A NOTE ON HOMOTOPIES OF RATIONAL MATRIX INNER FUNCTIONS

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Abstract. We show that when $m > n$, the space of $m \times n$-matrix-valued rational inner functions is path connected.

A matrix-valued rational function is an $m \times n$ matrix $W(z)$ each of whose entries is a rational function $w_{ij}(z)$ of the complex variable z. Thus $W(z)$ is an $m \times n$ matrix valued function defined at all but (at most) finitely many points of the complex plane $\mathbb{C}$.

We let $\|W\|_\infty$ denote the supremum of $\|W(z)\|$ over the open unit disk $|z| < 1$, here $\|W(z)\|$ is the usual operator norm of the linear transformation $W(z)$ acting between the Euclidean spaces $\mathbb{C}^n$ and $\mathbb{C}^m$. If W is rational and $\|W\|_\infty < \infty$, then W extends continuously to the closed disk $|z| \leq 1$, and conversely. (Evidently this occurs if and only if W has no poles in $|z| \leq 1$, we will be working only with such functions.) We say an $m \times n$ rational matrix function is *inner* if $\|W\|_\infty \leq 1$ and $W(e^{i\theta})^* W(e^{i\theta}) = I_n$ for all $\theta \in [0, 2\pi]$. (Note that this condition forces $m \geq n$.) We will let $\mathcal{RIF}(m, n)$ denote the set of all $m \times n$ matrix rational inner functions. The set $\mathcal{RIF}(m, n)$ is equipped with the (metric) topology induced by the norm $\| \cdot \|_\infty$, which it inherits as a subset of the continuous $m \times n$ matrix valued functions in the disk, this coincides with the topology of uniform convergence in the closed disk $|z| \leq 1$. The purpose of this note is to prove the following:

Theorem. *If $m > n$ then the metric space $\mathcal{RIF}(m, n)$ is path connected.*

Remark: It is easy to see that in the square case, $\mathcal{RIF}(m, m)$ is not path connected. Indeed, by considering the winding number of the function $\det W(e^{i\theta})$ about the origin, one sees that, for example, $W(z) = zI_m$ cannot be joined to I_m by a path lying within $\mathcal{RIF}(m, m)$.

Proof. Since we are assuming $m > n$, it will be helpful to write elements of $\mathcal{RIF}(m, n)$ in block form as columns

$$W(z) = \begin{pmatrix} X(z) \\ Y(z) \end{pmatrix}$$

where $X(z)$ is an $n \times n$ rational matrix function and $Y(z)$ is $(m - n) \times n$. The fact that W is inner is then expressed by the condition $X(e^{i\theta})^* X(e^{i\theta}) + Y(e^{i\theta})^* Y(e^{i\theta}) \equiv I_n$.

We will prove that every $W = \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathcal{RIF}(m, n)$ can be joined to $\begin{pmatrix} I_n \\ O_{(m-n)\times n} \end{pmatrix}$ by a path in $\mathcal{RIF}(m, n)$, this evidently proves the theorem. This in turn is accomplished in two steps: first we prove that for any $W \in \mathcal{RIF}(m, n)$, there is a square matrix rational inner function $\Phi(z) \in \mathcal{RIF}(n, n)$ such that there is a path in $\mathcal{RIF}(m, n)$ joining W to $\begin{pmatrix} \Phi \\ O \end{pmatrix}$. (Here O is the $(m - n) \times n$ zero matrix, henceforth we will drop the size subscripts when they are clear from context.) Then we will show that any such $\begin{pmatrix} \Phi \\ O \end{pmatrix}$ can be joined to $\begin{pmatrix} I \\ O \end{pmatrix}$ in $\mathcal{RIF}(m, n)$.

Since $W^* W \equiv I$ on the circle, the matrix $W(e^{i\theta})$ has full rank n for each $\theta \in [0, 2\pi)$. In particular, the matrix $W(1)$ has n linearly independent rows, and by continuity this same set of rows is independent in $W(e^{i\theta})$ for θ in a neighborhood of 0. Multiplying W on the left by an $m \times m$ permutation matrix, we may arrange that these are the first n rows. Since the unitary group $\mathcal{U}(m)$ is path connected, and a unitary times a matrix *RIF* is again a *RIF*, it follows that the new W with permuted rows is connected by a path in

