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shift invariant subspace \( M = \theta H^2 \subset H^2 \), \( \theta \) inner, \( \theta(0) = 0 \)

\( K_\theta := (\theta H^2)^\perp \)

Consider \( X = S^*|_{K_\theta} : \)

\[
\ker X = [1] \quad \text{coker} X = [S^*\theta] \quad (\text{note } S^*\theta \perp M) \quad (1)
\]

Fix \( |\alpha| = 1. \) Define unitary \( U_\alpha : K_\theta \to K_\theta : \)

\[
U^*_\alpha := S^*|_{K_\theta} + \alpha^*(S^*\theta \otimes 1) \quad (2)
\]

The vector 1 is cyclic for \( U_\alpha \)…spectral measure \( \mu_\alpha : \)

\[
\int_T \zeta^n \, d\mu_\alpha(\zeta) = \langle U_\alpha^n 1, 1 \rangle, \quad n \in \mathbb{Z} \quad (3)
\]
Theorem (Clark)

\( \mu_\alpha \) and \( \theta \) are related by

\[
\frac{1 + \alpha^* \theta(z)}{1 - \alpha^* \theta(z)} = \int_T \frac{1 + \zeta^* z}{1 - \zeta^* z} \, d\mu_\alpha(\zeta)
\]  \hspace{1cm} (4)

and \( \mu_\alpha \perp m \).

Moreover, the operator

\[
(V_\alpha f)(z) := (1 - \alpha^* \theta(z)) \int_T \frac{f(\zeta)}{1 - \zeta^* z} \, d\mu_\alpha(\zeta)
\]  \hspace{1cm} (5)

is unitary from \( L^2(\mu_\alpha) \) to \( K_\theta \), and

\[
U_\alpha V_\alpha = V_\alpha M_\zeta.
\]  \hspace{1cm} (6)
The NC setting:

$H$ a Hilbert space

Consider row unitaries $U : H^d \rightarrow H$

$$U = (U_1, \ldots U_d)(h_1, \ldots h_d)^T = \sum_{j=1}^{d} U_j h_j$$ (7)

$U$ is unitary iff $U_j$ satisfy the Cuntz relations

$$U_i^* U_j = \delta_{ij} I, \quad \sum_{j=1}^{d} U_j U_j^* = I$$ (8)
Row unitaries from backward shifts:

\[ \mathbf{F}_d \] – set of all NC words in letters \{1, \ldots, d\} (including empty word \( \emptyset \))

\[ w = i_1 i_2 \cdots i_m, \quad |w| = m, \quad |\emptyset| = 0 \quad (9) \]

\[ F^2_d := \ell^2(\mathbf{F}_d) \text{ (full Fock space) } \text{ o.n. basis } \{\xi_w : w \in \mathbf{F}_d\} \]

Shifts on \( F^2_d \): for \( j = 1, \ldots, d \)

\[ L_j \xi_w = \xi_{jw} \quad (10) \]

The \( L_j \) satisfy

\[ L_i^* L_j = \delta_{ij} I, \quad \sum_{j=1}^{d} L_j L_j^* = I - \xi_{\emptyset} \otimes \xi_{\emptyset} \quad (11) \]

\( \mathbf{L} := (L_1, \ldots, L_d) \) – NC \( d \)-shift
Row unitaries from backward shifts:

Suppose \( M \subset F_d^2 \) is \( \mathbf{L} \)-invariant \( (L_j M \subset M \text{ all } j) \) ...and cyclic.

Then (Davidson/Pitts, Popescu) there is a wandering vector \( \theta \in M \) such that

\[ M = \text{span}\{L^w \theta : w \in F_d\} \| \cdot \| \]  \hspace{1cm} (12)

Write

\[ \theta = \sum \hat{\theta}(w) \xi_w, \quad \sum_w |\hat{\theta}(w)|^2 = 1 \]  \hspace{1cm} (13)
Row unitaries from backward shifts:

If $X_1, \ldots X_d$ are operators in some $B(K)$ with $\| \sum X_j X_j^* \| < 1$, then (Popescu)

$$\theta(X) := \sum_w \hat{\theta}(w) X^w$$

is norm convergent and $\| \theta(X) \| \leq 1$ (so $\theta$ is a free holomorphic function).

Since also $\sum_w |\hat{\theta}(w)|^2 = 1$, call $\theta$ a free inner function.
Motivation:

"repeated interaction model" for atom-photon interactions (Gohm et al.)

