Function-theoretic aspects of Schur class mappings of the unit ball

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Definition

A function $\varphi : \mathbb{B}^d \to \mathbb{C}^d$ belongs to the Schur class on $\mathbb{B}^d$ if the Hermitian kernel

$$\frac{1 - \langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle}$$

is positive semidefinite.

If $\varphi$ is Schur class, then $\varphi$ is automatically holomorphic and bounded by 1.

The converse holds when $d = 1$ but fails when $d > 1$.

E.g. $\varphi(z_1, z_2) = (2z_1z_2, 0)$ is bounded by 1, but NOT Schur class.

Every linear fractional map of $\mathbb{B}^d$ is Schur class.

$$\varphi(z) = \frac{Az + B}{\langle z, C \rangle + D}$$
A (commuting) row contraction is a $d$-tuple of commuting operators

$$T = (T_1, \ldots, T_d)$$

such that

$$I - T_1 T_1^* - \cdots - T_d T_d^* \geq 0$$

To say that $\varphi = (\varphi_1, \ldots, \varphi_d)$ belongs to the Schur class roughly means that

$$\varphi(T) := (\varphi_1(T), \ldots \varphi_d(T))$$

is a row contraction whenever $T$ is.
Define

\[ C_\varphi f := f \circ \varphi \]

**Theorem (Littlewood, 1925)**

*Let \( \varphi \) be a holomorphic self-map of \( \mathbb{D} \) with \( \varphi(0) = 0 \). Then

\[
\int_0^{2\pi} |f(\varphi(e^{i\theta}))|^2 \, d\theta \leq \int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta
\]

for all \( f \in H^2 \). Equivalently,

\[ \| C_\varphi \| \leq 1. \]

The analogous result is utterly false when \( d > 1 \); in fact \( C_\varphi \) is not even bounded on \( H^2(\mathbb{B}^d) \) in general, and even for very nice \( \varphi \).

E.g. \( \varphi(z_1, z_2) = (2z_1z_2, 0) \)
Proof 1 (Littlewood, 1925):

Two observations:

- \( zH^2 \perp \mathbb{C}1 \)
- \( \| \varphi f \|_2 \leq \| f \|_2 \) for all \( f \in H^2 \)

We may assume \( f \) is a polynomial.

\[
\| f \circ \varphi \|_2^2 = \| a_0 + a_1 \varphi + \cdots + a_n \varphi^n \|_2^2 \\
= |a_0|^2 + \| a_1 \varphi + \cdots + a_n \varphi^n \|_2^2 \\
\leq |a_0|^2 + \| a_1 + a_2 \varphi + \cdots + a_n \varphi^{n-1} \|_2^2 \\
\leq \ldots \text{iterate} \ldots \\
\leq |a_0|^2 + |a_1|^2 + \cdots + |a_n|^2 \\
= \| f \|_2^2
\]
Proof 2 (J., 2007):

Again, two observations:

- Since $|\varphi(z)| < 1$ for all $z \in \mathbb{D}$, the kernel

\[ \frac{1}{1 - \varphi(z)\varphi(w)} \]

is positive semidefinite.

- Since $\varphi$ is a contractive multiplier of $H^2$, the kernel

\[ k_\varphi(z, w) = \frac{1 - \varphi(z)\varphi(w)}{1 - zw} \]

is positive semidefinite; and since $\varphi(0) = 0$ the kernel

\[ k_\varphi(z, w) - \frac{k_\varphi(z, 0)k_\varphi(0, w)}{k_\varphi(0, 0)} = \frac{1 - \varphi(z)\varphi(w)}{1 - zw} - 1 \]

is also positive (Schur complement theorem)
Proof 2, continued.

Consider the Szegő kernel

\[ k_w(z) = \frac{1}{1 - zw} \]

Note that \( C^* \phi_1 k_w = k_{\phi(w)} \).

