Analysis on Varieties

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Consider a polynomial $p \in \mathbb{C}[z, w]$. Its zero variety is the set

$$\{(z, w) \in \mathbb{C}^2 : p(z, w) = 0\}$$

Say p defines a distinguished variety if

$$\text{for all } (z, w) \in Z_p, |z| = 1 \text{ if and only if } |w| = 1. \qquad \text{(DV)}$$

Examples:

- $p(z, w) = z - w$ (boring)
- $p(z, w) = z^2 - w^2 = (z - w)(z + w)$ (mostly boring)
- $p(z, w) = z^3 - w^2$ (not boring)

The zero variety in the last example

$$p(z,w) = z^3 - w^2$$

is called the Neil parabola. Since

$$\nabla p = \left(\frac{\partial p}{\partial z}, \frac{\partial p}{\partial w}\right) = (3z^2, -2w),$$

the origin $(0,0)$ is a singular point (in this case, a cusp).

Mostly we are interested in analytic functions on

$$V = Z_p \cap \mathbb{D}^2.$$

Notice $\partial V \subset \partial\mathbb{D} \times \partial\mathbb{D} \subsetneq \partial(\mathbb{D} \times \mathbb{D})$.

Definition

Let V be a distinguished variety. A function $f : V \to \mathbb{C}$ is holomorphic at $(z, w) \in V$ if there exist:

- a neighborhood Ω of (z, w) in $\mathbb{C}^2$, and
- a holomorphic function $F : \Omega \to \mathbb{C}$

such that

$$F|_{V \cap \Omega} = f|_{V \cap \Omega}.$$

FACT (H. Cartan): If f is holomorphic on V then there exists an F holomorphic on $\mathbb{D}^2$ such that $F|_V = f$.

Example: what do the holomorphic functions on the Neil parabola $\mathcal{N}$ look like?

The Neil parabola

$$\mathcal{N} = \{(z, w) \in \mathbb{D}^2 : z^3 = w^2\}$$

is paramaterized by the disk:

$$\psi : t \to (t^2, t^3)$$

is a holomorphic bijection of $\mathbb{D}$ with $\mathcal{N}$. Take $f \in Hol(\mathcal{N})$, extend to $\mathbb{D}^2$ (Cartan):

$$f = F|_V, \quad F(z, w) = \sum_{n,m \geq 0} a_{mn} z^m w^n.$$

Then

$$F(\psi(t)) = \sum a_{mn} t^{2m+3n} = \sum_{j \neq 1} b_j t^j.$$

is holomorphic in $\mathbb{D}$.

Conversely for each $j \neq 1$ choose m, n with $j = 2m + 3n$. If $f(t) = \sum_{j \neq 1} b_j t^j$ is holomorphic in $\mathbb{D}$, then

$$F(z, w) = \sum_{m,n \geq 0} b_{2m+3n} z^m w^n$$

is holomorphic on $\mathbb{D}^2$, and $F(\psi(t)) = f(t)$.

Conclusion: the map $f \to f \circ \psi$ is an isomorphism of $Hol(\mathcal{N})$ with

$$\{f \in Hol(\mathbb{D}) : f'(0) = 0\}.$$

This was a special case of a general fact:

Theorem (Agler-McCarthy,2007)

Let $V \subset \mathbb{D}^2$ be a distinguished variety. Then there exist:

- *a finite Riemann surface R,*
- *a holomorphic map $\psi : R \to V$, and*
- *a finite codimension subalgebra $\mathcal{A} \subset Hol(R)$*

such that

$$f \to f \circ \psi$$

is an isomrophism of $Hol(V)$ with $\mathcal{A}$.

The map ψ has the form

$$t \to (\psi_1(t), \psi_2(t))$$

with $|\psi_1| = |\psi_2| = 1$ on ∂R, and $p(\psi_1, \psi_2) = 0$. In other words, an algebraic pair of inner functions on R.

von Neumann's inequality

Consider $\mathbb{C}^n$ with its usual inner product (and Euclidean norm):

$$\langle \vec{z}, \vec{w} \rangle = \sum_{j=1}^{n} z_j \overline{w_j}, \quad \|z\| = \left(\sum_{j=1}^{n} |z_j|^2 \right)^{1/2}$$

An $n \times n$ matrix T is contractive if

$$\|T\vec{z}\| \leq \|\vec{z}\|, \qquad \text{for all } \vec{z} \in \mathbb{C}^n,$$

that is, $\|T\| \leq 1$.

