $p_n$ *such that* $p_n \to f$ *uniformly on compact subsets of* $\mathcal{V}$ *and* $\|p_n\|_\infty \leq \|f\|_\infty$ *for all* $n$.

**Corollary 1.13.** $H^\infty_{\mathcal{K}}(\mathcal{V}) = H^\infty(\mathcal{V})$ *isometrically.*

Theorem 1.6 is then immediate from Theorem 1.8 and Corollary 1.13.

1.3. **Readers Guide.** The remainder of the paper is organized as follows. Section 2 considers a number of examples of the Pick interpolation theorem. The result for $H^\infty_{\mathcal{K}}$, Theorem 1.8, is proved in Section 3 by verifying the collection of admissible kernels satisfies the conditions of the abstract interpolation theorem from [JKM]. Section 4 develops facts about admissible pairs, admissible kernels and determinantal representations needed for the sequel. Section 5 treats the pairs of operators that play the role of bundle shifts on $\mathcal{V}$. It contains proofs of Theorems 1.9 and 1.10. Theorem 1.12 is proved in Section 6. The proof here is function-theoretic and independent of the other sections. Finally, in Section 7 we prove Corollary 1.13, and in particular show that elements of $H^\infty_{\mathcal{K}}(\mathcal{V})$ are actually holomorphic on $\mathcal{V}$.

## 2. Examples

It is instructive to consider a couple of examples related to existing Pick interpolation theorems.

2.1. **The Neil Parabola.** The Neil parabola $\mathcal{N}$ is the distinguished variety determined by the polynomial $p(z,w) = z^3 - w^2$. Note the singularity at the origin. It is easily checked that the pair

$$Q(z,w) = \begin{pmatrix} 1 & w \end{pmatrix}$$
$$P(z,w) = \begin{pmatrix} 1 & z & z^2 \end{pmatrix}$$

is an admissible pair with corresponding reproducing kernel

$$\frac{1+w\overline{\eta}}{1-z\overline{\zeta}} = K((z,w),(\zeta,\eta)) = \frac{1+z\overline{\zeta}+z^2\overline{\zeta}^2}{1-w\overline{\eta}}.$$

Again, we emphasize that the equalities hold on $\mathcal{V}\times\mathcal{V}$. Similarly, the pair

$$Q(z,w) = \begin{pmatrix} z & w \end{pmatrix}$$
$$P(z,w) = \begin{pmatrix} w & z & z^2 \end{pmatrix}$$

is admissible. The corresponding kernel vanishes at $((0,0),(0,0))$.

The next proposition identifies the algebra $H^\infty_{\mathcal{K}}(\mathcal{N})$ with $\mathcal{A} := \{f \in H^\infty(\mathbb{D}) : f'(0) = 0\}$.

**Proposition 2.1.** *The parametrization $\Psi : \mathbb{D} \to \mathcal{N}$ of the Neil parabola given by $\Psi(t) = (t^2, t^3) = (z(t), w(t))$ induces an isometric isomorphism $\Psi^* : H^\infty_{\mathcal{K}}(\mathcal{N}) \to \mathcal{A}$ defined by $\Psi^* f(t) = f(\Psi(t))$.*

*Proof.* Since every $f \in H^\infty_{\mathcal{K}}(\mathcal{N})$ is bounded, the mapping $\Psi^*$ does map into $H^\infty(\mathbb{D})$. On the other hand, $(f\circ\Psi)'(0) = Df(\Psi(0))\cdot D\Psi(0) = 0$. Thus $\Psi^*$ maps into $\mathcal{A}$. Moreover, if $f \in H^\infty_{\mathcal{K}}(\mathcal{N})$, then

$$\|f\|_{\mathcal{N}} \geq \|f\|_\infty \geq \|\Psi^* f\|$$

and thus $\Psi^*$ is contractive. It remains to show that $\Psi^*$ is isometric and onto. We first prove this for polynomials; the general case will follow by approximation.

So, let $p \in \mathcal{A}$ be a polynomial. Then $p(t) = p_0 + \sum_{j=2}^N p_j t^j$. Each $j \geq 2$ can be written as $j = 2\alpha + 3\beta$ for non-negative integers $\alpha, \beta$. (Of course this representation is not unique; we fix one such for each $j$.) Let $q(z,w) = p_0 + \sum_{\alpha,\beta} p_{2\alpha+3\beta} z^\alpha w^\beta$. Thus $(\Psi^* q)(t) = p(t)$. If we write $\|p\|$ for the supremum of $|p|$ over the unit disk, it follows that $\|q\|_\infty = \|p\|$. By Corollary 1.11, $\|q\|_{\mathcal{N}} = \|p\|$.

Now suppose that $g \in \mathcal{A}$ is arbitrary. There exists a sequence of polynomials $p_n \in \mathcal{A}$ such that $\|p_n\| \leq \|g\|$ and $p_n$ converges pointwise to $g$ (e.g., one can take $p_n$ to be the $n^{th}$ Cesáro mean of the Fourier series of $g$). For each $n$ there is a polynomial $q_n$ such that $\Psi^*(q_n) = p_n$ and $\|q_n\|_{\mathcal{N}} = \|p_n\|$. Then $(q_n)$ is Cauchy (as $(p_n)$ is) and hence converges pointwise and in norm to a function $f \in H^\infty_{\mathcal{K}}$ (see Proposition 7.1) satisfying $\Psi^* f = g$.By construction, $\|f\|_{\mathcal{N}} = \|g\|$. Thus $\Psi^*$ is isometric and onto, and the proof is complete.

Note that the argument in this proof directly establishes Corollary 1.13 for the Neil parabola. □

Interpolation on the Neil parabola is thus equivalent to the constrained Pick interpolation in the algebra $\mathcal{A}$, which was considered by [DPRS] and also by [BBtH, P]. In [DPRS] it is shown that for the Pick interpolation problem in $\mathcal{A}$, it suffices to consider the family of kernels

$$k_{a,b}(s,t) = (a+bs)\overline{(a+bt)} + \frac{s^2\overline{t^2}}{1-s\bar{t}} \tag{11}$$

over all complex numbers $a, b$ with $|a|^2 + |b|^2 = 1$. On the other hand, Proposition 2.1 and Theorem 1.6 say that it suffices to consider the family of kernels obtained by pulling back admissible kernels on $\mathcal{N}$ under the map $\Psi$, that is, all kernels on $\mathbb{D} \times \mathbb{D}$ of the form

$$K(s,t) = \frac{Q(s^2,s^3)Q(t^2,t^3)^*}{1-s^2\overline{t^2}} = \frac{P(s^2,s^3)P(t^2,t^3)^*}{1-s^3\overline{t^3}} \tag{12}$$

where $P, Q$ are an admissible pair. It is not hard to see that the latter family contains the former (up to conjugacy). Indeed, given $a, b$, define

$$\begin{aligned} Q(z,w) &= (az+bw \quad \bar{b}z^2 - \bar{a}zw), \\ P(z,w) &= (az+bw \quad \bar{b}zw - \bar{a}w^2 \quad z^2). \end{aligned}$$

This is an admissible pair, and if $K$ is then defined by (12), then

$$K(s,t) = s^2 k_{a,b}(s,t)\bar{t}^2.$$

In forthcoming work we show that fairly generally for distinguished varieties that it is necessary to consider only the scalar kernels and that moreover in the case of the Neil parabola we then obtain the result of [DPRS].

2.2. **The annulus.** Fix $0 < r < 1$ and consider the annulus $\mathbb{A} := \{z \in \mathbb{C} : r < |z| < 1\}$. Let $A(\mathbb{A})$ denote the algebra of functions analytic in $\mathbb{A}$ and continuous in $\overline{\mathbb{A}}$, and $H^\infty(\mathbb{A})$ the algebra of all bounded analytic functions in $\mathbb{A}$, both equipped with the uniform norm. By a theorem of Pavlov and Fedorov [PF], there exists an algebraic pair of inner functions $\theta_0, \theta_1$ in $A(\mathbb{A})$ such that the polynomials in $\theta_0, \theta_1$ are dense in $A(\mathbb{A})$. Here "algebraic" means that there is a polynomial $p$ such that $p(\theta_0, \theta_1) = 0$. Since the $\theta_j$ are inner, this $p$ must define a distinguished variety, which we denote $\mathcal{V}$. Now, if $g \in H^\infty(\mathbb{A})$, it is not hard to prove that there exists a sequence of functions $f_n \in A(\mathbb{A})$ such that $f_n \to g$ pointwise and $\|f_n\| \le \|g\|$ for all $n$. Evidently these $f_n$ may be taken to be polynomials in $\theta_0, \theta_1$. Imitating the proof of Proposition 2.1 (with $\theta_0, \theta_1$ in place of the inner functions $t^2, t^3$) gives

**Proposition 2.2.** *The spaces $H^\infty_{\mathcal{K}}(\mathcal{V})$ and $H^\infty(\mathbb{A})$ are isometrically isomorphic.*

Of course, this can also be obtained as a special case of Corollary 1.13. The prototype of Pick interpolation theorems involving a family of kernels is the interpolation Theorem of Abrahamse on multiply connected domains [Ab]. (See also [S65].) Thus while Theorem 1.6 (actually its refinement to just scalar admissible kernels) gives an interpolation condition for the annulus, we do not know if it is the same as that of Abrahamse. More generally, the results of Raghupathi [R2] and Davidson-Hamilton [DH] can be applied to obtain an interpolation theorem on distinguished varieties, by first lifting to the desingularizing Riemann surface and then uniformizing this surface as the quotient of the disk by a Fuchsian group. The interpolation theorem of [R2] does indeed reduce to that of Abrahamse in the case of the annulus. So, more generally, the question is open whether or not Theorem 1.6 gives the same conditions as those of [R2].

