

AGLER INTERPOLATION FAMILIES OF KERNELS

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ABSTRACT. An abstract Pick interpolation theorem for a family of positive semi-definite kernels on a set X is formulated. The result complements those in [Ag] and [AM02] and will subsequently be applied to Pick interpolation on distinguished varieties [JKM].

1. INTRODUCTION

Let $s(z, w)$ denote Szegő's kernel; i.e.,

$$s(z, w) = \frac{1}{1 - z\bar{w}},$$

for complex numbers z and w . The kernel s is the reproducing kernel for the Hardy space $H^2(\mathbb{D})$ of functions analytic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with square summable power series. Thus, an analytic function $f : \mathbb{D} \rightarrow \mathbb{C}$ with power series expansion

$$f(z) = \sum_{n=0}^{\infty} f_n z^n$$

is, by definition, in $H^2(\mathbb{D})$ if and only if $\sum |f_n|^2$ converges. The Hardy space is a Hilbert space with inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} f_n \bar{g}_n.$$

Evidently, for a fixed w , the function $s_w(z) = s(z, w)$ is in $H^2(\mathbb{D})$ and earns the title of reproducing kernel because, for $f \in H^2(\mathbb{D})$,

$$f(w) = \langle f, s_w \rangle.$$

Szegő's kernel is indispensable to the statement of

Theorem 1.1 (Pick Interpolation). *Let n be a positive integer. Given points $w_1, \dots, w_n; v_1, \dots, v_n \in \mathbb{D}$, there exists an analytic function $f : \mathbb{D} \rightarrow \mathbb{D}$ such that $f(w_j) = v_j$ if and only if Pick's matrix,*

$$((1 - v_j \bar{v}_\ell) s(w_j, w_\ell))$$

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is positive semi-definite.

Extensions of the Pick interpolation theorem to domains and settings more general than the disc \mathbb{D} often involve replacing the Szegő kernel with a family of kernels. The references [AM02, AM03, R1, R2, DPRS, Ab, B, P, CLW, M, JK, MP, S65] represent only fraction of the results in this direction. For instance, in Abrahamse's [Ab] interpolation theorem on the annulus the Szegő kernel is replaced by a family of kernels $k^t(z, w)$ - parameterized by t in the unit circle \mathbb{T} - identified by Sarason [S65]. See also [AD]. In a similar vein, the recent constrained Pick interpolation results in [DPRS] [R1][R2] are stated in terms of a family of kernels over the disc canonically determined by the constraints.

The main result of this paper, Theorem 1.4 below, is a Pick theorem formulated, like the related results in [Ag] and [AM02], purely in terms of a collection of kernels. The result here has a natural operator algebraic interpretation which is exploited in the proof by using the fact that the quotient of an operator algebra by a two sided ideal is again an operator algebra. (This is a corollary of the Blecher-Ruan-Sinclair (BRS) theorem. See [Pa] for an exposition of the BRS theorem and the related topics of completely positive maps, Arveson's extension theorem, and Stinespring's representation theorem.) In forthcoming work [JKM], Theorem 1.4 is applied to produce a Pick interpolation theorem on distinguished varieties [AM05] [AM03].

The statement of the main result requires the notion of a (positive semi-definite) matrix-valued kernel. Let M_n denote the $n \times n$ matrices with complex entries. An M_n -valued kernel on a set X is a function $k : X \times X \rightarrow M_n$ which is positive semi-definite in the sense that, for every finite subset $F \subset X$, the (block) matrix

$$(k(x, y))_{x, y \in F}$$

is positive semi-definite.

Definition 1.2. Fix a set X and a sequence $\mathcal{K} = (\mathcal{K}_n)$ where each \mathcal{K}_n is a set of M_n -valued kernels on X .

