Rank one perturbations of row unitaries

Michael Jury

University of Florida

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shift invariant subspace $M = \theta H^2 \subset H^2$, θ inner, $\theta(0) = 0$

$K_\theta := (\theta H^2)^\perp$

Consider $X = S^*|_{K_\theta}$:

$$\ker X = [1] \quad \operatorname{coker} X = [S^*\theta] \quad (\text{note } S^*\theta \perp M) \tag{1}$$

Fix $|\alpha| = 1$. Define unitary $U_\alpha : K_\theta \to K_\theta$:

$$U_\alpha^* := S^*|_{K_\theta} + \alpha^*(S^*\theta \otimes 1) \tag{2}$$

The vector 1 is cyclic for U_α...spectral measure μ_α:

$$\int_{\mathbb{T}} \zeta^n \, d\mu_\alpha(\zeta) = \langle U_\alpha^n 1, 1 \rangle, \quad n \in \mathbb{Z} \tag{3}$$

Theorem (Clark)

μ_α and θ are related by

$$\frac{1+\alpha^*\theta(z)}{1-\alpha^*\theta(z)} = \int_{\mathbb{T}} \frac{1+\zeta^* z}{1-\zeta^* z}\, d\mu_\alpha(\zeta) \tag{4}$$

and $\mu_\alpha \perp m$.

Moreover, the operator

$$(V_\alpha f)(z) := (1-\alpha^*\theta(z)) \int_{\mathbb{T}} \frac{f(\zeta)}{1-\zeta^* z}\, d\mu_\alpha(\zeta) \tag{5}$$

is unitary from $L^2(\mu_\alpha)$ to K_θ, and

$$U_\alpha V_\alpha = V_\alpha M_\zeta. \tag{6}$$

The NC setting:

H a Hilbert space

Consider row unitaries $\mathbf{U} : H^d \to H$

$$\mathbf{U} = (U_1, \dots U_d)(h_1, \dots h_d)^T = \sum_{j=1}^{d} U_j h_j \tag{7}$$

$\mathbf{U}$ is unitary iff U_j satisfy the *Cuntz relations*

$$U_i^* U_j = \delta_{ij} I, \quad \sum_{j=1}^{d} U_j U_j^* = I \tag{8}$$

Row unitaries from backward shifts:

$\mathbf{F_d}$ – set of all NC words in letters $\{1, \dots d\}$ (including empty word $\varnothing$)

$$w = i_1 i_2 \cdots i_m, \quad |w| = m, \quad |\varnothing| = 0 \tag{9}$$

$F^2_d := \ell^2(\mathbf{F_d})$ (full Fock space) o.n. basis $\{\xi_w : w \in \mathbf{F_d}\}$

Shifts on F^2_d: for $j = 1, \dots d$

$$L_j \xi_w = \xi_{jw} \tag{10}$$

The L_j satisfy

$$L_i^* L_j = \delta_{ij} I, \quad \sum_{j=1}^{d} L_j L_j^* = I - \xi_\varnothing \otimes \xi_\varnothing \tag{11}$$

$\mathbf{L} := (L_1, \dots L_d)$ – NC d-shift

Row unitaries from backward shifts:

Suppose $M \subset F^2_d$ is **L**-invariant $\quad (L_j M \subset M$ all $j)$ ...and *cyclic*.

Then (Davidson/Pitts, Popescu) there is a wandering vector $\theta \in M$ such that

$$M = \overline{\text{span}\{L^w\theta : w \in \mathbf{F_d}\}}^{\|\cdot\|} \tag{12}$$

Write

$$\theta = \sum \widehat{\theta}(w)\xi_w, \quad \sum_w |\widehat{\theta}(w)|^2 = 1 \tag{13}$$

Row unitaries from backward shifts:

If $X_1, \dots X_d$ are operators in some $B(K)$ with $\|\sum X_j X_j^*\| < 1$, then (Popescu)

$$\theta(X) := \sum_w \widehat{\theta}(w) X^w \tag{14}$$

is norm convergent and $\|\theta(X)\| \leq 1$ (so θ is a *free holomorphic function*).

Since also $\sum_w |\widehat{\theta}(w)|^2 = 1$, call θ a *free inner function*.

Motivation:

"repeated interaction model" for atom-photon interactions (Gohm et al.)