## 3. Kernel structures

Our proof of Theorem 1.8 relies on an application of the main result from [JKM] which, for the reader's convenience, we outline in this section. This approach to Pick interpolation complements that in [Ag].

Let $M_n$ denote the set of $n \times n$ matrices with entries from $\mathbb{C}$. An $M_n$-valued kernel on a set $X$ is a positive semi-definite function $k : X \times X \to M_n$. Of course, our admissible kernels on $\mathcal{V}$ are examples. In what follows we use $z^*$ to denote the complex conjugate of a complex number $z$ (anticipating that the results are valid for matrix-valued functions).

**Definition 3.1.** *Fix a set $X$ and a sequence $\mathcal{K} = (\mathcal{K}_n)$ where each $\mathcal{K}_n$ is a set of $M_n$-valued kernels on $X$.*

*The collection $\mathcal{K}$ is an* Agler interpolation family of kernels *provided:*

(i) *if $k_1 \in \mathcal{K}_{n_1}$ and $k_2 \in \mathcal{K}_{n_2}$, then $k_1 \oplus k_2 \in \mathcal{K}_{n_1+n_2}$;*

(ii) *if $k \in \mathcal{K}_n$, $z \in X$, $\gamma \in \mathbb{C}^n$, and $\gamma^* k(z,z)\gamma \neq 0$, then there exists an $N$, a kernel $\kappa \in \mathcal{K}_N$, and a function $G : X \to M_{n,N}$ such that*

$$k'(x,y) := k(x,y) - \frac{k(x,z)\gamma\gamma^* k(z,y)}{\gamma^* k(z,z)\gamma} = G(x)\kappa(x,y)G(y)^*;$$

(iii) *for each finite $F \subset X$ and for each $f : F \to \mathbb{C}$, there is a $\rho > 0$ such that, for each $k \in \mathcal{K}$,*

$$F \times F \ni (x,y) \mapsto (\rho^2 - f(x)f(y)^*)k(x,y)$$

*is a positive semi-definite kernel on $F$; and*

(iv) *for each $x \in X$ there is a $k \in \mathcal{K}$ such that $k(x,x)$ is nonzero (and positive semi-definite).*

**Theorem 3.2.** *[JKM, Theorem 1.3] Suppose $\mathcal{K}$ is an Agler interpolation family of kernels on $X$. Further suppose $Y \subset X$ is finite and $g : Y \to \mathbb{C}$ and $\rho \geq 0$. If for each $k \in \mathcal{K}$ the kernel*

$$Y \times Y \ni (x,y) \mapsto (\rho^2 - g(x)g(y)^*)k(x,y) \tag{13}$$

*is positive semi-definite, then there exists $f : X \to \mathbb{C}$ such that $f|_Y = g$ and for each $k \in \mathcal{K}$ the kernel*

$$X \times X \ni (x,y) \mapsto (\rho^2 - f(x)f(y)^*)k(x,y) \tag{14}$$

*is positive semi-definite.*

That the collection $\mathcal{A}$ of admissible kernels on $\mathcal{V}$ is an Agler-interpolation family is proved in the following subsections. Theorem 1.8 then follows.

The direct sum of admissible kernels is evidently admissible and a result of [Kn2] says that there is an admissible kernel $K$ on $\mathcal{V}$ such that $K((z,w),(z,w))$ does not vanish on $\mathcal{V}$. (See Theorem 11.3 of [Kn2].) Hence $\mathcal{A}$ satisfies conditions (i) and (iv).

## 3.1. **Compression Stability.**

That $\mathcal{A}$ satisfies condition (ii) of Definition 3.1 is proved in this subsection.

We begin with the observation that condition (ii) in the definition of an admissible pair (Definition 1.1) implies a stronger version of itself; this will be needed in the proofs of both Theorem 1.8 and Lemma 4.1 below.

**Lemma 3.3.** *If $Q(z,w)$ is an $M_{\alpha,m\alpha}$-valued polynomial and $Q$ has full rank at some point of each irreducible component of $\mathcal{Z}_p$, then $Q$ has full rank at all but finitely many points of $\mathcal{Z}_p$.*

*Proof.* It suffices to assume that $\mathcal{Z}_p$ is irreducible. Choose $(z_0,w_0) \in \mathcal{Z}_p$ such that $Q(z_0,w_0)$ has full rank. In particular, there are $\alpha$ columns of $Q$ which form a linearly independent set when evaluated at $(z_0,w_0)$. Choose such a set of columns and let $R(z,w)$ denote the resulting $\alpha \times \alpha$ matrix-valued polynomial. The polynomial $q = \det(R)$ does not vanish at the point $(z_0,w_0)$, so the variety $U = \mathcal{Z}_p \cap \mathcal{Z}_q$ is a proper sub-variety of $\mathcal{Z}_p$. By Bezout's theorem, $U$ is a finite set, and by construction $Q$ has full rank off $U$. □

Now, we may begin the proof. Fix an admissible kernel $K$ corresponding to the rank $\alpha$ admissible pair $(P,Q)$, a point $u = (x,y) \in \mathcal{V}$, a vector $\gamma \in \mathbb{C}^\alpha$, and assuming $K(u,u)\gamma \neq 0$, let

$$K'((z,w),(\zeta,\eta)) = \|K(u,u)^{\frac{1}{2}}\gamma\|^2 K((z,w),(\zeta,\eta)) - K((z,w),u)\gamma\gamma^* K(u,(\zeta,\eta)).$$

Write

$$\delta = Q(x,y)^*\gamma \in \mathbb{C}^{m\alpha}$$

and note we are assuming $\delta \neq 0$. From the definition of $K'$ we have

$$K'((z,w),(\zeta,\eta)) = \left(\frac{Q(z,w)Q(\zeta,\eta)^*}{1-z\bar{\zeta}}\right)\left(\frac{\|\delta\|^2}{1-|x|^2}\right) - \frac{Q(z,w)\delta\delta^* Q(\zeta,\eta)}{(1-z\overline{x})(1-x\overline{\zeta})}.$$

Let $\varphi_x$ denote the Mobiüs map

$$\varphi_x(z) = \frac{z-x}{1-z\overline{x}};$$

and recall the identity

$$\frac{(1-z\bar{\zeta})(1-|x|^2)}{(1-z\bar{x})(1-x\bar{\zeta})} = 1 - \varphi_x(z)\overline{\varphi_x(\zeta)}.$$

Then we may rewrite $K'$ as

(15)
$$K'((z,w);(\zeta,\eta)) = \frac{Q(z,w)\left(\|\delta\|^2 - \delta\delta^* + \varphi_x(z)\overline{\varphi_x(\zeta)}\delta\delta^*\right)Q(\zeta,\eta)^*}{(1-z\bar{\zeta})(1-|x|^2)}.$$

Now let $P_\delta$ denote the orthogonal projection of $\mathbb{C}^{m\alpha}$ onto the one-dimensional subspace spanned by $\delta$. Define

$$B(z) := P_\delta^\perp + \varphi_x(z)P_\delta.$$

Observe that $B$ is analytic, contraction-valued in the unit disk, and unitary on the unit circle. We then have

$$\|\delta\|^2 - \delta\delta^* + \varphi_x(z)\overline{\varphi_x(\zeta)}\delta\delta^* = \|\delta\|^2 B(z)B(\zeta)^*. \tag{16}$$

Finally, define

$$Q'(z,w) := (1-w\overline{y})\frac{\|\delta\|}{\sqrt{1-|x|^2}}(1-z\overline{x})Q(z,w)B(z) \tag{17}$$

Combining (15), (16), and (17), we get

$$(1-w\overline{y})(1-y\overline{\eta})(1-z\overline{x})(1-x\overline{\zeta})K' = \frac{Q'(z,w)Q'(\zeta,\eta)^*}{1-z\overline{\zeta}}.$$

An analogous construction produces a $P'$ so that

$$(1-w\overline{y})(1-y\overline{\eta})(1-z\overline{x})(1-x\overline{\zeta})K' = \frac{P'(z,w)P'(\zeta,\eta)^*}{1-w\overline{\eta}}.$$

From the construction of $Q'$ it is of rank at most $m\alpha$ and is a polynomial in $(z,w)$; and similarly for $P'$. Also, the rank of $Q'$ is the same as the rank of $Q$, except at the point $(x,y)$. Hence by Lemma 3.3, $Q'$ has full rank at some point on each irreducible subvariety of $\mathcal{Z}_p$ (indeed, at all but finitely many points). Thus

$$\kappa((z,w),(\zeta,\eta)) = \frac{Q'(z,w)Q'(\zeta,\eta)^*}{1-z\overline{\zeta}} = \frac{P'(z,w)P'(\zeta,\eta)^*}{1-w\overline{\eta}}$$

is an admissible kernel and

$$K' = \frac{1}{(1-z\overline{x})(1-w\overline{y})}\kappa\frac{1}{(1-x\overline{\zeta})(1-y\overline{\eta})}.$$

3.2. **Existence of interpolants.** Finally, we verify condition (iii) of Definition 3.1. Fix a finite set

$$X = \{(z_1,w_1),\dots(z_N,w_N)\} \subset \mathcal{V}$$

and let $f : X \to \mathbb{C}$ be given. Since polynomials separate points of $\mathbb{C}^2$, for each $j = 1,\dots N$, we can choose a polynomial $p_j(z,w)$ such that $p_j(z_k,w_k) = \delta_{jk}$ for each $k = 1,\dots N$. Now define

$$q(z,w) = \sum_{j=1}^{N} f(z_j,w_j)p_j(z,w)$$

Then $q|_X = f$. Fix an admissible kernel $K$ and let $S = M_z, T = M_w$. As noted in the remark following Definition 1.3, $S$ and $T$ are contractions, so by applying Ando's inequality to $q$, we find that $\|q(S,T)\| = \|M_q\|_{B(H^2(K))}$ is bounded by the supremum of $|q|$ over $\mathbb{D}^2$, and in particular is bounded independently of $K$. It then follows from equation (10) that $q \in H^\infty_{\mathcal{K}}(\mathcal{V})$, and Definition 3.1(iii) holds with $\rho = \sup_{(z,w)\in\mathbb{D}^2}|q(z,w)|$.