The collection \mathcal{K} is an *Agler interpolation family of kernels* provided:

- (i) if $k_1 \in \mathcal{K}_{n_1}$ and $k_2 \in \mathcal{K}_{n_2}$, then $k_1 \oplus k_2 \in \mathcal{K}_{n_1+n_2}$;
- (ii) if $k \in \mathcal{K}_n$, $z \in X$, $\gamma \in \mathbb{C}^n$, and $\gamma^*k(z, z)\gamma \neq 0$, then there exists an N , a kernel $\kappa \in \mathcal{K}_N$, and a function $G : X \rightarrow M_{n, N}$ such that

$$k'(x, y) := k(x, y) - \frac{k(x, z)\gamma\gamma^*k(z, y)}{\gamma^*k(z, z)\gamma} = G(x)\kappa(x, y)G(y)^*;$$

- (iii) for each finite $F \subset X$ and for each $f : F \rightarrow \mathbb{C}$, there is a $\rho > 0$ such that, for each $k \in \mathcal{K}$,

$$F \times F \ni \mapsto (\rho^2 - f(x)f(y)^*)k(x, y)$$

is a positive semi-definite kernel on F ; and

- (iv) for each $x \in X$ there is a $k \in \mathcal{K}$ such that $k(x, x)$ is nonzero.

Remark 1.3. Given $Y \subset X$ and a kernel $k : X \times X \rightarrow M_n$, let $k|_Y = k|_{Y \times Y}$. Thus $k|_Y$ is a kernel on Y . If \mathcal{K} is an Agler interpolation family of kernels (on X), then \mathcal{K}_Y , the collection of kernels of the form $k|_Y$ for $k \in \mathcal{K}$, is an Agler interpolation family of kernels (on Y).

Theorem 1.4. *Suppose \mathcal{K} is an Agler interpolation family of kernels on X . Further suppose $Y \subset X$ is finite, $g : Y \rightarrow \mathbb{C}$ and $\rho \geq 0$. If for each $k \in \mathcal{K}$ the kernel*

$$(1.1) \quad Y \times Y \ni (x, y) \rightarrow (\rho^2 - g(x)g(y)^*)k(x, y)$$

is positive semi-definite, then there exists $f : X \rightarrow \mathbb{C}$ such that $f|_Y = g$ and for each $k \in \mathcal{K}$ the kernel

$$(1.2) \quad X \times X \ni (x, y) \rightarrow (\rho^2 - f(x)f(y)^*)k(x, y)$$

is positive semi-definite.

Remark 1.5. Theorem 1.4 is stated for scalar-valued interpolation; i.e., the functions f and g take values in \mathbb{C} as opposed to M_n . In this case it suffices to consider a collection of scalar kernels canonically associated with \mathcal{K} giving a result more in line with that found in [Ag] and [AM02]. Some details are provided in Section 5.

In the forthcoming paper [JKM], Theorem 1.4 is applied to yield a Pick interpolation theorem for distinguished varieties. There are similarities to interpolation on multiply connected domains and the case of the annulus is discussed in Section 6, where the role of item (ii) of Definition 1.2 becomes apparent.

2. OPERATOR THEORETIC PRELIMINARIES

The operator theoretic approach to interpolation associates to a positive semi-definite matrix-valued kernel k a Hilbert space $H^2(k)$. Functions satisfying, for this given k , the positivity condition of item (iii) of Definition 1.2 determine bounded operators on $H^2(k)$.

2.1. The Hilbert Space $H^2(k)$. To a positive semi-definite kernel $k : X \times X \rightarrow M_n$, there is associated a Hilbert space $H^2(k)$ so that in the case that k is positive definite and X is finite, $H^2(k)$ is, as a set, all functions $F : X \rightarrow \mathbb{C}^n$. To construct $H^2(k)$, define a semi-inner product on functions $F, G : X \rightarrow \mathbb{C}^n$ of the form

$$F = \sum_{x \in X} k(\cdot, x)F_x,$$

$$G = \sum_{x \in X} k(\cdot, x)G_x,$$

by

$$\langle F, G \rangle = \sum_{x, y \in X} \langle k(x, y)F_y, F_x \rangle.$$

Let $H^2(k)$ denote the Hilbert space obtained by quotienting out null vectors and then forming the completion of the resulting pre-Hilbert space. When X is finite the quotient is finite dimensional and hence already complete. If moreover, k is positive definite, then the set of null vectors is trivial.

