Function Spaces on Varieties

Michael Dritschel (Newcastle), Greg Knese (U Alabama), Scott McCullough (U Florida)

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Consider a polynomial $p \in \mathbb{C}[z, w]$. Its zero variety is the set

$$\mathcal{Z}_p =: \{(z, w) \in \mathbb{C}^2 : p(z, w) = 0\}$$

Say p defines a **distinguished variety** if

$$\text{for all } (z, w) \in \mathcal{Z}_p, |z| = 1 \text{ if and only if } |w| = 1. \qquad \text{(DV)}$$

Examples:

- $p(z, w) = z - w$ (boring)
- $p(z, w) = z^2 - w^2 = (z - w)(z + w)$ (mostly boring)
- $p(z, w) = z^3 - w^2$ (not boring)

The zero variety in the last example

$$p(z, w) = z^3 - w^2$$

is called the **Neil parabola**. Since

$$\operatorname{grad}_{\mathbb{C}} p = \left(\frac{\partial p}{\partial z}, \frac{\partial p}{\partial w} \right) = (3z^2, -2w),$$

the origin $(0, 0)$ is a **singular point** (in this case, a **cusp**).

Mostly we are interested in analytic functions on

$$\mathcal{V} = Z_p \cap \mathbb{D}^2.$$

Notice $\partial\mathcal{V} \subset \partial\mathbb{D} \times \partial\mathbb{D} \subsetneq \partial(\mathbb{D} \times \mathbb{D})$.

More examples:

$$z^m = \prod_{j=1}^{n} \frac{a_j - w}{1 - a_j^* w}, \text{ with } a_j \text{ distinct, nonzero}$$

The set

$$\mathcal{V} = \left\{ (z, w) \in \mathbb{D}^2 : z^m = B_n(z) \right\}$$

is a smooth distinguished variety; in fact it is a finite Riemann surface.

$$\mathcal{V} = \{z^m = \prod_{j=1}^{n} \frac{a_j - w}{1 - a_j^* w}\} \cap \mathbb{D}^2$$

defines a finite Riemann surface of genus

$$g = \frac{(m-1)(n-1) - (k-1)}{2}$$

with $k = gcd(m, n)$ disks removed.

- $m = n = 2$: annulus, every annulus arises this way (Bell)
- every finitely connected planar domain is a variety, but not smooth if 2 or more holes (Rudin, Fedorov)

Definition

Let $\mathcal{V}$ be a distinguished variety. A function $f : \mathcal{V} \to \mathbb{C}$ is holomorphic at $(z, w) \in V$ if there exist:

- a neighborhood Ω of (z, w) in $\mathbb{C}^2$, and
- a holomorphic function $F : \Omega \to \mathbb{C}$

such that

$$F|_{\mathcal{V} \cap \Omega} = f|_{\mathcal{V} \cap \Omega}.$$

FACT (H. Cartan): If f is holomorphic on $\mathcal{V}$ then there exists an F holomorphic on $\mathbb{D}^2$ such that $F|_V = f$.

Example: what do the holomorphic functions on the Neil parabola $\mathcal{N}$ look like?

The Neil parabola

$$\mathcal{N} = \{(z, w) \in \mathbb{D}^2 : z^3 = w^2\}$$

is paramaterized by the disk: the map

$$\psi : t \to (t^2, t^3)$$

is a holomorphic bijection of $\mathbb{D}$ with $\mathcal{N}$.

$f \to f \circ \psi$ is an isomorphism of $Hol(\mathcal{N})$ with

$$\{f \in Hol(\mathbb{D}) : f'(0) = 0\}.$$

$$\mathcal{N} = \{(z, w) \in \mathbb{D}^2 : z^3 = w^2\}$$
$$\psi(t) = (t^2, t^3)$$

Take f holmorphic on $\mathcal{N}$, extend to $\mathbb{D}^2$ (Cartan): then

$$f(\psi(t)) = \sum_{m,n=0}^{\infty} a_{mn}(t^2)^m(t^3)^n = \sum_{k\neq 0} b_k t^k$$

Conversely for $k \neq 1$ write $t^k = (t^2)^m(t^3 n)$, then

$$\sum_{k\neq 0} b_k t^k = \sum_{m,n} b_{2m+3n}(t^2)^m(t^3)^n$$

Conclusion:

$$\mathrm{Hol}(\mathcal{N}) \cong \{f \in \mathrm{Hol}(\mathbb{D}) : f'(0) = 0\}$$

This was a special case of a general fact:

Theorem (Agler-McCarthy,2007)

Let $\mathcal{V} \subset \mathbb{D}^2$ be a distinguished variety. Then there exist:

- *a finite Riemann surface R,*
- *a holomorphic map $\psi : R \to \mathcal{V}$, and*
- *a finite codimension subalgebra $\mathcal{A} \subset Hol(R)$*

such that

$$f \to f \circ \psi$$

is an isomrophism of $Hol(\mathcal{V})$ with $\mathcal{A}$.