**Remark 3.4.** In the verification of Definition 3.1(iii), the bound on $\|q\|_{\mathcal{V}}$ coming from the above argument is quite crude; if we appeal instead to Corollary 1.11 we obtain the sharp value $\|q\|_{\mathcal{V}} = \|q\|_\infty$. We have arranged the proof this way only to make the proof of the interpolation theorem for $H^\infty_{\mathcal{K}}$ independent of the later dilation results.

## 4. Admissible Pairs

Recall that the variety $\mathcal{Z}_p$ is the zero set of a square free polynomial $p(z,w)$, of bidegree $(n,m)$, as in the introduction and $\mathcal{V} = \mathcal{Z}_p \cap \mathbb{D}^2$.

**Lemma 4.1.** *If $P, Q$ is an $\alpha$-admissible pair, then for all but finitely many $\lambda \in \mathbb{D}$ there exist distinct points $\mu_1, \dots, \mu_m \in \mathbb{D} \setminus \{0\}$ such that $(\lambda, \mu_j) \in \mathcal{V}$ and the $m\alpha \times m\alpha$ matrices*

$$\begin{pmatrix} Q(\lambda,\mu_1)^* & \dots & Q(\lambda,\mu_m)^* \end{pmatrix}, \qquad \begin{pmatrix} Q(\frac{1}{\bar\lambda},\frac{1}{\bar\mu_1})^* & \dots & Q(\frac{1}{\bar\lambda},\frac{1}{\bar\mu_m})^* \end{pmatrix}$$

*are invertible.*

The proof of the lemma in turn uses a very modest generalization of the construction found in [Kn1] with essentially identical proof involving the lurking isometry. See Lemma 4.3 below. We record some preliminary observations.

**Lemma 4.2.** *For each $z \in \mathbb{C}$ the set $\{w : (z,w) \in \mathcal{Z}_p\}$ has cardinality at most $m$. Conversely, for all but at most finitely many $z$, this set has $m$ elements.*

*Proof.* For fixed $z$, the polynomial $q(w) = p(z,w)$ has degree less than or equal to $m$ in $w$. Thus, $q(w)$ has at most $m$ zeros or is identically zero. However, $q$ can't be identically zero since $\mathcal{Z}_p \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$.

Conversely, let us prove there are only finitely many $z$ at which $\{w : (z,w) \in \mathcal{Z}_p\} \neq m$. First, note that there are only finitely many $z$ at which $q(w) = p(z,w)$ has degree strictly less than $m$ (because the leading coefficient of $q$ is a polynomial in $z$). Next, using the fact that $p$ is square free, it is not hard to show that $\partial p/\partial w$ and $p$ have no common factors. Therefore, $p$ and $\partial p/\partial w$ have finitely many common zeros. Thus, as long as we avoid the finitely many $z$ at which $q(w) = p(z,w)$ has degree less than $m$ and the finitely many $z$ corresponding to (the first coordinate of) common roots of $p$ and $\partial p/\partial w$, $q$ will be a polynomial of degree $m$ with no multiple roots. This proves the claim. □

Let $U$ be a unitary matrix of size $(m+n)\alpha$ written in block $2 \times 2$ form as

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with respect to the orthogonal sum $\mathbb{C}^{m\alpha} \oplus \mathbb{C}^{n\alpha}$. To $U$ we associate the linear fractional, or transfer, function

$$\Phi(z) = A^* + C^*(I - zD^*)^{-1} z B^*.$$

Very standard calculations show that $\Phi$ is a rational matrix function with poles outside $\mathbb{D}$; is contractive-valued in $\mathbb{D}$; and unitary-valued (except for possibly finitely many points) on the boundary of $\mathbb{D}$. Indeed, by Cramer's rule the entries of $(I - zD^*)^{-1}$ are rational functions of $z$, and since $\|D\| \le 1$ they are analytic in $\mathbb{D}$. Moreover, a short calculation using the fact that $U$ is unitary shows that

$$I - \Phi(z)^*\Phi(z) = (1-|z|^2)B(I-\overline{z}D)^{-1}(I-zD^*)^{-1}B^* \tag{18}$$

which is a positive matrix when $|z|<1$ (showing $\Phi(z)$ is contractive in $\mathbb{D}$) and 0 when $|z|=1$ (showing that $\Phi$ is unitary on the circle, except at the finitely many points where $(I-zD^*)$ may fail to be invertible).

**Lemma 4.3.** *If $(P,Q)$ is an $\alpha$-admissible pair, then there exists an $m\alpha \times m\alpha$ matrix-valued transfer function such that*

$$\Phi(z)^*Q(z,w)^* = w^*Q(z,w)^*$$

*for all $(z,w) \in V$ and for all except finitely many $(z,w) \in \mathcal{Z}_p$.*

*Moreover, $p$ divides* $\det(\Phi(z) - wI)$.

*Proof.* First, we rearrange the relation

$$\frac{Q(z,w)Q(\zeta,\eta)^*}{1-z\overline{\zeta}} = \frac{P(z,w)P(\zeta,\eta)^*}{1-w\overline{\eta}}$$

for $(z,w),(\zeta,\eta) \in \mathcal{V}$ into the isometric form

$$QQ^* + z\overline{\zeta}PP^* = w\overline{\eta}QQ^* + PP^*. \tag{19}$$

Let

$$\begin{aligned}\mathcal{E} =&\text{span}\{\begin{pmatrix} Q(\zeta,\eta)^*\gamma \\ \overline{\zeta}P(\zeta,\eta)^*\gamma\end{pmatrix} : (\zeta,\eta)\in\mathcal{V}, \gamma\in\mathbb{C}^\alpha\}\\ \mathcal{F} =&\text{span}\{\begin{pmatrix} \overline{\eta}Q(\zeta,\eta)^*\gamma \\ P(\zeta,\eta)^*\gamma\end{pmatrix} : (\zeta,\eta)\in\mathcal{V}, \gamma\in\mathbb{C}^\alpha\}.\end{aligned}$$

Equation (19) implies that the mapping

$$U\begin{pmatrix} Q(\zeta,\eta)^*\gamma \\ \overline{\zeta}P(\zeta,\eta)^*\gamma\end{pmatrix} = \begin{pmatrix} \overline{\eta}Q(\zeta,\eta)^*\gamma \\ P(\zeta,\eta)^*\gamma\end{pmatrix} \tag{20}$$

determines a well defined isometry $U : \mathcal{E} \to \mathcal{F}$. Since we are in finite dimensions, it follows that $U$ extends to a unitary (which we still denote $U$) on $\mathbb{C}^{(m+n)\alpha}$. Write

$$U = \begin{pmatrix} A & B \\ C & D\end{pmatrix} : \begin{matrix}\mathbb{C}^{m\alpha}\\ \oplus \\ \mathbb{C}^{n\alpha}\end{matrix} \to \begin{matrix}\mathbb{C}^{m\alpha}\\ \oplus \\ \mathbb{C}^{n\alpha}\end{matrix}$$

and define

$$\Phi(\zeta)^* = A + \overline{\zeta}B(I-\overline{\zeta}D)^{-1}C.$$

By definition of $U$,

$$\begin{aligned} AQ(\zeta,\eta)^*\gamma + B\overline{\zeta}P(\zeta,\eta)^*\gamma &= \overline{\eta}Q(\zeta,\eta)^*\gamma \\ CQ(\zeta,\eta)^*\gamma + D\overline{\zeta}P(\zeta,\eta)^*\gamma &= P(\zeta,\eta)^*\gamma\end{aligned}$$

which implies

$$(I-\overline{\zeta}D)^{-1}CQ(\zeta,\eta)^*\gamma = P(\zeta,\eta)^*\gamma$$

and therefore

$$AQ(\zeta,\eta)^*\gamma+\overline{\zeta}B(I-\overline{\zeta}D)^{-1}CQ(\zeta,\eta)^*\gamma = \Phi(\zeta)^*Q(\zeta,\eta)^*\gamma = \overline{\eta}Q(\zeta,\eta)^*\gamma.$$

for all $\gamma \in \mathbb{C}^\alpha$. So, we indeed have

$$\Phi(z)^*Q(z,w)^* = w^*Q(z,w)^*$$

everywhere on $\mathcal{Z}_p$, excluding the finitely many points $(z,w)$ where $\Phi(z)$ may not be defined. □

Note in passing that we also have:

$$\begin{aligned}\frac{1}{\overline{\zeta}}P^* &= [D + C(\overline{\eta}-A)^{-1}B]P^* \\ \frac{1}{\overline{\eta}}Q(\zeta,\eta)^* &= [A^* + C^*(\overline{\zeta}-D^*)^{-1}B^*]Q^* \\ \overline{\zeta}P^* &= [D^* + \overline{\eta}(I-\overline{\eta}A^*)^{-1}C^*]P^*.\end{aligned}$$

*Proof of Lemma 4.1.* Choose $\lambda \in \mathbb{D}$ such that

(i) there are $m$ *distinct* points $\mu_1,\ldots,\mu_m \in \mathbb{D}$ such that $(\lambda,\mu_j) \in \mathcal{V}$, and
(ii) for each $j = 1,\ldots m$, the matrices $Q(\lambda,\mu_j)$ and $Q(\frac{1}{\overline{\lambda}},\frac{1}{\overline{\mu}_j})$ have rank $\alpha$.