NC stationary process

$\Longrightarrow$ Cuntz scattering system (Ball/Vinnikov)

$\Longrightarrow$ free inner function θ

Row unitaries from backward shifts:

Recall

$$M = \overline{\mathrm{span}\{L^w\theta : w \in \mathbf{F_d}\}}^{\|\cdot\|} \tag{15}$$

Write $K_\theta = M^\perp$. Assume $\widehat{\theta}(\varnothing) = 0$. Then $\xi_\varnothing \in K_\theta$. Wandering property of θ implies

$$L_j^*\theta \in M^\perp \quad \text{all } j \tag{16}$$

Form the column vector $\mathbf{y} \in K_\theta^d$

$$\mathbf{y} = (L_1^*\theta, \ldots L_d^*\theta)^T \tag{17}$$

Since $\theta(0) = 0$,

$$\|\mathbf{y}\|^2 = \langle \sum_{j=1}^d L_j L_j^*\theta, \theta\rangle = \|\theta\|^2 = 1. \tag{18}$$

Conclusion: $\mathbf{L}^*|_{K_\theta}$ is a column contraction with rank one defects. Identify defect spaces to get a row unitary:

$$U_j^{\alpha *} := L_j^*|_{K_\theta} + \alpha^* y_j \otimes \xi_\varnothing \tag{19}$$

Theorem (J)

For each $|\alpha| = 1$, $\mathbf{U}^\alpha := (U_1^\alpha, \ldots U_d^\alpha)$ is a cyclic row unitary on K_θ with cyclic vector $\xi_\varnothing$.

So: every free inner function determines a family of cyclic row unitaries.

Q: Which row unitaries arise this way?

Theorem (J)

A cyclic row unitary (cyclic vector x) arises as above if and only if for

$$\mathbf{T} := (P_x^{\perp} U_1, \ldots P_x^{\perp} U_d) \tag{20}$$

satisfies

$$\lim_{n\to\infty} \sum_{|w|=n} T^{w*} = 0 \quad (SOT) \tag{21}$$

Proof.

The NC characteristic function $\theta_{\mathbf{T}}$ is free inner (Popescu) □

States (NC measures) and a Clark theorem:

Recall the NC d-shift $\mathbf{L}$ and form the *Cuntz-Toeplitz operator system*

$$\mathcal{S} = \text{span}\{L^w, L^{v*} : w, v \in \mathbf{F_d}\} \tag{22}$$

A "free" Herglotz formula:

Theorem (Popescu)

If b is a contractive free holomorphic function, then there is a unique positive functional $\mu : \mathcal{S} \to \mathbb{C}$ such that

$$(I+b(X))(I-b(X))^{-1} = (id\otimes\mu)((I+\sum X_j\otimes L_j^*)(I-\sum X_j\otimes L_j)^{-1})$$

for all $X = (X_1, \ldots X_d) \in B(H)$ with $\|\sum X_j X_j^*\| < 1$

States from cyclic row unitaries:

Given cyclic $(\mathbf{U}, x)$, define a state μ on $\mathcal{S}$ by

$$\mu(L^w) := \langle U^w x, x\rangle \tag{23}$$

A Clark-type theorem:

Theorem (J)

Suppose θ is a free inner function with associated row unitaries $\mathbf{U}^\alpha$ and states μ_α. Then

$$(I+\theta(X))(I-\theta(X))^{-1} = (id\otimes\mu_\alpha)\left((I+\sum X_j\otimes L_j^*)(I-\sum X_j\otimes L_j^*)^{-1}\right)$$

Example ("point masses"): Fix a point $\eta = (\eta_1, \ldots \eta_d)$ on the unit sphere in $\mathbb{C}^d$. Define

$$\delta_\eta(L^w) = \eta^w = \eta_{i_1}\eta_{i_2}\cdots\eta_{i_m} \tag{24}$$

(the *Cuntz states*). These arise from the free inner function

$$\theta_\eta(X) = \sum_{j=1}^{d} \eta_j^* X_j. \tag{25}$$

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$$\theta(X_1, X_2) = I - (I - X_1)\left(I - \frac{X_1 + X_2}{2}\right)^{-1}(I - X_2)$$

is free inner (coming from $\frac{1}{2}(\delta_{e_1} + \delta_{e_2})$)

(a sort of free version of $\frac{x_1 + x_2 - 2x_1x_2}{2 - x_1 - x_2}$)

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Q: Is the set of quasi-singular states on $\mathcal{S}$ convex???