The map ψ has the form

$$t \to (\psi_1(t), \psi_2(t))$$

with $|\psi_1| = |\psi_2| = 1$ on ∂R, and $p(\psi_1, \psi_2) = 0$. In other words, an **algebraic pair of inner functions** on R.

Determinantal representations:

Theorem (Agler-McCarthy 2005, Knese 2009)

If Φ is a rational matrix inner function, then

$$\det(wI - \Phi(z)) = 0$$

defines a distinguished variety; conversely every distinguished variety may be put in this form.

Example:

$$\Phi(z) = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}$$

then

$$\det(wI_2 - \Phi(z)) = \det \begin{pmatrix} w & -z \\ -z^2 & w \end{pmatrix} = w^2 - z^3,$$

the Neil parabola.

Idea of proof:

Let $\mathcal{V} = \mathcal{Z}_q \cap \mathbb{D}^2$ and choose a nice measure $d\mu$ on $\partial\mathcal{V}$ (e.g. push down harmonic measure from ∂R.

Operators

$$M_z, \quad M_w \quad \in B(H^2(\mu)$$

are commuting isometries with $q(M_z, M_w) = 0$. The pair has a Sz.-Nagy–Foais model

$$M_z \cong S \otimes I_m, \quad M_w \cong \Phi(S)$$

for a matrix inner function Φ. Or we could take

$$M_z \cong \Psi(S), \quad M_w \cong S \otimes I_n$$

Then

$$\det(zI_m - \Phi(w)) = \det(wI_n - \Psi(z)) = 0$$

defines $\mathcal{V}$.

Application: Nevanlinna-Pick interpolation on V

General question: given points

$$s_1, \ldots s_n$$

in a (domain, surface, variety) Ω, and points

$$t_1, \ldots t_n$$

in the unit disk $\mathbb{D}$, when does there exist a holomorphic function $f : \Omega \to \mathbb{D}$ with

$$f(s_i) = t_i, \quad \text{all } i = 1, \ldots n \text{ ?}$$

Theorem (Pick(1916), Nevanlinna(1919),...)

For $\Omega = \mathbb{D}$, the interpolation problem has a solution if and only if the $n \times n$ matrix

$$\frac{1 - t_i t_j^*}{1 - s_i s_j^*}$$

is positive semidefinite.

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$$\frac{1 - t_i t_j^*}{1 - s_i s_j^*} = (1 - t_i t_j^*) k(s_i, s_j)$$

is positive semidefinite.

Determinantal representation $\implies$ reproducing kernels:

For each matrix inner function giving a determinantal representation

$$\mathcal{V} = \{\det(wI - \Phi(z)) = 0\},$$

there is an analytic (on V) family of "eigenvectors" $Q(z,w)$

$$Q(z,w)\Phi(z) = wQ(z,w)$$

for all $(z,w) \in \mathcal{V}$.

Example:

$$\begin{pmatrix} w & z \end{pmatrix} \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} = \begin{pmatrix} z^3 & wz \end{pmatrix} = w \begin{pmatrix} w & z \end{pmatrix}$$

since $z^3 = w^2$ on $\mathcal{N}$

In addition to

$$Q(z,w)\Phi(z) = wQ(z,w)$$

we have $P(z,w)$ so that for the "companion" matrix inner function Ψ,

$$P(z,w)\Psi(w) = zP(z,w)$$

For $(z,w) \in V$, $(\zeta,\eta) \in V$, we have the identity

$$\frac{Q(z,w)Q(\zeta,\eta)^*}{1 - z\overline{\zeta}} = \frac{P(z,w)P(\zeta,\eta)^*}{1 - w\overline{\eta}}$$

Conversely, if P, Q are vector polynomials satisfying the identity, the come from a determinantal representation.

$$\frac{Q(z,w)Q(\zeta,\eta)^*}{1-z\overline{\zeta}} = \frac{P(z,w)P(\zeta,\eta)^*}{1-w\overline{\eta}}$$

Call the above expression $K((z,w),(\zeta,\eta))$. Then

Theorem (Knese, McCullough, J.)