(This is possible by combining Lemmas 4.2 and 3.3.) Let $\Phi$ denote the rational function from Lemma 4.3. For $\gamma \in \mathbb{C}^\alpha$,

$$\Phi(\lambda)^*Q(\lambda,\mu_j)^*\gamma = \overline{\mu}_jQ(\lambda,\mu_j)^*\gamma.$$

Thus $Q(\lambda,\mu_j)^*\gamma$ is in the eigenspace of $\Phi(\lambda)^*$ corresponding to the eigenvalue $\overline{\mu}_j$, and for each $j$ this eigenspace has dimension $\alpha$. It follows that the matrix $(Q(\lambda,\mu_j)^*)_j$ has rank $m\alpha$; similarly for $(Q(\frac{1}{\overline{\lambda}},\frac{1}{\overline{\mu}_j})^*)_j$. □

## 5. Bundle Shifts

Recall that $(S,T) = (M_z,M_w)$ on $H^2(K)$ and $(\tilde{S},\tilde{T}) = (M_{1/z},M_{1/w})$ on $H^2(\tilde{K})$. See Definition 1.3.

**Theorem 5.1.** *Suppose $(P,Q)$ is a rank $\alpha$-admissible pair. The operators $S,\tilde{S},T,\tilde{T}$ are all pure isometries. Moreover, the Taylor spectra of $(S,T)$ and $(\tilde{S}^*,\tilde{T}^*)$ are contained in $\mathcal{Z}_p \cap \overline{\mathbb{D}^2}$.*

That $p(S,T)=0$ is immediate, since $p(S,T)=M_p$ and

$$M_p^* K_{(z,w)} = \overline{p(z,w)} K_{(z,w)} = 0$$

for all $(z,w) \in \mathcal{V}$. For the claim about the Taylor spectrum, note that since both $S$ and $T$ are contractions, $\sigma_{Tay}(S,T) \subset \overline{\mathbb{D}^2}$. Further, by Taylor's mapping theorem we have

$$\{0\} = \sigma(p(S,T)) = p(\sigma_{Tay}(S,T))$$

so $\sigma_{Tay}(S,T) \subset \mathcal{Z}_p \cap \overline{\mathbb{D}^2}$.

Let $\check{p}(z,w) = \overline{p(\overline{z},\overline{w})}$. Thus $\check{p}$ is obtained from $p$ by conjugating the coefficients of $p$. A computation like that above gives

$$\begin{aligned} \check{p}(\tilde{S},\tilde{T})^* \tilde{K}_{(\zeta,\eta)} =& \overline{\check{p}(\frac{1}{\zeta},\frac{1}{\eta})} \tilde{K}_{(\zeta,\eta)} \\ =& p(\frac{1}{\overline{\zeta}},\frac{1}{\overline{\eta}}) \tilde{K}_{(\zeta,\eta)} \\ =& 0. \end{aligned}$$

Thus $\check{p}(\tilde{S},\tilde{T}) = 0$. Equivalently, $p(\tilde{S}^*,\tilde{T}^*) = 0$.

Our proof that $S$ is an isometry begins with a lemma. From the general theory of reproducing kernel Hilbert spaces (see [AM02] for instance), if $K$ is an $\alpha \times \alpha$ matrix-valued kernel, $g$ is a $\mathbb{C}^\alpha$-valued function on $\mathcal{V}$, and

$$K((z,w),(\zeta,\eta)) - g(z,w)g(\zeta,\eta)^*$$

is positive semi-definite, then $g \in H^2(K)$, so in particular

$$\langle g, K_{(\zeta,\eta)}\gamma \rangle = \langle g(\zeta,\eta), \gamma \rangle_{\mathbb{C}^\alpha}.$$

Given the admissible pair $(P,Q)$, let

$$K_Q = Q(z,w)Q(\zeta,\eta)^*.$$

It is immediate that $K - K_Q$ is a positive kernel. Hence, if $\gamma$ is a unit vector, then

$$K - Q(z,w)\gamma\gamma^* Q(\zeta,\eta)^*$$

is also a positive kernel and thus $Q(z,w)\gamma \in H^2(K)$. The next lemma develops this observation further.

**Lemma 5.2.** *Let* $K_Q((z,w),(\zeta,\eta)) = Q(z,w)Q(\zeta,\eta)^*$ *and let* $\mathcal{Q}$ *denote the span of* $\{K_Q(\cdot,(\zeta,\eta))\gamma : (\zeta,\eta) \in \mathcal{V},\ \gamma \in \mathbb{C}^\alpha\}$. *Then* $K_Q$ *is the reproducing kernel for* $\mathcal{Q}$ *with respect to the inner product on* $H^2(K)$*; i.e.,* $K_Q \in \mathcal{Q}$ *and if* $g \in \mathcal{Q}$*, then*

$$g(\zeta,\eta) = \langle g, K_Q(\cdot,(\zeta,\eta)) \rangle.$$

*Thus,*

$$P_{\mathcal{Q}}K(\cdot,(\zeta,\eta)) = K_Q(\cdot,(\zeta,\eta))$$

*and*

$$K(\cdot,(\zeta,\eta)) - P_{\mathcal{Q}}K(\cdot,(\zeta,\eta)) = \overline{\zeta}SK(\cdot,(\zeta,\eta)).$$

Let $\mathcal{D} = \{K_{(\zeta,\eta)} : (\zeta,\eta) \in \mathcal{V},\ \zeta \neq 0\}$. The last identity in the Lemma implies the set $S\mathcal{D}$ is orthogonal to $\mathcal{Q}$. Since the span of $\mathcal{D}$ is dense in $H^2(K)$, it follows that $SH^2(K)$ is orthogonal to $\mathcal{Q}$. Using the Lemma and the fact that for $f \in H^2(K)$ and $(\zeta,\eta) \in \mathcal{V}$,

$$\begin{aligned}\langle Sf, K(\zeta,\eta)\rangle =& \langle Sf, \overline{\zeta}SK(\cdot,(\zeta,\eta))\rangle + \langle Sf, P_{\mathcal{Q}}K(\cdot,(\zeta,\eta))\rangle \\ =& \langle Sf, \overline{\zeta}SK(\cdot,(\zeta,\eta))\rangle.\end{aligned}$$

Consequently,

$$\begin{aligned}\zeta\langle f, K(\cdot,(\zeta,\eta))\rangle =& \zeta f(\zeta,\eta) = \langle Sf, K(\cdot,(\zeta,\eta))\rangle \\ =& \langle Sf, \overline{\zeta}SK(\cdot,(\zeta,\eta))\rangle.\end{aligned}$$

It follows that $S$ is an isometry.

*Proof of Lemma 5.2.* The claim of the lemma may be understood as follows: the finite-dimensional space $\mathcal{Q}$ can be made into a reproducing kernel Hilbert space in two ways. On the one hand, since $\mathcal{Q} \subset H^2(K)$ we can simply restrict the norm from $H^2(K)$. On the other hand, we can define the kernel $K_Q$ as in the statement of the lemma and give $\mathcal{Q}$ the norm coming from the resulting inner product. The kernel in the first case is $P_{\mathcal{Q}}K$, and the kernel in the second case is of course $K_Q$. The lemma says that the two kernels are in fact equal, which is equivalent to saying that the associated Hilbert space norms are equal (since the norm determines inner product and the inner product determines the kernel). To prove this, it suffices to prove that the identity map of $\mathcal{Q}$ is contractive in both directions. This may be proved by inspecting the kernels: the identity is contractive from the $H^2(K_Q)$ norm to the (restricted) $H^2(K)$ norm if and only if

$$K \succeq K_Q, \tag{21}$$

while the map is contractive from the $H^2(K)$ norm to the $H^2(K_Q)$ norm if and only if for any $g \in \mathcal{Q}$

$$K \succeq gg^* \text{ implies } K_Q \succeq gg^*. \tag{22}$$

(Recall from subsection 1.2.1 that "$\succeq$" represents an inequality in the sense of positive semi-definite kernels; e.g. (21) says $K - K_Q$ is a positive semi-definite kernel.) These equivalences follow from the fact that for any reproducing kernel Hilbert space on a set $X$ with kernel

$L$, a function $f : X \to \mathbb{C}$ lies in the unit ball of $H^2(L)$ if and only if $L \succeq ff^*$.

