Function-theoretic aspects of Schur class mappings of the unit ball

Michael Jury

University of Florida

July 23, 2008

Definition

A function $\varphi : \mathbb{B}^d \to \mathbb{C}^d$ belongs to the Schur class on $\mathbb{B}^d$ if the Hermitian kernel

$$\frac{1 - \langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle}$$

is positive semidefinite.

If φ is Schur class, then φ is automatically holomorphic and bounded by 1.

The converse holds when $d = 1$ but fails when $d > 1$.

E.g. $\varphi(z_1, z_2) = (2z_1 z_2, 0)$ is bounded by 1, but NOT Schur class.

Every linear fractional map of $\mathbb{B}^d$ is Schur class.

$$\varphi(z) = \frac{Az + B}{\langle z, C \rangle + D}$$

A (commuting) row contraction is a d-tuple of commuting operators

$$T = (T_1, \ldots, T_d)$$

such that

$$I - T_1 T_1^* - \cdots - T_d T_d^* \geq 0$$

To say that $\varphi = (\varphi_1, \ldots, \varphi_d)$ belongs to the Schur class roughly means that

$$\varphi(T) := (\varphi_1(T), \cdots \varphi_d(T))$$

is a row contraction whenever T is.

Define

$$C_\varphi f := f \circ \varphi$$

Theorem (Littlewood, 1925)

Let φ be a holomorphic self-map of $\mathbb{D}$ with $\varphi(0) = 0$. Then

$$\int_0^{2\pi} |f(\varphi(e^{i\theta}))|^2 \, d\theta \leq \int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta$$

for all $f \in H^2$. Equivalently,

$$\|C_\varphi\| \leq 1.$$

The analogous result is utterly false when $d > 1$; in fact C_φ is not even bounded on $H^2(\mathbb{B}^d)$ in general, and even for very nice φ.

E.g. $\varphi(z_1, z_2) = (2z_1z_2, 0)$

Proof 1 (Littlewood, 1925):

Two observations:

- $zH^2 \perp \mathbb{C}1$
- $\|\varphi f\|_2 \leq \|f\|_2$ for all $f \in H^2$

We may assume f is a polynomial.

$$
\begin{aligned}
\|f \circ \varphi\|^2 &= \|a_0 + a_1\varphi + \cdots + a_n\varphi^n\|^2 \\
&= |a_0|^2 + \|a_1\varphi + \cdots + a_n\varphi^n\|^2 \\
&\leq |a_0|^2 + \|a_1 + a_2\varphi + \cdots + a_n\varphi^{n-1}\|^2 \\
&\leq \text{....iterate....} \\
&\leq |a_0|^2 + |a_1|^2 + \cdots + |a_n|^2 \\
&= \|f\|^2
\end{aligned}
$$

$\square$

Proof 2 (J., 2007):

Again, two observations:

- Since $|\varphi(z)| < 1$ for all $z \in \mathbb{D}$, the kernel

$$\frac{1}{1 - \varphi(z)\overline{\varphi(w)}}$$

  is positive semidefinite.

- Since φ is a contractive multiplier of H^2, the kernel

$$k_\varphi(z, w) = \frac{1 - \varphi(z)\overline{\varphi(w)}}{1 - z\overline{w}}$$

  is positive semidefinite; and since $\varphi(0) = 0$ the kernel

$$k_\varphi(z, w) - \frac{k_\varphi(z, 0)k_\varphi(0, w)}{k_\varphi(0, 0)} = \frac{1 - \varphi(z)\overline{\varphi(w)}}{1 - z\overline{w}} - 1$$

  is also positive (Schur complement theorem)

Proof 2, continued.

Consider the Szegő kernel

$$k_w(z) = \frac{1}{1 - z\overline{w}}$$

Note that $C_\varphi^* k_w = k_{\varphi(w)}$.
We must show $\|C_\varphi\| \leq 1$; or equivalently $I - C_\varphi C_\varphi^* \geq 0$. Test against k:

$$\begin{aligned}
\langle (I - C_\varphi C_\varphi^*) k_w, k_z \rangle &= \langle k_w, k_z \rangle - \langle k_{\varphi(w)}, k_{\varphi(z)} \rangle \\
&= \frac{1}{1 - z\overline{w}} - \frac{1}{1 - \varphi(z)\overline{\varphi(w)}} \\
&= \left(\frac{1}{1 - \varphi(z)\overline{\varphi(w)}} \right) \cdot \left(\frac{1 - \varphi(z)\overline{\varphi(w)}}{1 - z\overline{w}} - 1 \right)
\end{aligned}$$

which is positive, so done. $\square$

Let $H^2_{d,m}$ denote the RKHS with kernel

$$k_m(z,w) = \frac{1}{(1-\langle z,w\rangle)^m} \quad m = 1,2,\ldots$$

Proof 2 generalizes immediately to the ball, provided φ belongs to the Schur class:

Theorem (J., 2007)

Let φ be a Schur class mapping of $\mathbb{B}^d$ and $\varphi(0)=0$. Then $\|C_\varphi\| \leq 1$ on $H^2_{d,m}$. In particular (for $m=d$)

$$\int_{\partial\mathbb{B}^d} |f\circ\varphi|^2\, d\sigma \leq \int_{\partial\mathbb{B}^d} |f|^2\, d\sigma$$

for all f in the classical Hardy space $H^2(\mathbb{B}^d)$.
(σ = surface measure on $\partial\mathbb{B}^d$)

the case $m = 1$.

As in the disk case, the kernels

$$\frac{1}{1 - \langle\varphi(z), \varphi(w)\rangle}, \qquad \frac{1 - \langle\varphi(z), \varphi(w)\rangle}{1 - \langle z, w\rangle} - 1$$

are positive (since φ is Schur class!!!)

Thus

$$\langle (I - C_\varphi C_\varphi^*) k_w, k_z\rangle = \left(\frac{1}{1 - \langle\varphi(z), \varphi(w)\rangle}\right) \cdot \left(\frac{1 - \langle\varphi(z), \varphi(w)\rangle}{1 - \langle z, w\rangle} - 1\right)$$

is a positive kernel. □

If 0 is not fixed, we have:

Theorem

Let φ be a Schur class mapping of $\mathbb{B}^d$. Then C_φ is bounded on each of the spaces $H^2_{d,m}$, and

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{m/2} \leq \|C_\varphi\|_m \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{m/2}$$

Two nice things:

- both sides are roughly the same size, $\sim (1-|\varphi(0)|)^{-m/2}$
- the inequality iterates...

The norm inequality

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{m/2} \leq \|C_\varphi\|_m \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{m/2}$$

iterates to give

$$\|C_\varphi^n\|_m \sim (1-|\varphi_n(0)|)^{-m/2}$$

[here $\varphi_n = \varphi \circ \cdots \circ \varphi$, n times], and hence

Corollary (Spectral radius)

Let φ be a Schur class mapping of the ball. The spectral radius of C_φ acting on $H^2_{d,m}$ is

$$r(C_\varphi) = \lim_{n\to\infty} (1-|\varphi_n(0)|)^{-m/2n}$$

Can we evaluate this limit?

Theorem (MacCluer, 1983)

Let φ be a holomorphic self-map of $\mathbb{B}^d$. Then:

1. *There exists a unique point $\zeta \in \overline{\mathbb{B}^d}$ (the Denjoy-Wolff point) such that*

$$\varphi_n(z) \to \zeta$$

locally uniformly in $\mathbb{B}^d$.

2. *If $\zeta \in \partial\mathbb{B}^d$, then*

$$0 < \liminf_{z\to\zeta} \frac{1-|\varphi(z)|^2}{1-|z|^2} = \alpha \leq 1$$

If the Denjoy-Wolff point ζ lies in $\mathbb{B}^d$, then φ is called elliptic.

If $\zeta \in \partial\mathbb{B}^d$, the number α is called the dilatation coefficient of φ.

The map φ is called parabolic if $\alpha = 1$, and hyperbolic if $\alpha < 1$.

We want to evaluate

$$\lim_{n\to\infty} (1 - |\varphi_n(0)|)^{-1/2n}$$

If φ is elliptic or parabolic, it is not hard to show this limit is 1.

In one dimension we have:

Theorem (C. Cowen, 1983)

If φ is an elliptic or parabolic self-map of $\mathbb{D}$, then the spectral radius of C_φ (on H^2) is 1.

If φ is hyperbolic with dilatation coefficient α, then the spectral radius is $\alpha^{-1/2}$.

Goal: Extend this theorem to Schur class mappings of $\mathbb{B}^d$.

The elliptic and parabolic cases go through (with identical proofs). The hyperbolic case takes work...