We must show \( \| C_{\phi} \| \leq 1 \); or equivalently \( I - C_{\phi} C^* \geq 0 \). Test against \( k \):

\[
\langle (I - C_{\phi} C^*) k_w, k_z \rangle = \langle k_w, k_z \rangle - \langle k_{\phi(w)}, k_{\phi(z)} \rangle
\]

\[
= \frac{1}{1 - zw} - \frac{1}{1 - \phi(z)\phi(w)}
\]

\[
= \left( \frac{1}{1 - \phi(z)\phi(w)} \right) \cdot \left( \frac{1 - \phi(z)\phi(w)}{1 - zw} - 1 \right)
\]

which is positive, so done.
Let $H^2_{d,m}$ denote the RKHS with kernel

$$k_m(z, w) = \frac{1}{(1 - \langle z, w \rangle)^m} \quad m = 1, 2, \ldots$$

Proof 2 generalizes immediately to the ball, provided $\varphi$ belongs to the Schur class:

**Theorem (J., 2007)**

Let $\varphi$ be a Schur class mapping of $\mathbb{B}^d$ and $\varphi(0) = 0$. Then $\|C_\varphi\| \leq 1$ on $H^2_{d,m}$. In particular (for $m = d$)

$$\int_{\partial \mathbb{B}^d} |f \circ \varphi|^2 \, d\sigma \leq \int_{\partial \mathbb{B}^d} |f|^2 \, d\sigma$$

for all $f$ in the classical Hardy space $H^2(\mathbb{B}^d)$. ($\sigma =$ surface measure on $\partial \mathbb{B}^d$)
the case $m = 1$.

As in the disk case, the kernels

$$\frac{1}{1 - \langle \varphi(z), \varphi(w) \rangle}, \quad \frac{1 - \langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle} - 1$$

are positive (since $\varphi$ is Schur class!!!)

Thus

$$\langle (I - C_\varphi C_\varphi^*) k_w, k_z \rangle = \left( \frac{1}{1 - \langle \varphi(z), \varphi(w) \rangle} \right) \cdot \left( \frac{1 - \langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle} - 1 \right)$$

is a positive kernel.
If 0 is not fixed, we have:

**Theorem**

Let $\varphi$ be a Schur class mapping of $\mathbb{B}^d$. Then $C_\varphi$ is bounded on each of the spaces $H^2_{d,m}$, and

$$
\left(\frac{1}{1 - |\varphi(0)|^2}\right)^{m/2} \leq \|C_\varphi\|_m \leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\right)^{m/2}
$$

Two nice things:
- both sides are roughly the same size, $\sim (1 - |\varphi(0)|)^{-m/2}$
- the inequality iterates...
The norm inequality

\[
\left( \frac{1}{1 - |\varphi(0)|^2} \right)^{m/2} \leq \| C_\varphi \|_m \leq \left( \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{m/2}
\]

iterates to give

\[
\| C^n_\varphi \|_m \sim (1 - |\varphi_n(0)|)^{-m/2}
\]

[here \( \varphi_n = \varphi \circ \cdots \circ \varphi, \ n \text{ times} \)], and hence

**Corollary (Spectral radius)**

Let \( \varphi \) be a Schur class mapping of the ball. The spectral radius of \( C_\varphi \) acting on \( H^2_{d,m} \) is

\[
r(C_\varphi) = \lim_{n \to \infty} (1 - |\varphi_n(0)|)^{-m/2n}
\]

Can we evaluate this limit?
Theorem (MacCluer, 1983)

Let $\varphi$ be a holomorphic self-map of $\mathbb{B}^d$. Then:

1. There exists a unique point $\zeta \in \overline{\mathbb{B}}^d$ (the Denjoy-Wolff point) such that
   \[ \varphi_n(z) \to \zeta \]
   locally uniformly in $\mathbb{B}^d$.

2. If $\zeta \in \partial \mathbb{B}^d$, then
   \[ 0 < \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} = \alpha \leq 1 \]

If the Denjoy-Wolff point $\zeta$ lies in $\mathbb{B}^d$, then $\varphi$ is called elliptic.

If $\zeta \in \partial \mathbb{B}^d$, the number $\alpha$ is called the dilatation coefficient of $\varphi$.

The map $\varphi$ is called parabolic if $\alpha = 1$, and hyperbolic if $\alpha < 1$. 
We want to evaluate

$$
\lim_{n \to \infty} (1 - |\varphi_n(0)|)^{-1/2n}
$$

If $\varphi$ is elliptic or parabolic, it is not hard to show this limit is 1.

In one dimension we have:

**Theorem (C. Cowen, 1983)**

*If $\varphi$ is an elliptic or parabolic self-map of $\mathbb{D}$, then the spectral radius of $C_\varphi$ (on $H^2$) is 1.*

*If $\varphi$ is hyperbolic with dilatation coefficient $\alpha$, then the spectral radius is $\alpha^{-1/2}$.***

Goal: Extend this theorem to Schur class mappings of $\mathbb{B}^d$.