We may now begin the proof of the lemma. First, it is clear that $K \succeq K_Q$ since

$$K - K_Q = \bar{\zeta} SK.$$

For the other direction, suppose

$$g = \sum_{j=1}^{m} Q_j(z,w)\gamma_j = Q(z,w)\gamma$$

is in $\mathcal{Q}$ where $\gamma = (\gamma_1 \dots \gamma_m)^t \in \mathbb{C}^{m\alpha}$ and $K - gg^* \succeq 0$. The inequality $K - gg^* \succeq 0$ is equivalent to positive semi-definiteness of the kernel

$$L(z,w) = Q(z,w)[\frac{1}{1 - z\bar{\zeta}} I_{m\alpha} - \gamma\gamma^*]Q(\zeta,\eta)^*. \tag{23}$$

Fix $\lambda$ satisfying the conditions of Lemma 4.1. In particular, there are $m$ distinct points $\mu_1, \dots, \mu_m \in \mathbb{D}$ such that $(\lambda, \mu_1), \dots, (\lambda, \mu_m) \in \mathcal{V} \cap \mathbb{D}^2$. The block matrix

$$(L((\lambda,\mu_j),(\lambda,\mu_k)))_{j,k} = Z[\frac{1}{1-|\lambda|^2} - \gamma\gamma^*]Z^*$$

is positive definite, where

$$Z^* = \begin{pmatrix} Q(\lambda,\mu_1)^* & \dots & Q(\lambda,\mu_m)^* \end{pmatrix}$$

(see Lemma 4.1). Since $Z$ is invertible, it follows that

$$\left(\frac{1}{1-|\lambda|^2} I_{m\alpha} - \gamma\gamma^*\right) \geq 0.$$

Since, by Lemma 4.1, there is a sequence $\lambda_n \to 0$ for which this last inequality holds,

$$(I_{m\alpha} - \gamma\gamma^*) \geq 0.$$

It now follows that

$$Q(z,w)(I_{m\alpha} - \gamma\gamma^*)Q(\zeta,\eta)^* \succeq 0.$$

This last inequality is equivalent to

$$K_Q \succeq gg^*.$$

Thus, we have proved that $K_Q$ is the reproducing kernel for $\mathcal{Q}$. Since $P_{\mathcal{Q}} K(\cdot, (\zeta,\eta))$ is also the reproducing kernel for $\mathcal{Q}$, the second identity in the lemma follows. The last statement follows from the identity

$$K((z,w),(\zeta,\eta)) - K_Q((z,w),(\zeta,\eta)) = z\bar{\zeta}K((z,w),(\zeta,\eta)).$$

□

To see that $S$ is pure, note that for any given $(\zeta,\eta)$ and vector $\gamma$,

$$S^{*j}k(\cdot,(\zeta,\eta))\gamma = \overline{\zeta}^j k(\cdot,(\zeta,\eta))\gamma.$$

Thus $S^{*j}f$ converges to 0 for each $f$ in the span of $\{K(\cdot,(\zeta,\eta))\gamma : (\zeta,\eta)\in\mathcal{V},\ \gamma\in\mathbb{C}^\alpha\}$. Since this set is dense in $H^2(K)$ and since the sequence $S^{*j}$ is norm bounded, it follows that $S^{*j}$ converges to 0 in the SOT. Consequently, $S$ is a pure shift.

Similar arguments show that $\tilde{S}$, $T$, and $\tilde{T}$ are also pure isometries. The following proposition identifies their defect spaces.

**Proposition 5.3.** *The kernel of $S^*$ is $\mathcal{Q}$. Moreover, if $\lambda,\mu_1,\dots,\mu_m$ satisfy the conditions of Lemma 4.1, then $\mathcal{Q}$ is equal to the span of $\{Q(\cdot)Q(\lambda,\mu_j)^*\gamma : \gamma\in\mathbb{C}^\alpha,\ 1\le j\le m\}$.*

*Proof.* As noted already, the subspace $\mathcal{Q}$ is orthogonal to $SH^2(K)$. Hence $\mathcal{Q}$ is a subspace of the kernel of $S^*$. On the other hand, $\mathcal{Q}+SH^2(K)=H^2(K)$, since

$$K(\cdot,(\zeta,\eta)) = K_Q(\cdot,(\zeta,\eta)) + \overline{\zeta}SK(\cdot,(\zeta,\eta))$$

and the first conclusion of the lemma follows.

The dimension of $\mathcal{Q}$ is at most $m\alpha$. On the other hand, under the hypothesis of the moreover part of the lemma the span of $\{Q(\cdot)Q(\lambda,\mu_j)^*\gamma\}$ has dimension $m\alpha$. □

5.1. **Proof of Theorem 1.10.** In this subsection we give a proof of Theorem 1.10 based upon knowledge of the commutant of a pure shift. We sketch a second geometric proof which identifies the extension in terms of the operators $\tilde{S}$ and $\tilde{T}$ canonically associated to $S$ and $T$ via the reflected kernel $\tilde{K}$.

The pure shift $S$ has multiplicity $m\alpha$ and thus can be modeled as multiplication by the coordinate function on a vector valued Hardy space $H^2\otimes\mathbb{C}^{m\alpha}$. Since $T$ commutes with $S$ and is itself a pure isometry of multiplicity $n\alpha$, it is multiplication by a matrix valued rational inner function, say $\Phi$, on $H^2\otimes\mathbb{C}^{m\alpha}$. Therefore, the pair $(S,T)$ can be thought of as the pair

$$(M_z, M_\Phi) : H^2\otimes\mathbb{C}^{m\alpha}\to H^2\otimes\mathbb{C}^{m\alpha}.$$

We will necessarily have

$$p(zI,\Phi(z))=0$$

since $p(M_z,M_w)=0$.

This pair extends to a pair of unitary multiplication operators on $L^2\otimes\mathbb{C}^{m\alpha}$. The resulting pair of unitaries will still satisfy the polynomial $p$ which defines the distinguished variety in question.

Next we sketch our geometric proof. As in the proof of Lemma 5.2, let $K_Q(\cdot,(\zeta,\eta)) = Q(\cdot)Q(\zeta,\eta)^*$ and let $\mathcal{Q}$ denote the span of $\{K_Q(\cdot,(\zeta,\eta))\gamma : (\zeta,\eta) \in \mathcal{V},\ \gamma \in \mathbb{C}^\alpha\}$. If $g \in \mathcal{Q}$, then

$$\langle g, K_Q(\cdot,(\zeta,\eta))\gamma\rangle = \langle g(\zeta,\eta),\gamma\rangle.$$

Since both sides are defined and analytic in $\mathcal{Z}_p$, it follows that the identity is valid for $(\zeta,\eta) \in \tilde{\mathcal{V}}$ too. In particular, if also $\gamma' \in \mathbb{C}^\alpha$, then

$$\langle K_Q(\cdot,(\zeta',\eta'))\gamma', K_Q(\cdot,(\zeta,\eta))\gamma\rangle = K_Q((\zeta,\eta),(\zeta',\eta'))\gamma',\gamma\rangle,$$

for $(\zeta,\eta),(\zeta',\eta') \in \tilde{\mathcal{V}}$.

By analogy with $K_Q$ and $\mathcal{Q}$, let $\tilde{K_Q}((z,w),(\zeta,\eta)) = Q(z,w)Q(\zeta,\eta)^*$ for $(z,w),(\zeta,\eta) \in \tilde{\mathcal{V}}$ and $\tilde{\mathcal{Q}}$ denote the span of $\{\tilde{K_Q}(\cdot,(\zeta,\eta))\gamma : (\zeta,\eta) \in \tilde{\mathcal{V}},\ \gamma \in \mathbb{C}^\alpha\}$.

Define $\Sigma,\Gamma : H^2(\tilde{K}) \to H^2(K)$ by,

$$\begin{aligned}\Sigma\tilde{K}(\cdot,(\zeta,\eta)) =& \frac{1}{\bar{\zeta}}Q(\cdot)Q(\zeta,\eta)^*\\ \Gamma\tilde{K}(\cdot,(\zeta,\eta)) =& \frac{1}{\bar{\eta}}P(\cdot)P(\zeta,\eta)^*.\end{aligned}$$

Of course at this point $\Sigma$ and $\Gamma$ are only densely defined. The computations below show that $\Sigma^*\Sigma = P_{\tilde{\mathcal{Q}}}$, and similarly $\Gamma^*\Gamma$, is a projection.

Note that the functions on the left hand side are defined on $\tilde{\mathcal{V}}$ and those on the right are defined on $\mathcal{V}$.

With these definitions of $\Sigma$ and $\Gamma$, the operators $X, Y$ on $H^2(K) \oplus H^2(\tilde{K})$ from Theorem 1.10 are given by

$$X = \begin{pmatrix} S & \Sigma \\ 0 & \tilde{S}^* \end{pmatrix} \qquad Y = \begin{pmatrix} T & \Gamma \\ 0 & \tilde{T}^* \end{pmatrix}$$

and it is now our task to prove $X$ and $Y$ are commuting unitaries satisfying $p(X,Y) = 0$.

Compute, for $(\zeta,\eta),(\zeta',\eta') \in \tilde{\mathcal{V}}$ and $\gamma,\gamma' \in \mathbb{C}^\alpha$,

$$\begin{aligned}\langle \Sigma^*\Sigma\tilde{K}(\cdot,(\zeta',\eta'))\gamma', \tilde{K}(\cdot,(\zeta,\eta))\gamma\rangle =& \frac{1}{\zeta'\bar{\zeta}}\langle \tilde{K_Q}((\zeta,\eta),(\zeta',\eta'))\gamma',\gamma\rangle \\ =& \langle P_{\tilde{\mathcal{Q}}}\tilde{K}(\cdot,((\zeta',\eta'))\gamma', \tilde{K}(\cdot,(\zeta,\eta))\gamma\rangle.\end{aligned}$$

Thus, $\Sigma^*\Sigma = P_{\tilde{\mathcal{Q}}}$. Hence, by Proposition 5.3, $\Sigma^*\Sigma$ is the projection onto the kernel of $\tilde{S}^*$ and in particular

$$I = \tilde{S}\tilde{S}^* + \Sigma^*\Sigma \tag{24}$$

Since the range of $\Sigma$ is in $\mathcal{Q}$,

$$S^*\Sigma = 0. \tag{25}$$

Using equations (24) and (25) and the fact that $S$ is an isometry, it follows that the $X$ in Theorem 1.10 is an isometry; i.e., $X^*X = I$.