Let $\mathcal{V}$ be an irreducible distinguished variety. There exists $f \in H^\infty(\mathcal{V})$ with

$$f(z_i,w_i) = t_i \quad \text{and} \quad \|f\|_\infty \leq 1$$

if and only if the matrices

$$(1 - t_i t_j^*)K((z_i,w_i),(z_j,w_j))$$

are all positive semidefinite, over all choices of determinantal representation.

For the Neil parabola, we had

$$H^\infty(\mathcal{N}) \cong \{f \in H^\infty(\mathbb{D}) : f'(0) = 0\}.$$

Interpolation theorem recovers result of Davidson-Paulsen-Raghupathi-Singh

Ingredients of proof:

Step 1) The kernels

$$\frac{Q(z,w)Q(\zeta,\eta)^*}{1-z\overline{\zeta}} = \frac{P(z,w)P(\zeta,\eta)^*}{1-w\overline{\eta}} \tag{1}$$

over all determinantal representations form a complete Nevanlinna-Pick family: fix one such kernel k with associated RKHS $H^2(k)$. Fix $(z_0, w_0) \in \mathcal{V}$. The kernel for the subspace

$$\{f \in H^2(k) : f(z_0, w_0) = 0\}$$

has the form

$$\frac{1}{(1-zz_0^*)(1-ww_0^*)}\widetilde{K}((z,w),(\zeta,\eta))\frac{1}{(1-z_0\zeta^*)(1-w_0\eta^*)}$$

Step 2) Define a new norm on polynomials by

$$\|p\|_{\mathcal{K}} = \sup_{K}\{\|M_p\|_{B(H^2(K))}\}$$

Now by Step 1, interpolation problem is solved for $H^\infty_{\mathcal{K}}$ norm by a general interpolation theorem for kernel families (Knese-McCullough-J)

Step 3) Finally, prove the $\mathcal{K}$ norm equals the supremum norm on $\mathcal{V}$–this follows from

Lemma

If $f \in H^\infty(\mathcal{V})$ and $\|f\|_\infty \leq 1$, there is a sequence of polynomials $p_n \in \mathbb{C}[z, w]$ such that

- $|p_n| \leq 1$ *on* $\mathcal{V}$, *and*
- $p_n \to f$ *locally uniformly on* V

Dilations & spectral sets:

Let $q(z, w)$ define a distinguished variety $\mathcal{V}$.

Question: given commuting contractive operators $S, T \in B(H)$ with $q(S, T) = 0$, when do there exist commuting unitary operators $U, V \in B(K)$ with $q(U, V) = 0$ and

$$S^m T^n = P_H U^m V^n|_H \text{ ?}$$

Necessary: for all polynomials p,

$$\|p(S, T)\| \leq \|p\|_{\mathcal{V},\infty}$$

A homomorphism

$$\pi : \mathcal{A}(\mathcal{V}) \to B(H)$$

is **contractive** if

$$\|\pi(f)\|_{B(H)} \leq \|f\|_\infty$$

for all $f \in \mathcal{A}(V)$,
and **completely contractive** if

$$\|[\pi(f_{ij})]\|_{B(H)} \leq \|[f_{ij}\|_\infty$$

for all matrices $f \in M_n(\mathcal{A}(\mathcal{V}))$.

Arveson: S, T dilate if and only if the contractive homomorphism

$$\pi(p) = p(S, T)$$

is completely contractive.

Say **rational dilation holds on $\mathcal{V}$** if every contractive π is completely contractive.

Rational dilation holds on the disk (Sz.-Nagy dilation)
...holds on the annulus (Agler)
...fails on a two-holed planar domain (Dritschel-McCullough, Agler-Harman-Raphael)

Theorem (Dritschel, McCullough, J.)

Rational dilation fails on the Neil parabola $\mathcal{N}$.

In fact: there is a representation

$$\pi : \mathcal{A}(\mathcal{N}) \to M_{12\times 12}(\mathbb{C})$$

contractive but not 2-contractive.