NC stationary process

\[ \rightarrow \text{ Cuntz scattering system (Ball/Vinnikov)} \]

\[ \rightarrow \text{ free inner function } \theta \]
Row unitaries from backward shifts:

Recall

\[ M = \text{span}\{ L^w \theta : w \in F_d \} \| \cdot \| \]  \hspace{1cm} (15)

Write \( K_\theta = M^\perp \). Assume \( \hat{\theta}(\emptyset) = 0 \). Then \( \xi_\emptyset \in K_\theta \). Wandering property of \( \theta \) implies

\[ L_j^* \theta \in M^\perp \quad \text{all } j \]  \hspace{1cm} (16)

Form the column vector \( y \in K^d_\theta \)

\[ y = (L_1^* \theta, \ldots, L_d^* \theta)^T \]  \hspace{1cm} (17)

Since \( \theta(0) = 0 \),

\[ \|y\|^2 = \langle \sum_{j=1}^{d} L_j L_j^* \theta, \theta \rangle = \|\theta\|^2 = 1. \]  \hspace{1cm} (18)
Conclusion: $L^*|_{K_\theta}$ is a column contraction with rank one defects. Identify defect spaces to get a row unitary:

$$U_j^{\alpha*} := L_j^*|_{K_\theta} + \alpha^* y_j \otimes \xi_\emptyset$$

(19)

Theorem (J)

For each $|\alpha| = 1$, $U^\alpha := (U_1^\alpha, \ldots, U_d^\alpha)$ is a cyclic row unitary on $K_\theta$ with cyclic vector $\xi_\emptyset$.

So: every free inner function determines a family of cyclic row unitaries.

Q: Which row unitaries arise this way?
Theorem (J)

A cyclic row unitary (cyclic vector $x$) arises as above if and only if for

$$ T := (P_x^\perp U_1, \ldots P_x^\perp U_d) $$

(20)
satisfies

$$ \lim_{n \to \infty} \sum_{|w| = n} T^{w*} = 0 \quad (SOT) $$

(21)

Proof.

The NC characteristic function $\theta_T$ is free inner (Popescu)
States (NC measures) and a Clark theorem:

Recall the NC $d$-shift $L$ and form the \textit{Cuntz-Toeplitz operator system}

\[ S = \text{span}\{L^w, L^v^*: w, v \in F_d\} \quad \text{(22)} \]

A "free" Herglotz formula:

\begin{quote}
\textbf{Theorem (Popescu)}

\textit{If $b$ is a contractive free holomorphic function, then there is a unique positive functional $\mu : S \to \mathbb{C}$ such that}

\[ (I + b(X))(I - b(X))^{-1} = (id \otimes \mu)((I + \sum X_j \otimes L_j^*)(I - \sum X_j \otimes L_j)^{-1}) \]

\textit{for all $X = (X_1, \ldots X_d) \in B(H)$ with $\|\sum X_j X_j^*\| < 1$}
\end{quote}
States from cyclic row unitaries:

Given cyclic \((U, x)\), define a state \(\mu\) on \(S\) by

\[
\mu(L^w) := \langle U^w x, x \rangle
\]  

(23)

A Clark-type theorem:

**Theorem (J)**

Suppose \(\theta\) is a free inner function with associated row unitaries \(U^\alpha\) and states \(\mu^\alpha\). Then

\[
(I + \theta(X))(I - \theta(X))^{-1} = (id \otimes \mu^\alpha) \left( (I + \sum X_j \otimes L^*_j) (I - \sum X_j \otimes L^*_j)^{-1} \right)
\]
Example ("point masses"): Fix a point $\eta = (\eta_1, \ldots, \eta_d)$ on the unit sphere in $\mathbb{C}^d$. Define

$$\delta_{\eta}(L^w) = \eta^w = \eta_1 \eta_2 \cdots \eta_m$$

(24) (the Cuntz states). These arise from the free inner function

$$\theta_\eta(X) = \sum_{j=1}^{d} \eta_j^* X_j.$$ 

(25)
Q: Which states give free inner functions? (call these \textit{quasi-singular})
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Example: by brute force, convex combinations of pairs of Cuntz states give free inner functions. So e.g.

\[ \theta(X_1, X_2) = I - (I - X_1) \left( I - \frac{X_1 + X_2}{2} \right)^{-1} (I - X_2) \]

is free inner (coming from \( \frac{1}{2}(\delta_e_1 + \delta_e_2) \))

(a sort of free version of \( \frac{x_1 + x_2 - 2x_1x_2}{2 - x_1 - x_2} \))
Q: Which states give free inner functions? (call these *quasi-singular*)

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Q: *Is the set of quasi-singular states on \( S \) convex???