Since $\Sigma\Sigma^*$ is a projection of rank $m\alpha$ (same as the rank of $\Sigma^*\Sigma$) with range in the kernel of $S^*$, we conclude $SS^* + \Sigma\Sigma^* = I$. Since also $\tilde{S}^*\tilde{S} = I$ and $XX^* \leq I$, it follows that $XX^* = I$. Hence $X$ is unitary. A similar argument shows $Y$ is unitary.

The commutation relation $XY = YX$ is equivalent to

$$S\Gamma - \Gamma\tilde{S}^* = T\Sigma - \Sigma\tilde{T}^*.$$

To see that this is indeed the case, compute,

$$\begin{aligned}&\langle[S\Gamma{-}\Gamma\tilde{S}^*]\tilde{K}(\cdot,(\zeta,\eta)\gamma,\tilde{K}(\cdot,(z,w))\delta\rangle\\ &=\langle(\frac{z}{\overline{\eta}}-\frac{1}{\overline{\zeta\eta}})P(z,w)P(\zeta,\eta))^*\gamma,\delta\rangle\\ &=\langle(\frac{z\overline{\zeta}-1}{\overline{\zeta\eta}}P(z,w)P(\zeta,\eta)^*\gamma,\delta\rangle.\end{aligned}$$

Similarly,

$$\begin{aligned}&\langle[T\Sigma{-}\Sigma\tilde{T}^*]\tilde{K}(\cdot,(\zeta,\eta)\gamma,\tilde{K}(\cdot,(z,w))\delta\rangle\\ &=\langle(\frac{w\overline{\eta}-1}{\overline{\zeta\eta}})Q(z,w)Q(\zeta,\eta)^*\gamma,\delta\rangle.\end{aligned}$$

The commutation relation thus follows as a consequence of the fact that $(P,Q)$ is an admissible pair.

For the statement about the spectrum, it is a property of the Taylor spectrum that, given the upper triangular structure of the pair $(X,Y)$ that

$$\sigma_T(X,Y) \subset \sigma_T(S,T) \cup \sigma_T(\tilde{S}^*,\tilde{T}^*).$$

The sets on the right hand side both lie in closure($\mathcal{V}$). On the other hand, the projection property of the Taylor spectrum implies,

$$\sigma_T(X,Y) \subset \sigma(X)\times\sigma(Y) \subset \mathbb{T}\times\mathbb{T}.$$

Putting the last two inclusions together it follows that $\sigma_T(X,Y) \subset \partial\mathcal{V}$.

## 6. Polynomial approximation on $\mathcal{V}$

This section proves the fundamental and function-theoretic Theorem 1.12. It is largely independent from other sections.

Suppose $p \in \mathbb{C}[z,w]$ defines a distinguished variety $\mathcal{V} = \mathcal{Z}_p \cap \mathbb{D}^2$, where $\mathcal{Z}_p$ is the zero set of $p$. Let $R$ be the Riemann surface desingularizing $\mathcal{Z}_p$, with map $h : R \to \mathcal{Z}_p$. Let $S \subset R$ be the bordered Riemann surface $h^{-1}(V)$, so $h : S \to V$ is a *holomap* in the sense of [AM07].

If $W$ is any surface or variety, write $O(W)$ for the holomorphic functions on $W$. (In particular, we recall that to say $f$ is holomorphic at $(z,w) \in \mathcal{V}$ means "$f$ extends to be holomorphic in a neighborhood of $(z,w)$ in $\mathbb{C}^2$.") If $W$ is a surface or variety with (always assumed smooth) boundary $\partial W$, then $A(W)$ denotes those functions continuous on $\overline{W} = W \cup \partial W$ and holomorphic on $W$. Finally, $H^\infty(W)$ denotes the algebra of bounded analytic functions on $W$. We remark that if $W$ is a Riemann surface with smooth boundary, and $\omega$ is harmonic measure on $\partial W$, then $H^\infty(W)$ coincides with $H^\infty(\omega)$ as defined in the theory of uniform algebras (as the weak-* closure of $A(W)$ in $L^\infty(\omega)$). This precise result is found in [GL] Theorem 3.10, page 171.

We recall some terminology and a theorem from [AM07].

**Definition 6.1.** *If $S$ is a bordered Riemann surface, a linear functional on $O(S)$ is called* local *if it comes from a finitely supported distribution, i.e. has the form*

$$\Lambda(f) = \sum_{i=1}^{m} \sum_{j=0}^{n} c_{ij} f^{(j)}(\alpha_i).$$

*It is assumed that for each $i$, some $c_{ij} \neq 0$. The set $\{\alpha_1, \dots \alpha_m\}$ is then called the* support *of $\Lambda$.*

*A* connection *$\Gamma$ supported in $\{\alpha_1, \dots \alpha_m\}$ is a finite set of local functionals $\Lambda$ supported in $\{\alpha_1, \dots \alpha_m\}$. Write $\Gamma^\perp = \bigcap_{\Lambda \in \Gamma} \ker\Lambda$. Say $\Gamma$ is* algebraic *if $\Gamma^\perp$ is an algebra, and* irreducible *if every $f \in \Gamma^\perp$ is constant on the support of $\Gamma$.*

A theorem of Gamelin [G2] says that the finite codimension subalgebras of $O(S)$ are exactly the $\Gamma^\perp$'s for algebraic connections $\Gamma$. Moreover, each connection is the union of finitely many irreducible connections with disjoint supports. Finally, each finite codimension subalgebra $A \subset O(S)$ has a filtration $A_n \subsetneq A_{n-1} \cdots \subsetneq A_1 = O(S)$ where each $A_j$ has codimension 1 in the next and $A_{j+1}$ is obtained either as the kernel of a point derivation on $A_j$ or by identifying two points of the maximal ideal space of $A_j$.

The main step in our proof will be an appeal to the following, which is Theorem 2.8(i) of [AM07]. For us, $V$ will always be the intersection of $\mathcal{Z}_p$ with a bidisk $U$ centered at $(0,0)$ (of some radius) and $S$ will always be the piece of the disingularization living over $V$. Note that an algebraic curve intersected with a bounded domain in $\mathbb{C}^n$ is what is called a *hyperbolic algebraic curve* and this is a special case of the hyperbolic analytic curves defined in [AM07].

**Theorem 6.2.** *If $h : S \to V \subset U$ is a holomap from a Riemann surface $S$ onto a hyperbolic analytic curve $V \subset U$, then*

$$A_h := \{F \circ h : F \in O(V)\}$$

*is a finite codimension subalgebra of $O(S)$.*

We can now prove the approximation theorem.

*Proof of Theorem 1.12.* We allow that $\mathcal{V}$ may have singularities on $\mathbb{T}^2$. First we extend $\mathcal{V}$ slightly: choose $r > 1$ so that $\mathcal{V}_r := \mathcal{Z}_p \cap r\mathbb{D}^2$ has no additional singularities. Let $S_r$ be the piece of the desingularization lying over $\mathcal{V}_r$. Then $S_r$ is also a bordered Riemann surface, and $\overline{S}$ is compactly contained in $S_r$. From the theory of hypo-Dirichlet algebras [G1], every function in $H^\infty(S)$ can be contractively locally uniformly approximated on $S$ by functions in $O(S_r)$. (In particular, from [G1, Theorem IV.8.1] every function in $A(S)$ can be uniformly approximated on $\overline{S}$ by functions in $O(S_r)$, and from [G1, Theorem VI.5.2] each $f \in H^\infty(S)$ can be approximated pointwise on $S$ (and hence locally uniformly) with functions $f_n \in A(S)$, satisfying $\|f_n\|_S \leq \|f\|_S$.)

Fix the function $f \in H^\infty(\mathcal{V})$ that we would like to approximate with polynomials. We may assume $\|f\|_\infty = 1$. Then $f \circ h$ belongs to $H^\infty(S)$, and so $f \circ h$ is approximated on $\overline{S}$ by functions which extend to be holomorphic on $S_r$. On the other hand, let $O_h(S_r)$ denote the subalgebra of functions $\{F \circ h : F \in O(\mathcal{V}_r)\}$; by Theorem 6.2, this is a finite codimension subalgebra of $O(S_r)$, and hence by Gamelin's theorem is of the form $\Gamma^\perp$ for some connection $\Gamma$ on $S_r$. The idea of the proof is to "correct" the approximants from $O(S_r)$ so that they belong to $O_h(S_r)$. It then follows from Theorem 6.2 that the corrected approximants can be pushed down to holomorphic functions on $\mathcal{V}_r$. This process is straightforward for the portion of $\Gamma$ supported in the interior of $S$, but when the support of $\Gamma$ meets $\partial S$, it seems that some care is needed (this is the case when $\mathcal{V}$ has singularities on its boundary in $\mathbb{T}^2$). For the Neil parabola and the annulus discussed in Section 2 there are no singularities on the boundary which explains why it is possible to give simple proofs that $H^\infty_{\mathcal{K}}$ and $H^\infty$ are isometric in these cases. On the other hand, when a triply connected domain is realized as a distinguished variety, there are singularities on the boundary [Ru].