Definition

Given a point $\zeta \in \partial\mathbb{B}^d$ and a real number $c > 0$, the Koranyi region $D_c(\zeta)$ is the set

$$D_c(\zeta) = \left\{ z \in \mathbb{B}^d : |1 - \langle z, \zeta \rangle| \leq \frac{c}{2}(1 - |z|^2) \right\}$$

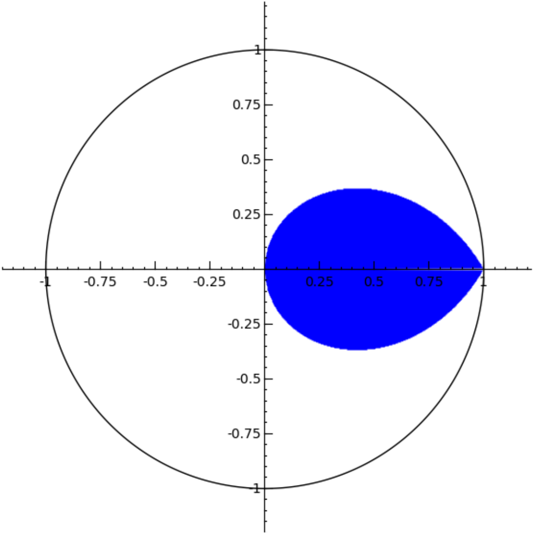
A function f has K-limit equal to L at ζ if

$$\lim_{z \to \zeta} f(z) = L$$

whenever $z \to \zeta$ within a Koranyi region.

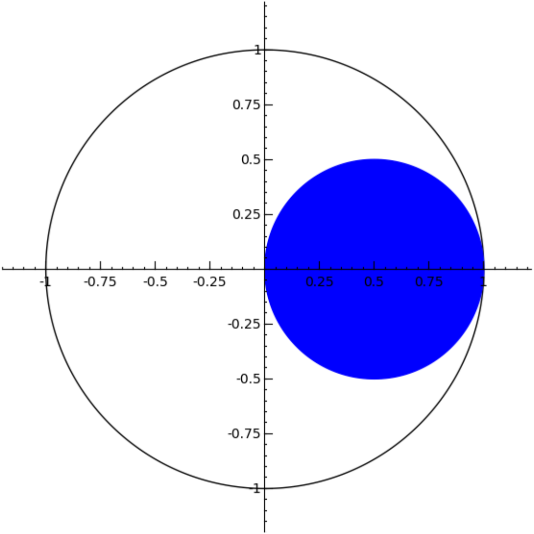
When $d = 1$ (the disk), K-limit is the same as non-tangential limit. Not so in the ball...

Slice of a Koranyi region with vertex at $(1, 0)$ in $\mathbb{B}^2$:



$z_2 = 0$

Slice of a Koranyi region with vertex at $(1, 0)$ in $\mathbb{B}^2$:



$\mathrm{Im} z_1 = \mathrm{Im} z_2 = 0$

$$\alpha = \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} < \infty. \qquad \text{(C)}$$

Theorem (Rudin, 1980)

Suppose $\varphi = (\varphi_1, \dots \varphi_d)$ is a holomorphic mapping from $\mathbb{B}^d$ to itself satisfying condition (C)at e_1. The following functions are then bounded in every Koranyi region with vertex at e_1:

(i) $\frac{1-\varphi_1(z)}{1-z_1}$

(ii) $(D_1\varphi_1)(z)$

(iii) $\frac{1-|\varphi_1(z)|^2}{1-|z_1|^2}$

(iv) $\frac{1-|\varphi(z)|^2}{1-|z|^2}$

Moreover, each of these functions has restricted K-limit α at e_1.

What is a restricted K-limit?

Fix a point $\zeta \in \partial\mathbb{B}^d$ and consider a curve $\Gamma : [0, 1) \to \mathbb{B}^n$ such that $\Gamma(t) \to \zeta$ as $t \to 1$. Let $\gamma(t) = \langle \Gamma(t), \zeta \rangle \zeta$ be the projection of Γ onto the complex line through ζ. The curve Γ is called special if

$$\lim_{t \to 1} \frac{|\Gamma - \gamma|^2}{1 - |\gamma|^2} = 0 \tag{1}$$

and restricted if it is special and in addition

$$\frac{|\zeta - \gamma|}{1 - |\gamma|^2} \leq A \tag{2}$$

for some constant $A > 0$.

Definition

We say that a function $f : \mathbb{B}^d \to \mathbb{C}$ has restricted K-limit L at ζ if $\lim_{z \to \zeta} f(z) = L$ along every restricted curve.

We have

K-limit $\implies$ restricted K-limit $\implies$ non-tangential limit

and each implication is strict when $d > 1$.

One more fact:

Theorem

Let φ be a hyperbolic, holomorphic self-map of $\mathbb{B}^d$ with Denjoy-Wolff point ζ. Then

$$\varphi_n(z_0) \to \zeta$$

within a Koranyi region for every $z_0 \in \mathbb{B}^d$.

C. Cowen 1981 ($d = 1$)
Bracci, Poggi-Corradini 2003 ($d > 1$)

If we knew that some orbit $\{\varphi_n(z_0)\}$ approached the Denjoy-Wolff point restrictedly, this combined with Rudin's Julia-Caratheodory theorem would imply

$$\lim_{n\to\infty} (1 - |\varphi_n(z_0)|)^{-1/2n} = \alpha^{-1/2}$$

It is not known if such orbits always exist. (Yes, if φ is an LFT.)