Theorem (von Neumann, 1949)

For every contractive matrix T and every polynomial $p \in \mathbb{C}[z]$,

$$\|p(T)\| \leq \|p\|_{\mathbb{D}} := \sup_{|z| \leq 1} |p(z)|$$

Same thing in 2 variables:

Theorem (Ando, 1963)

If S, T are commuting contractive matrices and $p \in \mathbb{C}[z, w]$, then

$$\|p(S, T)\| \leq \|p\|_{\mathbb{D}^2} := \sup_{|z|,|w|\leq 1} |p(z, w)|.$$

Once S and T are <u>fixed</u>, this can be sharpened...

Suppose S, T are simultaneously diagonalizable matrices with common eigenvectors $v_1, \dots v_n$:

$$Sv_i = \lambda_i v_i, \quad Tv_i = \mu_i v_i.$$

We also assume all $|\lambda_1| < 1, |\mu_i| < 1$. The set

$$\Lambda = \{(\lambda_i, \mu_i) : i = 1, \dots n\}$$

is called the joint spectrum of S, T.

Theorem (Agler-McCarthy, 2005)

Let , T be as above. Then there exists a disguished variety V such that $\Lambda \subset V$ and for all $p \in \mathbb{C}[z, w]$,

$$\|p(S, T)\| \leq \|p\|_V := \sup_{(z,w)\in V} |p(z, w)|.$$

In other words, in Ando we can sup over just V, instead of all of $\mathbb{D}^2$.

Since S is contractive, the matrix

$$\langle (I - S^*S)v_j, v_i\rangle = (1 - \lambda_i\overline{\lambda_j})\langle v_j, v_i\rangle$$

is positive semidefinite; same for T. So, factor:

$$(1 - \overline{\lambda_i}\lambda_j)\langle v_j, v_i\rangle = \langle x_j, x_i\rangle \quad (1)$$
$$(1 - \overline{\mu_i}\mu_j)\langle v_j, v_i\rangle = \langle y_j, y_i\rangle \quad (2)$$

with $x_i \in \mathbb{C}^{d_1}$, $y_i \in \mathbb{C}^{d_2}$.
Multiply first equation by $(1 - \overline{\mu_i}\mu_j)$, second by $(1 - \overline{\lambda_i}\lambda_j)$, get

$$(1 - \overline{\lambda_i}\lambda_j)\langle y_j, y_i\rangle = (1 - \overline{\mu_i}\mu_j)\langle x_j, x_i\rangle$$

We have

$$(1-\overline{\lambda_i}\lambda_j)\langle y_j, y_i\rangle = (1-\overline{\mu_i}\mu_j)\langle x_j, x_i\rangle$$

Rearrange to get

$$\langle y_j, y_i\rangle + \overline{\mu_i}\mu_j\langle x_j, x_i\rangle = \langle x_j, x_i\rangle + \overline{\lambda_i}\lambda_j\langle y_j, y_i\rangle$$

Or:

$$\left\langle \begin{pmatrix} \mu_j x_j \\ y_j \end{pmatrix}, \begin{pmatrix} \mu_i x_i \\ y_i \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} x_j \\ \lambda_j y_j \end{pmatrix}, \begin{pmatrix} x_i \\ \lambda_i y_i \end{pmatrix} \right\rangle$$

From

$$\left\langle \begin{pmatrix} \mu_j x_j \\ y_j \end{pmatrix}, \begin{pmatrix} \mu_i x_i \\ y_i \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} x_j \\ \lambda_j y_j \end{pmatrix}, \begin{pmatrix} x_i \\ \lambda_i y_i \end{pmatrix} \right\rangle$$

there exists a unitary matrix

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathbb{C}^{d_1} \oplus \mathbb{C}^{d_2} \to \mathbb{C}^{d_1} \oplus \mathbb{C}^{d_2}$$

such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_i \\ \lambda_i y_i \end{pmatrix} = \begin{pmatrix} \mu_i x_i \\ y_i \end{pmatrix}$$