Consider a sequence $(q_n) \subset O(S_r)$ converging uniformly to $f \circ h$ on compact subsets of $S$, with each $q_n$ bounded by 1 on $S$. By Gamelin's theorem, $O_h(S_r) = \Gamma^\perp$ for some algebraic connection $\Gamma$. Since $\mathcal{V}$ meets the boundary of $\mathbb{D}^2$ only in $\mathbb{T}^2$, it follows that each irreducible component of $\Gamma$ is supported either entirely in the interior of $S$ or entirely in the boundary of $S$ (points in the interior of $S$ cannot be identified with points in the boundary of $S$ when we push forward to $\mathcal{V}$). Decompose $\Gamma = \Gamma_1 \cup \Gamma_2$ into its interior and boundary pieces. We first correct the $q_n$ to lie in $\Gamma_1^\perp$, then correct these functions to lie in $\Gamma_2^\perp$ as well.

Let

$$\Gamma_1^\perp := A_m \subsetneq A_{m-1} \cdots \subsetneq A_1 = O(S_r)$$

be a Gamelin filtration. We show by induction that for each $k = 1, \dots m$ there exists a sequence $(q_n^k) \subset A_k$ approximating $f$ in the required way. We already have $q_n^1 = q_n$. Suppose $(q_n^k)$ is given. Now $A_{k+1}$ is obtained from $A_k$ as $A_{k+1} = \ker \gamma_{k+1}$, where $\gamma_{k+1}$ is either a point derivation or identifies two points. In either case, choose $a_k \in A_k$ such that $\gamma_{k+1}(a_k) = 1$. Define

$$q_n^{k+1} = q_n^k - \gamma_{k+1}(q_n^k) a_k.$$

By construction, $q_n^{k+1}$ lies in $A_{k+1}$ and converges locally uniformly to $f$; since $\gamma_{k+1}(q_n^k) \to \gamma_{k+1}(f) = 0$, the sup norms of the $q_n^{k+1}$ converge to 1, so after normalization the $q_n^{k+1}$ work.

To accomplish the modification on the boundary, we multiply the functions $q_n$ by functions $G_n$ that converge to 1 pointwise in $S$ and "zero out" the boundary relations. The $G_n$ are constructed using two lemmas:

**Lemma 6.3.** *Let $S, \mathcal{V}, \Gamma$ be as above. Let $\alpha_1, \dots \alpha_m$ be the interior points of $S$ belonging to the support of $\Gamma$, let $\beta_1, \dots \beta_l$ be the boundary points in the support of $\Gamma$, and let an integer $N \geq 1$ be given. Then there exists a function $b$, holomorphic in a neighborhood of $\overline{S}$, such that*

i) *$b$ is inner (that is, $|b| = 1$ on $\partial S$),*
ii) *$b$ vanishes to order $N$ at each $\alpha_j$, and*
iii) *$b$ is 1 at each $\beta_j$.*

*Proof.* Write $h = (h_1, h_2)$ and consider the projection $h_1 : \overline{S_r} \to \overline{r\mathbb{D}}$. It is straightforward to construct a finite Blaschke product $B$ which vanishes to order $N$ at each of the points $h_1(\alpha_j)$, and takes the value 1 at the points $h_1(\beta_j)$ on the unit circle. By shrinking $r$ if necessary, $b = B \circ h_1$ does the job.

□

**Lemma 6.4.** *There exists a sequence of functions $g_n$ in the unit disk such that:*

i) *Each $g_n$ is holomorphic in some neighborhood of $\overline{\mathbb{D}}$ and bounded by 1 in $\mathbb{D}$,*
ii) *$g_n(1) = 0$ for all $n$, and*
iii) *$g_n \to 1$ uniformly on compact subsets of $\mathbb{D}$.*

*Proof of lemma.* To construct the $g_n$, let $c_n = 1 - n^{-2}$ and define

$$h_n(z) = \exp\left(-\frac{1}{n}\left(\frac{1+c_n z}{1-c_n z}\right)\right) - \exp\left(-\frac{1}{n}\left(\frac{1+c_n}{1-c_n}\right)\right)$$

It is evident that $h_n$ is holomorphic on $\overline{\mathbb{D}}$ and that $h_n(1) = 0$ for all $n$. Moreover it is readily verified that $\|h_n\|_\infty \leq 1 + o(1)$ as $n \to \infty$ and $h_n \to 1$ locally uniformly in $\mathbb{D}$. Taking $g_n = h_n/\|h_n\|_\infty$ works. □

Now we combine the two lemmas. For each $n$, we may shrink the domain of $b$ further (but so that it still contains $\overline{S}$) so that $b$ maps into the domain of $g_n$. We may then form the composition $g_n \circ b$. Now, $b$ is bounded by 1 in $\overline{S}$, and $g_n$ is given by a uniformly convergent power series on $D$, and vanishes at each $\beta_j$. So by taking a suitably high power $G_n = (g_n \circ b)^N$, we see that each $G_n$ is annihilated by $\Gamma$ (it vanishes to high order at the boundary points, and satisfies the interior relations because $b$ does). Thus, if we call $S_n$ the domain of $G_n$, then each $G_n$ belongs to $O_h(S_n)$. By construction the $G_n$ are all bounded by 1 in $S$, and $G_n \to 1$ pointwise on $S$.

We can now use the $G_n$ to correct the sequence $q_n$ converging to $f \circ h$. In particular, by construction the product $G_n q_n$ belongs to $O_h(S_n)$, since the relations on the boundary are zeroed out by the $G_n$. Setting $\mathcal{V}_n = h(S_n)$, from Theorem 6.2 there is an analytic function $p_n$ on $\mathcal{V}_n$ satisfying $p_n \circ h = G_n q_n$. So the $p_n$ are each holomorphic on a neighborhood of $\mathcal{V}$ in $\mathbb{C}^2$, bounded by 1 on $\mathcal{V}$, and converge to $f$ uniformly on compact subsets of $\mathcal{V}$. Finally, since $\overline{\mathcal{V}}$ is polynomially convex, the Oka-Weil theorem says that each $p_n$ is uniformly approximable on $\overline{\mathcal{V}}$ by polynomials, and thus $f$ is approximable by polynomials as desired. □

## 7. Bounded Analytic Functions on $\mathcal{V}$

In this section we prove Corollary 1.13. By Corollary 1.11, every polynomial belongs to $H^\infty_{\mathcal{K}}(\mathcal{V})$, with norm equal to the supremum norm over $\mathcal{V}$. The first step is an elementary completeness result for $H^\infty_{\mathcal{K}}(\mathcal{V})$.

**Proposition 7.1.** *The algebra $H^\infty_{\mathcal{K}}(\mathcal{V})$ is closed both in norm and under pointwise bounded convergence.*

Since the result is standard (see for instance [AM02]), we only sketch a proof.

*Proof.* Let $(f_n)$ be a given sequence from $H^\infty_{\mathcal{K}}(\mathcal{V})$ and suppose there is a $C$ such that $\|f_n\| \leq C$ independent of $n$. Further, assume $f_n$ converges pointwise on $\mathcal{V}$. It follows that for every finite subset $F$ of $\mathcal{V}$, every admissible kernel $K$ and every $n$, the (block) matrix

$$\big((C^2 - f_n(x)\overline{f_n(y)})K(x,y)\big)_{x,y\in F}$$

is positive semidefinite. Thus,

$$\big((C^2 - f(x)\overline{f(x)})K(x,y)\big)_{x,y\in F}$$

is positive semi-definite and hence $f \in H^\infty_{\mathcal{K}}(\mathcal{V})$.

Now suppose $(f_n)$ is Cauchy in $H^\infty_{\mathcal{K}}(\mathcal{V})$. Since $\|f\|_{\mathcal{V}}$ dominates $\|f\|_\infty$, the sequence converges pointwise to some $f$. It follows that $f \in H^\infty_{\mathcal{K}}(\mathcal{V})$ and moreover $\|f\| \le C$. It remains to verify that $(f_n)$ converges to $f$ in $H^\infty_{\mathcal{K}}(\mathcal{V})$.

Let $\epsilon > 0$ be given. There is an $N$ so that if $m,n \ge N$, then $\|f_m - f_n\|_{\mathcal{V}} < \epsilon$. From what has already been proved, it now follows that

$$\|f - f_n\|_{\mathcal{V}} \le \epsilon.$$

□

*Proof of Corollary 1.13.* Suppose $f \in H^\infty(\mathcal{V})$. Then by Theorem 1.12 there exist polynomials $p_n \to f$ pointwise with $\|p_n\|_\infty \le \|f\|_\infty$. By Corollary 1.11, each $p_n$ belongs to $H^\infty_{\mathcal{K}}(\mathcal{V})$, and $\|p_n\|_\infty = \|p_n\|_{\mathcal{V}}$. It follows that $f \in H^\infty_{\mathcal{K}}(\mathcal{V})$ and $\|f\|_{\mathcal{V}} \le \|f\|_\infty$ by Proposition 7.1.