Indeed, if we know some orbit $z_n := \varphi_n(z_0)$ approaches ζ restrictedly, then

$$\lim_{n\to\infty} \frac{1-|\varphi(z_n)|}{1-|\varphi(z_{n-1})|} = \alpha$$

and hence

$$\begin{aligned}\lim_{n\to\infty}(1-|\varphi_n(z_0)|)^{1/n} &= \lim_{n\to\infty}\left(\prod_{k=1}^{n}\frac{1-|\varphi(z_k)|}{1-|\varphi(z_{k-1})|}\right)^{1/n} \\ &= \alpha\end{aligned}$$

$$\alpha = \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} < \infty. \tag{C}$$

Theorem (J., to appear)

Let φ be a Schur class map and $\zeta \in \partial\mathbb{B}^d$. Then the following are equivalent:

1. *Condition (C).*
2. *There exists $\xi \in \partial\mathbb{B}^d$ such that the function*

$$h(z) = \frac{1 - \langle \varphi(z), \xi \rangle}{1 - \langle z, \zeta \rangle}$$

   belongs to $\mathcal{H}(\varphi)$.

3. *Every $f \in \mathcal{H}(\varphi)$ has a finite K-limit at ζ.*

$d = 1$ case: Sarason 1994

Theorem (J., to appear)

Suppose $\varphi = (\varphi_1, \dots \varphi_d)$ is a ~~holomorphic~~ Schur class mapping from $\mathbb{B}^d$ to itself satisfying condition (C) at e_1. The following functions are then bounded in every Koranyi region with vertex at e_1:

(i) $\frac{1-\varphi_1(z)}{1-z_1}$

(ii) $(D_1\varphi_1)(z)$

(iii) $\frac{1-|\varphi_1(z)|^2}{1-|z_1|^2}$

(iv) $\frac{1-|\varphi(z)|^2}{1-|z|^2}$

Moreover, each of these functions (i)-(iii) has ~~restricted~~ K-limit α at e_1.

Unfortunately, our attempted argument works only if (iv) has a K-limit at e_1, which is not true in general even if φ is Schur class. Nonetheless, this theorem is sufficient to solve the spectral radius problem, in a more indirect way....

For $0 < \alpha < 1$, let θ_α denote the disk automorphism

$$\theta_\alpha(z) = \frac{z + \left(\frac{1-\alpha}{1+\alpha}\right)}{1 + \left(\frac{1-\alpha}{1+\alpha}\right) z}$$

Theorem (J., in progress)

Let φ be a hyperbolic Schur class self-map of $\mathbb{B}^d$ with dilatation coefficient α. Then there exists a nonconstant Schur class map $\sigma : \mathbb{B}^d \to \mathbb{D}$ such that

$$\sigma \circ \varphi = \theta_\alpha \circ \sigma$$

Proof needs[1] strengthened Julia-Caratheodory theorem.

($d = 1$: Valiron, 1931; also Pommerenke 1979, C. Cowen 1981)
($d > 1$, under different assumptions:
Bracci, Gentili, Poggi-Corradini, 2007)

[1] Probably.

Given a Schur class solution σ to the Abel-Schroeder equation

$$\sigma \circ \varphi = \theta_\alpha \circ \sigma$$

we can compute the spectral radius of C_φ via "transference:" Suppose F is holomorphic in $\mathbb{D}$ and $\lambda \in \mathbb{C}$ satisfies

$$F \circ \theta_\alpha = \lambda F$$

Then

$$\begin{aligned} F \circ \sigma \circ \varphi &= F \circ \theta_\alpha \circ \sigma \\ &= \lambda F \circ \sigma \end{aligned}$$

Formally, σ transfers eigenfunctions of C_{θ_α} to eigenfunctions of C_φ.

KEY FACT: If $F \in H^2(\mathbb{D})$ and σ Schur class, then $F \circ \sigma \in H^2_{d,1}$.

The transference technique then proves:

Theorem (J., in progress)

Let φ be a hyperbolic Schur class map of $\mathbb{B}^d$ with dilatation coefficient α. Then the spectral radius of C_φ acting on $H^2_{d,m}$ is $\alpha^{-m/2}$. Moreover every complex number λ in the annulus

$$\alpha^{m/2} < |\lambda| < \alpha^{-m/2}$$

is an eigenvalue of C_φ of infinite multiplicity.

($d = 1$ case: C. Cowen, 1983)

If $d > 1$ and φ is an automorphism, the closure of this annulus is equal to the spectrum of C_φ (MacCluer, 1984)