Given

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_i \\ \lambda_i y_i \end{pmatrix} = \begin{pmatrix} \mu_i x_i \\ y_i \end{pmatrix} \tag{3}$$

define

$$\Phi(z) = A + zB(I - zD)^{-1}C$$

Since U was unitary,

$$I - \Phi(z)^*\Phi(z) = (1 - |z|^2)C^*(I - zD)^{-1*}(I - zD)^{-1}C$$

so $\Phi(z)$ is unitary when $|z| = 1$. (Φ is a matrix inner function.)
Moreover, (3) says

$$\Phi(\lambda_i)x_i = \mu_i x_i.$$

Indeed, write out

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_i \\ \lambda_i y_i \end{pmatrix} = \begin{pmatrix} \mu_i x_i \\ y_i \end{pmatrix}$$

as

$$Ax_i + \lambda_i By_i = \mu_i x_i \tag{4}$$

$$Cx_i + \lambda_i Dy_i = y_i \tag{5}$$

Solve second equation to get

$$y_i = (I - \lambda_i D)^{-1} Cx_i,$$

substitute into first equation to get

$$\Phi(\lambda_i)x_i = \mu_i x_i.$$

The eigenvalue equation

$$\Phi(\lambda_i)x_i = \mu_i x_i.$$

says that each point (λ_i, μ_i) of the joint spectrum lies in the variety

$$\det(wI - \Phi(z)) = 0.$$

Arranging the algebra differently, there is another matrix inner function Ψ so that

$$V := \{\det(wI_{d_1} - \Phi(z)) = 0\} = \{\det(zI_{d_2} - \Psi(w)) = 0\}.$$

This is a distinguished variety!

Theorem (Agler-McCarthy 2005, Knese 2009)

If Φ is a rational matrix inner function, then

$$\det(wI - \Phi(z)) = 0$$

defines a distinguished variety; conversely every distinguished variety may be put in this form.

For example, if we put

$$\Phi(z) = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}$$

then

$$\det(wI_2 - \Phi(z)) = \det \begin{pmatrix} w & -z \\ -z^2 & w \end{pmatrix} = w^2 - z^3,$$

the Neil parabola.

Nevanlinna-Pick interpolation on V

General question: given points

$$s_1, \ldots s_n$$

in a (domain, surface, variety) Ω, and points

$$t_1, \ldots t_n$$

in the unit disk $\mathbb{D}$, when does there exist a holomorphic function $f : \Omega \to \mathbb{D}$ with

$$f(s_i) = t_i, \quad \text{all } i = 1, \ldots n \text{ ?}$$

For $\Omega = \mathbb{D}$, the answer is

Theorem (Pick(1916), Nevanlinna(1919),...)

The interpolation problem has a solution if and only if the $n \times n$ matrix

$$\frac{1 - t_i \overline{t_j}}{1 - s_i \overline{s_j}}$$

is positive semidefinite.

On V? A family of matrices:
For each matrix inner function giving a determinantal representation

$$V = \{\det(wI - \Phi(z)) = 0\},$$

there is an analytic (on V) family of "eigenvectors" $Q(z, w)$

$$Q(z, w)\Phi(z) = wQ(z, w)$$

for all $(z, w) \in V$.
Example:

$$\begin{pmatrix} w & z \end{pmatrix} \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} z^3 & wz \end{pmatrix} = w \begin{pmatrix} w & z \end{pmatrix}$$

since $z^3 = w^2$ on $\mathcal{N}$

In addition to

$$Q(z,w)\Phi(z) = wQ(z,w)$$

we have $P(z,w)$ so that for the "companion" matrix inner function Ψ,

$$P(z,w)\Psi(w) = zP(z,w)$$

For $(z,w) \in V$, $(\zeta,\eta) \in V$, we have the identity

$$\frac{Q(z,w)Q(\zeta,\eta)^*}{1 - z\overline{\zeta}} = \frac{P(z,w)P(\zeta,\eta)^*}{1 - w\overline{\eta}}$$

$$\frac{Q(z,w)Q(\zeta,\eta)^*}{1-z\overline{\zeta}} = \frac{P(z,w)P(\zeta,\eta)^*}{1-w\overline{\eta}}$$

Call the above expression $K((z,w),(\zeta,\eta))$. Then

Theorem (Knese, McCullough, J.)

The interpolation problem on V

$$(z_i,w_i) \to t_i$$

has a solution if and only if the matrices

$$(1-t_i\overline{t_j})K((z_i,w_i),(z_j,w_j))$$

are all positive semidefinite, over all choices of determinantal representation.

For the Neil parabola, we had

$$H^\infty(\mathcal{N}) \cong \{f \in H^\infty(\mathbb{D}) : f'(0) = 0\}.$$

Interpolation theorem recovers result of Davidson-Paulsen-Raghupathi-Singh (2009).

Bigger picture: each K is a reproducing kernel for a Hilbert space of analytic functions on V; the operators

$$f \to zf, \quad f \to wf$$

on these spaces are commuting isometries with joint spectrum in ∂V; basically all such pairs arise this way.