We now turn to the proof that each function in $H^\infty_{\mathcal{K}}(\mathcal{V})$ is analytic on $\mathcal{V}$. It is proved in [Kn2] (Theorem 11.3) that every distinguished variety has an admissible kernel

$$K((z,w),(\zeta,\eta)) = \frac{Q(z,w)Q(\zeta,\eta)^*}{1 - z\overline{\zeta}}$$

where $K((z,w),(z,w)) \ne 0$ for all $(z,w) \in \mathcal{V}$. Indeed, it is shown that $Q$ can be chosen to be of the form

$$(1, w, \dots, w^{m-1})A(z)$$

where $A(z)$ is an $m \times m$ matrix polynomial which is invertible for every $z$ in $\mathbb{D}$. Let $f$ belong to the unit ball of $H^\infty_{\mathcal{K}}(\mathcal{V})$. Then the kernel

$$\Big(1 - f(z,w)\overline{f(\zeta,\eta)}\Big)\, \frac{Q(z,w)Q(\zeta,\eta)^*}{1 - z\overline{\zeta}}$$

is positive, and hence there exists a (vector-valued) function $\Gamma$ on $\mathcal{V}$ such that

$$(1 - f(z,w)\overline{f(\zeta,\eta)})Q(z,w)Q(\zeta,\eta)^* = \Gamma(z,w)(1 - z\overline{\zeta})\Gamma(\zeta,\eta)^*.$$

A straightforward lurking isometry argument produces a contractive $m \times m$ $H^\infty(\mathbb{D})$ matrix function $F$ such that

$$F(z)^*Q(z,w)^* = \overline{f(z,w)}Q(z,w)^* \tag{26}$$

for all $(z, w) \in \mathcal{V}$. Since $K$ (hence $Q$) does not vanish at $(z_0, w_0)$, some coordinate of $Q$ doesn't vanish in a neighborhood of $(z_0, w_0)$, say $q_j$. Writing out the $j^{th}$ coordinate of (26) and taking conjugates gives

$$f(z, w) = \frac{\sum_{i=1}^{m} q_i(z, w) F_{ij}(z)}{q_j(z, w)}.$$

The right-hand side extends to be analytic in a neighborhood of $(z_0, w_0)$ in $\mathbb{D}^2$, hence $f$ is holomorphic (as a function on $\mathcal{V}$) at $(z_0, w_0)$. Finally, as already noted, the inequality $\|f\|_\infty \leq \|f\|_{\mathcal{V}}$ is trivial. $\square$

## References


[Ab] M.B. Abrahamse, *The Pick interpolation theorem for finitely connected domains,* Michigan Math. J. 26 (1979), no. 2, 195–203.

[AD] M.B. Abrahamse and Ronald Douglas, *A class of subnormal operators related to multiply-connected domains,* Advances in Math. 19 (1976), no. 1, 106–148.

[Ag] Jim Agler, *Interpolation*, unpublished manuscript.

[AKM] Jim Agler, Greg Knese and John McCarthy, *Algebraic pairs of isometries,* Journal of Operator Theory 67 (2012), no. 1, 101–122.

[AM07] Jim Agler and John McCarthy, *Hyperbolic algebraic and analytic curves,* Indiana University Mathematics Journal, Vol. 56, No. 6 [2007] 2899–2933 .

[AM99] Jim Agler and John McCarthy, *Nevanlinna-Pick interpolation on the bidisk,* J. Reine Angew. Math. 506 (1999), 191–204.

[AM00] Jim Agler and John McCarthy, *Complete Nevanlinna-Pick kernels,* J. Funct. Anal. **175** (2000), no. 1, 111–124.

[AM02] Jim Agler and John McCarthy, Pick interpolation and Hilbert function spaces. Graduate Studies in Mathematics, 44. American Mathematical Society, Providence, RI, 2002. xx+308 pp. ISBN: 0-8218-2898-3.

[AM03] Jim Agler and John McCarthy, John, *Norm preserving extensions of holomorphic functions from subvarieties of the bidisk,* Ann. of Math. (2) **157** (2003), no. 1, 289–312.

[AM05] Jim Agler and John McCarthy, *Distinguished varieties,* Acta Math. 194 (2005), no. 2, 133–153.

[AMS] Jim Agler, John McCarthy, and Mark Stankus, *Local geometry of zero sets of holomorphic functions near the torus,* New York J. Math. 14 (2008), 517–538.

[AW] Herbert Alexander and John Wermer, Several Complex Variables and Banach Algebras, 3rd ed., Graduate Texts in Mathematics, 35. Springer-Verlag, New York, 1988.

[B] Joseph Ball, *Interpolation problems and Toeplitz operators on multiply connected domains,* Integral Equations Operator Theory 4 (1981), no. 2, 172–184.

[BB] Joseph Ball and Vladimir Bolotnikov, *Nevanlinna-Pick interpolation for Schur-Agler class functions on domains with matrix polynomial defining function in $\mathbb{C}^n$,* New York J. Math. 11 (2005), 247–290.

[BBtH] J. A. Ball, V. Bolotnikov and S. ter Horst, *A constrained Nevanlinna-Pick interpolation problem for matrix-valued functions,* Indiana Univ. Math. J. **59** (2010), no. 1, 15–51.

[BCV] Joseph Ball, Kevin Clancey, and Victor Vinnikov, *Concrete interpolation of meromorphic matrix functions on Riemann surfaces,* Interpolation theory, systems theory and related topics (Tel Aviv/Rehovot, 1999), 137–156, Oper. Theory Adv. Appl., 134, Birkhäuser, Basel, 2002.

[BTV] Joseph Ball, Tavan Trent, and Victor Vinnikov, *Interpolation and commutant lifting for multipliers on reproducing kernel Hilbert spaces,* Operator theory and analysis (Amsterdam, 1997), 89–138, Oper. Theory Adv. Appl., 122, Birkhäuser, Basel, 2001.

[CLW] Brian Cole, Keith Lewis, and John Wermer, *Pick conditions on a uniform algebra and von Neumann inequalities,* J. Funct. Anal. 107 (1992), no. 2, 235–254.

[DH] Kenneth R. Davidson and Ryan Hamilton, *Nevanlinna-Pick interpolation and factorization of linear functionals,* Integral Equations and Operator Theory 70 (2011), no. 1, 125–149.

[DPRS] Kenneth R. Davidson, Vern I. Paulsen, Mrinal Raghupathi, and Dinesh Singh, *A constrained Nevanlinna-Pick interpolation problem,* Indiana Univ. Math. J. 58 (2009), no. 2, 709–732.

[F] S. I. Fedorov, *Harmonic analysis in a multiply connected domain. I.* Matematicheskiĭ Sbornik 181 (1990), no. 6, 833-864.

[FF] Ciprian Foias and Arthur Frazho, The commutant lifting approach to interpolation problems. Operator Theory: Advances and Applications, 44. Birkhäuser Verlag, Basel, 1990. xxiv+632 pp. ISBN: 3-7643-2461-9

[G1] Theodore W. Gamelin, *Uniform algebras*, Prentice-Hall Inc. Englewood Cliffs, N. J., 1969

[G2] Theodore W. Gamelin, *Embedding Riemann surfaces in maximal ideal spaces*, J. Func. Anal., Vol. 2 [1968] 123–146.

[GL] Theodore W. Gamelin and Günter Lumer, *Theory of abstract Hardy spaces and the universal Hardy class.* Advances in Math. 2, 1968, 118-174.

[H] Ryan Hamilton, *Pick interpolation in several variables,* Proc. Amer. Math. Soc., to appear. http://arxiv.org/abs/1110.0445

[Kn1] Greg Knese, *Polynomials defining distinguished varieties*, Trans. Amer. Math. Soc. **362** (2010), 5635-5655.

[Kn2] Greg Knese, *Polynomials with no zeros on the bidisk*, Analysis & PDE, Vol.3, (2010), no. 2, 109-149.

[JKM] Micheal Jury, Greg Knese, and Scott McCullough *Agler Interpolation Families of Kernels*, Oper. Matrices **3** (2009), no.4, 571-587.

[MS] Paul S. Muhly and Baruch Solel, *Schur Class Operator Functions and Automorphisms of Hardy Algebras,* Documenta Mathematica 13 (2008) 365–411.

[Pa] Vern Paulsen, *Operator Algebras of Idempotents,* J. Funct. Anal. 181 (2001), no. 2, 209–226.

[PF] B. S. Pavlov and S. I. Fedorov, *Shift group and harmonic analysis on a Riemann surface of genus one,* Algebra i Analiz 1 (1989), no 2, 132–168.

[P] James Pickering, *Test Functions in Constrained Interpolation,* arXiv:0811.2191.

[R1] Mrinal Raghupathi, *Nevanlinna-Pick interpolation for* $\mathbb{C} + BH^\infty$*,* Integral Equations Operator Theory **63** (2009), no. 1, 103–125.

[R2] Mrinal Raghupathi, *Abrahamse's interpolation theorem and Fuchsian groups,* J. Math. Anal. Appl. 355 (2009), no. 1, 258–276.

[Ru] Walter Rudin, *Pairs of inner functions on finite Riemann surfaces,* Trans. Amer. Math. Soc. 140 (1969), 423–434.

[S] Donald Sarason, *Generalized interpolation in $H^\infty$,* Trans. Amer. Math. Soc. 127 1967 179–203.

[S65] Donald Sarason, *The $H^p$ spaces of an annulus,* Mem. Amer. Math. Soc. No. 56 1965 78 pp.

[T02] Joseph L. Taylor, *Several complex variables with connections to algebraic geometry and Lie groups.* Graduate Studies in Mathematics, 46. American Mathematical Society, Providence, RI, 2002. xvi+507 pp. ISBN: 0-8218-3178-X

Department of Mathematics, University of Florida, Box 118105, Gainesville, FL 32611-8105, USA
*E-mail address*: mjury@ufl.edu

Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487-0350
*E-mail address*: geknese@bama.ua.edu

Department of Mathematics, University of Florida, Box 118105, Gainesville, FL 32611-8105, USA
*E-mail address*: sam@ufl.edu