The elliptic and parabolic cases go through (with identical proofs). The hyperbolic case takes work...
Definition

Given a point $\zeta \in \partial \mathbb{B}^d$ and a real number $c > 0$, the Koranyi region $D_c(\zeta)$ is the set

$$D_c(\zeta) = \left\{ z \in \mathbb{B}^d : |1 - \langle z, \zeta \rangle| \leq \frac{c}{2} (1 - |z|^2) \right\}$$

A function $f$ has K-limit equal to $L$ at $\zeta$ if

$$\lim_{z \to \zeta} f(z) = L$$

whenever $z \to \zeta$ within a Koranyi region.

When $d = 1$ (the disk), K-limit is the same as non-tangential limit. Not so in the ball...
Slice of a Koranyi region with vertex at \((1, 0)\) in \(\mathbb{B}^2\): \[z_2 = 0\]
Slice of a Koranyi region with vertex at \((1, 0)\) in \(\mathbb{B}^2\):

\[ \text{Im} z_1 = \text{Im} z_2 = 0 \]
\[ \alpha = \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} < \infty. \]

\textbf{Theorem (Rudin, 1980)}

\textit{Suppose} \( \varphi = (\varphi_1, \ldots, \varphi_d) \) \textit{is a holomorphic mapping from} \( \mathbb{B}^d \) \textit{to itself satisfying condition (C) at} \( e_1 \). \textit{The following functions are then bounded in every Koranyi region with vertex at} \( e_1 \):}

\begin{enumerate}
  \item \( \frac{1 - \varphi_1(z)}{1 - z_1} \)
  \item \( (D_1 \varphi_1)(z) \)
  \item \( \frac{1 - |\varphi_1(z)|^2}{1 - |z_1|^2} \)
  \item \( \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \)
\end{enumerate}

\textit{Moreover, each of these functions has \textit{restricted K-limit} \( \alpha \) at} \( e_1 \). 

\textbf{What is a restricted K-limit?}
Fix a point $\zeta \in \partial \mathbb{B}^d$ and consider a curve $\Gamma : [0, 1) \to \mathbb{B}^n$ such that $\Gamma(t) \to \zeta$ as $t \to 1$. Let $\gamma(t) = \langle \Gamma(t), \zeta \rangle \zeta$ be the projection of $\Gamma$ onto the complex line through $\zeta$. The curve $\Gamma$ is called **special** if

$$\lim_{t \to 1} \frac{|\Gamma - \gamma|^2}{1 - |\gamma|^2} = 0$$

and **restricted** if it is special and in addition

$$\frac{|\zeta - \gamma|}{1 - |\gamma|^2} \leq A$$

for some constant $A > 0$.

**Definition**

We say that a function $f : \mathbb{B}^d \to \mathbb{C}$ has **restricted K-limit** $L$ at $\zeta$ if $\lim_{z \to \zeta} f(z) = L$ along every restricted curve.

We have

$$\text{K-limit} \implies \text{restricted K-limit} \implies \text{non-tangential limit}$$

and each implication is strict when $d > 1$. 
One more fact:

**Theorem**

Let $\varphi$ be a hyperbolic, holomorphic self-map of $\mathbb{B}^d$ with Denjoy-Wolff point $\zeta$. Then

$$\varphi_n(z_0) \to \zeta$$

within a Koranyi region for every $z_0 \in \mathbb{B}^d$.

C. Cowen 1981 ($d = 1$)

Bracci, Poggi-Corradini 2003 ($d > 1$)

If we knew that some orbit $\{\varphi_n(z_0)\}$ approached the Denjoy-Wolff point **restrictedly**, this combined with Rudin’s Julia-Caratheodory theorem would imply

$$\lim_{n \to \infty} (1 - |\varphi_n(z_0)|)^{-1/2n} = \alpha^{-1/2}$$

It is not known if such orbits always exist. (Yes, if $\varphi$ is an LFT.)
Indeed, if we know some orbit \( z_n := \varphi_n(z_0) \) approaches \( \zeta \) restrictedly, then
\[
\lim_{n \to \infty} \frac{1 - |\varphi(z_n)|}{1 - |\varphi(z_{n-1})|} = \alpha
\]
and hence
\[
\lim_{n \to \infty} (1 - |\varphi_n(z_0)|)^{1/n} = \lim_{n \to \infty} \left( \prod_{k=1}^{n} \frac{1 - |\varphi(z_k)|}{1 - |\varphi(z_{k-1})|} \right)^{1/n}
= \alpha
\]
\[ \alpha = \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} < \infty. \] (C)