Condition (ii) in Definition 1.2 has a natural interpretation in terms of $H^2(k)$: if \mathcal{N} is the subspace of $H^2(k)$ spanned by the nonzero vector $k(\cdot, z)\gamma$, then k' is the reproducing kernel for \mathcal{N}^\perp . Indeed, we have

$$P_{\mathcal{N}} = \frac{k(\cdot, z)\gamma(k(\cdot, z)\gamma)^*}{\langle k(\cdot, z)\gamma, k(\cdot, z)\gamma \rangle}.$$

Hence,

$$\begin{aligned} \langle P_{\mathcal{N}}k(\cdot, y)v, k(\cdot, x)u \rangle &= \frac{\langle k(\cdot, y)v, k(\cdot, z)\gamma \rangle \langle k(\cdot, z)\gamma, k(\cdot, x)u \rangle}{\langle k(z, z)\gamma, \gamma \rangle} \\ &= \frac{\langle k(z, y)v, \gamma \rangle \langle k(x, z)\gamma, u \rangle}{\langle k(z, z)\gamma, \gamma \rangle} \\ &= u^* \frac{k(x, z)\gamma\gamma^*k(z, y)}{\langle k(z, z)\gamma, \gamma \rangle} v. \end{aligned}$$

Thus, letting $\mathcal{M} = H^2(k) \ominus \mathcal{N}$ and using the notation of item (iii) in Definition 1.2,

$$\begin{aligned} \langle P_{\mathcal{M}}k(\cdot, y)v, k(\cdot, x)u \rangle &= \langle k(\cdot, y)v, k(\cdot, x)u \rangle - \frac{\langle k(x, z)\gamma\gamma^*k(z, y)v, u \rangle}{\langle k(z, z)\gamma, \gamma \rangle} \\ &= \langle k(x, y)v, u \rangle - \frac{\langle k(x, z)\gamma\gamma^*k(z, y)v, u \rangle}{\langle k(z, z)\gamma, \gamma \rangle} \\ &= \langle k'(x, y)v, u \rangle. \end{aligned}$$

Assuming k is a member of an Agler interpolation family \mathcal{K} , then, by item (iii) of Definition 1.2 there is an N , a $\kappa \in \mathcal{K}_N$, and a function $G : X \rightarrow M_{n, N}$ such that

$$\langle P_{\mathcal{M}}k(\cdot, y)v, k(\cdot, x)u \rangle = \langle G(x)\kappa(x, y)G(y)^*v, u \rangle.$$

Lemma 2.1. *Let \mathcal{K} be an Agler interpolation family of kernels on a finite set X . Suppose $k \in \mathcal{K}$, $Z \subset X$ and for each $z \in Z$ there is an associated subspace $\mathcal{J}_z \subset \mathbb{C}^n$. Let $\mathcal{G}_z = k(\cdot, z)\mathcal{J}_z$, let $\mathcal{N} = \sum \mathcal{G}_z \subset H^2(k)$, and let $\mathcal{M} = H^2(k) \ominus \mathcal{N}$. There is an N , a kernel $\kappa \in \mathcal{K}_N$, and a function $G : X \rightarrow M_{n, N}$ such that*

$$(2.1) \quad \langle P_{\mathcal{M}}k(\cdot, y)v, k(\cdot, x)u \rangle = \langle G(x)\kappa(x, y)G(y)^*u, v \rangle.$$

Moreover, there is a positive M_n -valued kernel k' such that, for $v, w \in \mathbb{C}^n$,

$$(2.2) \quad \langle k'(x, y)u, v \rangle = \langle P_{\mathcal{M}}k(\cdot, y)u, k(\cdot, x)v \rangle.$$

Finally, the mapping $W : \mathcal{M} \rightarrow H^2(\kappa)$ defined by

$$WP_{\mathcal{M}}k(\cdot, y)u = \kappa(\cdot, y)G(y)^*u$$

is (well defined and) an isometry.