As before identify $\mathcal{A}(\mathcal{N})$ with subalgebra of $\mathcal{A}(\mathbb{D})$:

$$\mathcal{A}(\mathcal{N}) = \{f \in \mathcal{A}(\mathbb{D}) \ : \ f'(0) = 0\}.$$

Convexity approach (Agler):

Let

$$\mathcal{P} = \left\{ \frac{1+f}{1-f} : f \in H^\infty(\mathcal{N}), f(0) = 1 \right\}$$

The set $\mathcal{P}$ is compact and convex, let

$$\mathcal{E} = \text{extreme points of } \mathcal{P}$$

For each $f \in \mathcal{P}$ we have a Choquet integral

$$\frac{1+f}{1-f} = \int_{\mathcal{E}} \frac{1+\phi_t}{1-\phi_t}\, d\mu(t)$$

The functions

$$\{\phi : \frac{1+\phi}{1-\phi} \in \mathcal{E}\}$$

are called test functions for $H^\infty(\mathcal{N})$.

Representing the unit ball of $H^\infty(\mathcal{N})$:

Rearringing the Choquet integral we have

$$1 - f(z)f(w)^* = \int_{\mathcal{E}} 1 - \phi_t(z)\phi_t(w)^* \, d\mu_{zw}(t)$$

where μ is a positive measure valued kernel on $\mathbb{D} \times \mathbb{D}$. To proceed we need this more explicitly...

$\mathbb{D}^* =$ one-point compactification of $\mathbb{D}$

$\psi_\lambda(z) = z^2 \frac{\lambda - z}{1 - \bar{\lambda} z}, \quad \psi_*(z) = z^2$ (test functions)

Theorem (Pickering)

If f lies in the unit ball of $\mathcal{A}(\mathcal{N})$ and is smooth across the boundary then

$$1 - f(z)f(w)^* = \int_{\mathbb{D}^*} (1 - \psi_t(z)\psi_t(w)^*)\, d\mu_{z,w}(t)$$

where μ is a positive $M(\mathbb{D}^*)$-valued kernel.

The test functions can be pushed back down to $\mathcal{N}$, we get

$$\phi_\lambda(z, w) = z\frac{\lambda z - w}{z - \lambda^* w}, \phi_*(z) = z$$

Pickering also shows no (closed) subcollection of test functions suffices.

Corollary: a pair of commuting, contractive, invertible matrices X, Y with $X^3 = Y^2$ give a contractive representation of $\mathcal{A}(\mathcal{N})$ if and only if

$$X(\lambda X - Y)(X - \lambda^* Y)^{-1}$$

is contractive for all $\lambda \in \mathbb{D}$.

Loosely, moving to matrix valued F, if F every $n \times n$ matrix function also has a representation

$$1 - F(z)F(w)^* = \int_{\mathbb{D}^*} (1 - \psi_t(z)\psi_t(w)^*)\, d\mu_{z,w}(t)$$

then one can pass (nice) representaitons inside the integral to conclude contractive implies completely contractive.

Conversely, let $\mathfrak{F}$ be a finite set and form the closed, convex cone

$$C_{\mathfrak{F}} = \left\{ H(z,w) = \int_{\mathbb{D}^*} (1 - \psi_t(z)\psi_t(w)^*)\, d\mu_{z,w}(t) \right\}$$

where $\mu_{z,w}$ are matrix-valued measures. If for some F we have

$$I - F(z)F(w)^* \notin C_{\mathfrak{F}}$$

then we can separate $I - F(z)F(w)^*$ from $C_{\mathfrak{F}}$ with a positive functional, apply GNS to get a representation that is contractive but NOT completely contractive.

Key step: if F is a matrix inner function in $M_2 \otimes \mathcal{A}(\mathcal{N})$, an integral representation for F imposes constraints on its zeroes...

$F(z) = z^2\Phi(z)$, $\quad\Phi$ rational, inner, degree 2,

Theorem

If F is representable as

$$I - F(z)F(w)^* = \int_{\mathbb{D}^*} (1 - \psi_t(z)\psi_t(w)^*)\, d\mu_{z,w}(t)$$

for z, w in a large finite set $\mathfrak{F}$, then either

$$\Phi \simeq \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} \quad or \quad \begin{pmatrix} 1 & 0 \\ 0 & \phi_1\phi_2 \end{pmatrix}$$