Theorem (J., to appear)

Let \( \varphi \) be a Schur class map and \( \zeta \in \partial B^d \). Then the following are equivalent:

1. **Condition (C).**
2. There exists \( \xi \in \partial B^d \) such that the function
   \[ h(z) = \frac{1 - \langle \varphi(z), \xi \rangle}{1 - \langle z, \zeta \rangle} \]
   belongs to \( \mathcal{H}(\varphi) \).
3. Every \( f \in \mathcal{H}(\varphi) \) has a finite K-limit at \( \zeta \).

\( d = 1 \) case: Sarason 1994
Theorem (J., to appear)

Suppose $\varphi = (\varphi_1, \ldots, \varphi_d)$ is a holomorphic Schur class mapping from $\mathbb{B}^d$ to itself satisfying condition (C) at $e_1$. The following functions are then bounded in every Koranyi region with vertex at $e_1$:

(i) \[ \frac{1 - \varphi_1(z)}{1 - \bar{z}_1} \]

(ii) \[ (D_1 \varphi_1)(z) \]

(iii) \[ \frac{1 - |\varphi_1(z)|^2}{1 - |z_1|^2} \]

(iv) \[ \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \]

Moreover, each of these functions (i)-(iii) has restricted K-limit $\alpha$ at $e_1$.

Unfortunately, our attempted argument works only if (iv) has a K-limit at $e_1$, which is not true in general even if $\varphi$ is Schur class. Nonetheless, this theorem is sufficient to solve the spectral radius problem, in a more indirect way....
For \( 0 < \alpha < 1 \), let \( \theta_\alpha \) denote the disk automorphism

\[
\theta_\alpha(z) = \frac{z + \left(\frac{1-\alpha}{1+\alpha}\right)}{1 + \left(\frac{1-\alpha}{1+\alpha}\right)z}
\]

**Theorem (J., in progress)**

Let \( \varphi \) be a hyperbolic Schur class self-map of \( \mathbb{B}^d \) with dilatation coefficient \( \alpha \). Then there exists a nonconstant Schur class map \( \sigma : \mathbb{B}^d \to \mathbb{D} \) such that

\[
\sigma \circ \varphi = \theta_\alpha \circ \sigma
\]

Proof needs\(^1\) strengthened Julia-Caratheodory theorem.

\(^1\) Probably.

\((d = 1: \) Valiron, 1931; also Pommerenke 1979, C. Cowen 1981\)

\((d > 1, \) under different assumptions:

\textbf{Bracci, Gentili, Poggi-Corradini, 2007} \)
Given a **Schur class** solution $\sigma$ to the Abel-Schroeder equation

$$\sigma \circ \varphi = \theta_\alpha \circ \sigma$$

we can compute the spectral radius of $C_\varphi$ via “transference.” Suppose $F$ is holomorphic in $\mathbb{D}$ and $\lambda \in \mathbb{C}$ satisfies

$$F \circ \theta_\alpha = \lambda F$$

Then

$$F \circ \sigma \circ \varphi = F \circ \theta_\alpha \circ \sigma$$

$$= \lambda F \circ \sigma$$

Formally, $\sigma$ transfers eigenfunctions of $C_{\theta_\alpha}$ to eigenfunctions of $C_\varphi$.

**KEY FACT:** If $F \in H^2(\mathbb{D})$ and $\sigma$ Schur class, then $F \circ \sigma \in H^2_{d,1}$. 
The transference technique then proves:

**Theorem (J., in progress)**

Let \( \varphi \) be a hyperbolic Schur class map of \( \mathbb{B}^d \) with dilatation coefficient \( \alpha \). Then the spectral radius of \( C_{\varphi} \) acting on \( H_{d,m}^2 \) is \( \alpha^{-m/2} \). Moreover every complex number \( \lambda \) in the annulus

\[
\alpha^{m/2} < |\lambda| < \alpha^{-m/2}
\]

is an eigenvalue of \( C_{\varphi} \) of infinite multiplicity.

\((d = 1 \text{ case: C. Cowen, 1983})\)

If \( d > 1 \) and \( \varphi \) is an automorphism, the closure of this annulus is equal to the spectrum of \( C_{\varphi} \) \((\text{MacCluer, 1984})\)