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$\mathcal{RIF}(m,n)$ to the original W. So, we may assume $W = \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathcal{RIF}(m,n)$ with $X(e^{i\theta})$ having full rank for θ in a neighborhood of 0. The rational matrix function X admits an inner-outer factorization $X = \Phi F$, where Φ is projection-valued on the circle and F is a matrix outer function satisfying $F^*F = X^*X$ on the unit circle; F will be unique if we additionally impose the condition that $F(0)$ be positive definite (which we do). From the theory of matrix inner-outer factorizations, F is also rational.[2, Section 6.8] Since $X(1)$ has full rank, it follows that $\Phi(1)$ has full rank n, but then by continuity $\operatorname{rank}(\Phi(e^{i\theta})) = \operatorname{trace}(\Phi(e^{i\theta})^*\Phi(e^{i\theta}))$ is constantly equal to n. Thus $\Phi \in \mathcal{RIF}(n,n)$. We may then write

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \Phi & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} F \\ Y \end{pmatrix}$$

Since $F^*F = X^*X$ on the circle, it follows that $V := \begin{pmatrix} F \\ Y \end{pmatrix}$ is inner, i.e. belongs to $\mathcal{RIF}(m,n)$. If we show that V can be joined to $\begin{pmatrix} I \\ 0 \end{pmatrix}$, then (since multiplication by $diag(\Phi, I)$ will carry $\mathcal{RIF}(m,n)$ into itself continuously) it will follow that W can be joined to $\begin{pmatrix} \Phi \\ 0 \end{pmatrix}$.

Now, for $0 \le t \le 1$ the $n \times n$ matrix function $Q_t(e^{i\theta}) = I - t^2 Y(e^{i\theta})^* Y(e^{i\theta})$ takes positive semidefinite values on the unit circle (in fact positive definite values when $0 \le t < 1$). Since Y is a rational matrix function, we can choose a polynomial p of minimal degree with the property that $\tilde{Y}(z) := p(z)Y(z)$ is a polynomial matrix function. (That is, p is a common denominator for the entries of Y.) Since Y has no poles in $|z| \le 1$, this minimal degree common denominator will have no zeroes in $|z| \le 1$, and we may normalize so that $p(0) > 0$. We then consider the nonnegative matrix-valued trigonometric polynomials Q_t given by

$$\tilde{Q}_t(e^{i\theta}) = \overline{p(e^{i\theta})}p(e^{i\theta})I_n - t^2 \tilde{Y}(e^{i\theta})^* \tilde{Y}(e^{i\theta}).$$

By the Fejer-Riesz theorem for matrix valued trigonometric polynomials [2, Section 6.6], there is an outer (analytic) polynomial matrix function $G_t(z)$, with $\deg G_t = \deg Q_t \le \max(\deg p, \deg Y)$, such that

$$\overline{p(e^{i\theta})}p(e^{i\theta})I_n - t^2 \tilde{Y}(e^{i\theta})^* \tilde{Y}(e^{i\theta}) = G_t(e^{i\theta})^* G_t(e^{i\theta}).$$

This G_t will be unique if we impose the requirement that $G_t(0)$ be positive definite. Doing this, in particular we will have $G_0(z) = p(z)I_n$ and $G_1(z) = p(z)F(z)$. Moreover, the outer factor G_t is determined as the unique matrix function H analytic in $|z| < 1$ which satisfies $H^*H \le \tilde{Q}_t$ and which maximizes $H(0)^*H(0)$ [2, Theorem C, Section 3.10]. In addition, since all the G_t have full rank and are outer, it follows that $\det G_t(z)$ is nonvanishing in $|z| < 1$ for all $0 \le t \le 1$. With these facts in hand we can prove that the map $t \to G_t$ is norm continuous on $[0,1]$. We must show that if $t_n \to t$ then $G_{t_n} \to G_t$ uniformly. Since the norms and degrees of the polynomials G_t are uniformly bounded, by compactness there will be a subsequence $G_{t_{n_k}}$ which converges uniformly in $|z| \le 1$ to some polynomial matrix function $H(z)$. Since $G_t(0)^*G_t(0) \ge G_1(0)^*G_1(0) = |p(0)|^2 F(0)^*F(0)$ for all t, and $F(0)$ is positive definite, it follows that $\det H(0) = \lim_k \det G_{t_{n_k}}(0) \ne 0$, and hence from Hurwitz's theorem that $\det H(z) = \lim \det G_{t_{n_k}}(z)$ is nonvanishing in $|z| < 1$, so (since H is polynomial) $H(z)$ is outer. But by uniform convergence it follows that $H(0) > 0$ and $\overline{p(e^{i\theta})}p(e^{i\theta})I_n - t^2 \tilde{Y}(e^{i\theta})^* \tilde{Y}(e^{i\theta}) = H(e^{i\theta})^* H(e^{i\theta})$ for all θ, so by uniqueness we must have $H = F_t$. Thus, for each fixed sequence $t_n \to t$, every subsequence of G_{t_n} has a subsequence converging to G_t, so the full sequence converges to G_t, and thus $t \to G_t$ is continuous. If we now put $F_t = p^{-1}G_t$, then each F_t is a rational matrix function satisfying

$$F_t(e^{i\theta})^* F_t(e^{i\theta}) + t^2 Y(e^{i\theta})^* Y(e^{i\theta}) \equiv I_n$$

