Operator-valued Herglotz kernels and functions of positive real part on the ball

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July 24, 2008

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Theorem (Riesz-Herglotz)

Let $f \in Hol(\mathbb{D})$. Then $\Re f \geq 0$ in $\mathbb{D}$ iff $\exists$ a postitive measure μ on $\partial\mathbb{D}$ such that

$$f(z) = \int_{\partial\mathbb{D}} \frac{1 + z\overline{\zeta}}{1 - z\overline{\zeta}}\, d\mu(\zeta) + i\Im f(0)$$

for all $z \in \mathbb{D}$.

Notation & definitions:

- $z = (z_1, \ldots z_d) \in \mathbb{C}^d$
- $\langle z, w\rangle = \sum z_j \overline{w_j}, \quad |z| = \sqrt{\langle z, z\rangle}$
- $\mathbb{B}^d = \{z \in \mathbb{C}^d : |z| < 1\}$
- $O = Hol(\mathbb{B}^d), \qquad O^+ = \{f \in O : \Re f \geq 0\}$
- H^2_d: the RKHS on $\mathbb{B}^d$ with kernel

$$k(z, w) = \frac{1}{1 - \langle z, w\rangle}$$

($H^2_d \subsetneq H^2(\mathbb{B}^d)$ when $d > 1$)

Definition

A *positive class* on $\mathbb{B}^d$ is a set of functions $\mathcal{P} \subset O^+$ with the following properties:

- $\mathcal{P}$ is a closed, convex cone in $O+$
- $\mathcal{P}$ is closed under dilations: if $f \in \mathcal{P}$ then

$$f_r(z) := f(rz)$$

  belongs to $\mathcal{P}$ for all $0 \leq r \leq 1$.

We are interested in certain positive classes admitting a "noncommutative Herglotz representation."

Examples of positive classes:

- $O^+ = \{f \in Hol(\mathbb{B}^d) : \Re f \geq 0\}$

- M^+:
$$f(z) = \int_{\partial\mathbb{B}^d} \frac{1 + \langle z, \zeta\rangle}{1 - \langle z, \zeta\rangle}\, d\mu(\zeta) + i\Im f(0),$$
μ a positive measure on $\partial\mathbb{B}^d$

- S^+: the set of $f \in Hol(\mathbb{B}^d)$ such that
$$\frac{f(z) + \overline{f(w)}}{1 - \langle z, w\rangle}$$
is a positive semidefinite kernel.

Write $H(z,\zeta)$ for the Herglotz kernel on $\mathbb{B}^d$:

$$H(z,\zeta) = \frac{1+\langle z,\zeta\rangle}{1-\langle z,\zeta\rangle}$$

Since

$$\frac{H(z,\zeta)+\overline{H(w,\zeta)}}{1-\langle z,w\rangle} = 2\left(\frac{1}{1-\langle z,\zeta\rangle}\right)\overline{\left(\frac{1}{1-\langle w,\zeta\rangle}\right)}\frac{1-\langle z,\zeta\rangle\overline{\langle w,\zeta\rangle}}{1-\langle z,w\rangle}$$

is a positive kernel for all $\zeta\in\partial\mathbb{B}^d$, it follows that

$$M^+ \subseteq S^+ \subseteq O^+$$

When $d=1$, all inclusions are equalities (Herglotz formula); when $d>1$ all inclusions are strict [McCarthy-Putinar '05].

Duality: let f and g have Taylor expansions $\sum c_\alpha z^\alpha$, $\sum d_\alpha z^\alpha$.

Definition

For $f, g \in Hol\mathbb{B}^d$ and $0 \leq r < 1$ define

$$Q_r(f,g) := \sum_\alpha c_\alpha \overline{d_\alpha} r^\alpha \frac{\alpha!}{|\alpha|!} + f(0)\overline{g(0)} \tag{1}$$

Motivation: the series defining Q_r converges for all $r < 1$, and if g is a Herglotz integral

$$g(z) = \frac{1}{2}\int_{\partial\mathbb{B}^d} \frac{1+\langle z,\zeta\rangle}{1-\langle z,\zeta\rangle}\, d\mu(\zeta)$$

then

$$Q_r(f,g) = \int_{\partial\mathbb{B}^d} f_r \, d\mu$$

Definition

For $\mathcal{C} \subset O$ define

$$\mathcal{C}^* = \{g \in O \mid \Re Q_r(f,g) \geq 0 \text{ for all } f \in \mathcal{C} \text{ and all } r \in [0,1)\}$$

It follows easily that

$$M^{+*} = O^+.$$

Main theorem on duality & positive classes:

Theorem (J. '07)

Let $M^+ \subset \mathcal{P} \subset O^+$ be a positive class. Then:

- *$\mathcal{P}^*$ is a positive class.*
- *$\mathcal{P}^{**} = \mathcal{P}$.*
- *If $\mathcal{P} \subset \mathcal{P}^*$ then $\exists$ a positive class $\mathcal{W}$ with*

$$\mathcal{P} \subset \mathcal{W} \subset \mathcal{P}^* \quad \text{and} \quad \mathcal{W} = \mathcal{W}^*.$$

Theorem (McCarthy-Putinar '05)

- $M^{+*} = O^{+}$
- $O^{+*} = M^{+}$
- $S^{+*} \subseteq S^{+}$

Question: Is $S^{+*} = S^{+}$*?*

It turns out the answer is "No," but we can identify S^{+*} explicitly...

Row contratctions and operator-valued Herglotz kernels:

Definition

A *row contraction* is a d-tuple of bounded operators $T = (T_1, \dots T_d)$ on a Hilbert space $\mathcal{H}$ such that

$$I - T_1 T_1^* - \cdots - T_d T_d^* \geq 0$$

If T is a row contraction, then for all $|z| < 1$ the operator

$$\langle z, T \rangle := z_1 T_1^* + \cdots + z_d T_d^*$$

is a strict contraction. Define

$$H(z, T) = (I + \langle z, T \rangle)(I - \langle z, T \rangle)^{-1}$$

Row contractions and positive classes:

$$\Re H(z,T) = 2(I - \langle z, T\rangle)^{-1}(I - \langle z, T\rangle\langle z, T\rangle^*)(I - \langle z, T\rangle^*)^{-1} \geq 0$$

This shows

Lemma

If ρ is a positive linear functional on the C-algebra generated by T, the holomorphic function*

$$\rho(H(z,T))$$

has positive real part on $\mathbb{B}^d$.

For many choices of T, the set

$$\mathcal{P}_T := \{\rho(H(z,T)) + i\lambda : \rho \text{ positive }, \lambda \in \mathbb{R}\}$$

is a positive class [Sufficient condition: T dilates rT for all $r < 1$]

Row contractions of interest:

- *Spherical contractions*: $Z = (Z_1, \dots Z_d)$

$$Z_j = \pi(\zeta_j), \qquad j = 1, \dots d$$

  where π is any representation of the commutative C*-algebra $C(\partial\mathbb{B}^d)$ on $\mathcal{B}(\mathcal{H})$

- *Cuntz isometries*: $V = (V_1, \dots V_d)$

$$V_i^* V_j = \delta_{ij} I; \qquad \sum_{j=1}^{d} V_j V_j^* = I$$

- *Coordinate multipliers on* H^2_d: $S = (S_1, \dots S_d)$

$$S_j = M_{z_j}$$

We have defined for each row contraction T

$$\mathcal{P}_T := \{\rho(H(z,T)) + i\lambda : \rho \text{ positive }, \lambda \in \mathbb{R}\}$$

For the row contractions Z, V, S we have:

Theorem

- $\mathcal{P}_Z = M^+$ *(definition of M^+, more or less)*
- $\mathcal{P}_V = S^+$ *[McCarthy-Putinar '05; Popescu '07]*
- $\mathcal{P}_S := R^+ = S^{+*}$ *[J. '07]*

Theorem (J. '07)

$$M^+ \subset R^+ \subset S^+ \subset O^+$$

and each inclusion is proper.

For a d-tuple $T = (T_1, \dots T_d)$ and a monomial z^α, define

$$(z^\alpha)^{sym}(T) := \frac{\alpha!}{|\alpha|!} \sum T_{i_1} T_{i_2} \cdots T_{i_{|\alpha|}}$$

Corollary

Let p be a d-variable polynomial. Then

$$\Re p^{sym}(T) \geq 0$$

for all row contractions T if and only if $p \in R^+$.

Compare: $p \in S^+$ iff

$$\Re p(T) \geq 0$$

for all *commuting* row contractions T.

Questions:

1. Given a positive class $\mathcal{P}$, let $\mathcal{P}_0$ denote the subclass of $\mathcal{P}$ for which $f(0) = 1$; this set is compact and convex. It is not hard to show that the Herglotz kernels

$$H(z, \zeta) = \frac{1 + \langle z, \zeta \rangle}{1 - \langle z, \zeta \rangle}$$

are extreme in S_0^+; but by Krein-Milman there must be others when $d > 1$. (The Herglotz kernels are *not* extreme in O^+ when $d > 1$ [Rudin].)

PROBLEM: find all extreme points of R_0^+, S_0^+.

2. Since $R^+ = S^{+*} \subsetneq S^+$, main theorem says there is a self-dual class in between.

PROBLEM: Find it!