(with $F_t(0)$ positive definite) for $0 \le t \le 1$, and the path $t \to F_t$ is continuous. By construction we have $F_0 = I_n$ and $F_1 = F$. Thus, the columns $\begin{pmatrix} F_t \\ tY \end{pmatrix}$ will belong to $\mathcal{RIF}(m,n)$, and form a path joining $\begin{pmatrix} F \\ Y \end{pmatrix}$ to

$\begin{pmatrix} I \\ 0 \end{pmatrix}$. Finally, if we put $X_t = \Phi F_t$, then $W_t := \begin{pmatrix} X_t \\ tY \end{pmatrix}$ is a continuous path in $\mathcal{RIF}(m,n)$ joining $W_0 = \begin{pmatrix} \Phi \\ 0 \end{pmatrix}$ to $W_1 = \begin{pmatrix} X \\ Y \end{pmatrix}$ as desired.

To carry out the second step of the proof, let $\Phi \in \mathcal{RIF}(n,n)$. By [1] Φ can be factored as a Blaschke-Potapov product

$$\Phi(z) = U\left(\prod_{k=1}^{N} (b_k(z)P_k + (I - P_k))\right)V$$

where U, V are constant unitary matrices, each $b_k(z)$ is a finite Blaschke product, and each P_k is a projection matrix. Each factor $b_k(z)P_k + (I - P_k)$ belongs to $\mathcal{RIF}(n,n)$. As noted above, since the unitary group is path connected we may assume $U = V = I_n$. Now let us write

$$\begin{pmatrix} \Phi(z) \\ 0 \end{pmatrix} = \begin{pmatrix} b_1(z)P_1 + (I - P_1) \\ 0 \end{pmatrix}\left(\prod_{k=2}^{N} (b_k(z)P_k + (I - P_k))\right)$$

Let us work with

$$\begin{pmatrix} b_1(z)P_1 + (I - P_1) \\ 0 \end{pmatrix} \tag{0.1}$$

Conjugating by a unitary we may assume $b_1(z)P_1 + (I - P_1)$ has the diagonal form

$$\begin{pmatrix} b_1(z) & & & & & \\ & \ddots & & & & \\ & & b_1(z) & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

Note that now, each column belongs to $\mathcal{RIF}(n,1)$. Within $\mathcal{RIF}(n+1,1)$ there is a path

$$t \to \begin{pmatrix} (1-t)b_1(z) + t \\ 0 \\ \vdots \\ 0 \\ (\sqrt{t - t^2})(1 - b_1(z)) \end{pmatrix}$$

joining $\begin{pmatrix} b_1(z) & 0 & \cdots & 0 & 0 \end{pmatrix}^T$ to $\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \end{pmatrix}^T$. Doing this in the first column of the matrix (0.1) leaves the other columns unaffected and the whole path will lie in $\mathcal{RIF}(m,n)$ (adding additional zeroes to the bottom of the column, if needed, to bring the size from $n+1$ up to m). We may thus successively move each diagonal entry $b_1(z)$ to 1. Thus, our original $\begin{pmatrix} \Phi \\ 0 \end{pmatrix}$ is now joined by a path in $\mathcal{RIF}(m,n)$ to

$$\begin{pmatrix} I_n \\ 0 \end{pmatrix}\left(\prod_{k=2}^{N} (b_k(z)P_k + (I - P_k))\right)$$

We may then absorb the next Blaschke-Potapov factor into the column:

$$\begin{pmatrix} b_2(z)P_2 + (I - P_2) \\ 0 \end{pmatrix}\left(\prod_{k=3}^{N} (b_k(z)P_k + (I - P_k))\right)$$

and repeat the process, so that in the end we see that $\begin{pmatrix} \Phi \\ 0 \end{pmatrix}$ is joined to $\begin{pmatrix} I_n \\ 0 \end{pmatrix}$ in $\mathcal{RIF}(m,n)$ as desired. $\square$

References

[1] V. P. Potapov. The multiplicative structure of J-contractive matrix functions. *Amer. Math. Soc. Transl. (2)*, pages 131–243, 1960.

[2] Marvin Rosenblum and James Rovnyak. *Hardy classes and operator theory*. Dover Publications, Inc., Mineola, NY, 1997. Corrected reprint of the 1985 original.

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