Proof. Equation (2.1) follows by an induction argument based on the computation preceding the proof. The right hand side of equation (2.2) determines a (positive semi-definite) kernel. Finally, that W is an isometry follows immediately from equation (2.1). \square

Lemma 2.2. *Suppose X is a finite set and $k : X \times X \rightarrow M_n$ is a (positive semi-definite) kernel. If for each $f : X \rightarrow \mathbb{C}$ there exists a $\rho > 0$ such that*

$$X \times X \ni (x, y) \mapsto (\rho^2 - f(x)f(y)^*)k(x, y)$$

is positive semi-definite, then, for each $x \in X$ the mapping

$$Q_x k(\cdot, y)v = \begin{cases} k(\cdot, x)v & y = x; \\ 0 & y \neq x. \end{cases}$$

determines a well defined mapping $Q_x : H^2(k) \rightarrow H^2(k)$.

Proof. Given x , let f denote the indicator function of the subset $\{x\}$ of X . By hypothesis, there exists $\rho > 0$ such that

$$A = ((\rho^2 - f(y)f(z)^*)K(y, z))_{y, z \in X}$$

is positive semi-definite. Consequently, if $\sum_y k(\cdot, y)v_y = 0$, then, letting v denote the vector with y -th entry v_y ,

$$0 \leq \langle Av, v \rangle = \left\| \sum k(\cdot, y)v_y \right\|^2 - \langle k(x, x)v_x, v_x \rangle \leq 0,$$

from which it follows that $k(\cdot, x)v_x = 0$. \square

2.2. The algebra $H^\infty(k)$. Let k be a positive semi-definite M_n -valued kernel on X and suppose for each $f : X \rightarrow \mathbb{C}$ there is a $\rho > 0$ such that

$$X \times X \ni (x, y) \mapsto (\rho^2 - f(x)f(y)^*)k(x, y)$$

is a positive semi-definite kernel on X . Let $H^\infty(k)$ denote the set of functions $f : X \rightarrow \mathbb{C}$ endowed with the norm,

$$\|f\|_k = \inf\{\rho > 0 : (\rho^2 - f(x)f(y)^*)k(x, y) \succeq 0 \text{ for all } k \in \mathcal{K}\}.$$

Here $\succeq 0$ means the relevant kernel is positive semi-definite.

An element f of $H^\infty(k)$ is identified with the operator $M_f : H^2(k) \rightarrow H^2(k)$ whose adjoint is determined by $M_k(f)^*k(\cdot, z)h = f(z)^*k(\cdot, z)h$. Indeed,

$$\|M_k(f)^*\|_k = \|f\|_k.$$

Hence $M_k : H^\infty(k) \rightarrow \mathcal{B}(H^2(k))$ defined by $f \mapsto M_k(f)$ is an isometric unital representation. Moreover, viewing $H^\infty(k)$ as a subalgebra of $\mathcal{B}(H^2(k))$ determines an operator algebra structure on $H^\infty(k)$.

Lemma 2.3. *Suppose X is finite. If \mathcal{H} is a Hilbert space and $\tau : H^\infty(k) \rightarrow \mathcal{B}(\mathcal{H})$ is a completely contractive unital representation, then there is a Hilbert space \mathcal{E} and an isometry $V : \mathcal{H} \rightarrow \mathcal{E} \otimes H^2(k)$ such that*

$$\tau(f) = V^*(I \otimes M_k(f))V.$$

Proof. Identify $H^\infty(k)$ with the subspace $\{M_k(f) : f \in H^\infty(k)\}$ of $\mathcal{B}(H^2(k))$. Since τ is completely contractive and unital, it extends to a completely contractive unital map $\Phi : \mathcal{B}(H^2(k)) \rightarrow \mathcal{B}(\mathcal{H})$. By Stinespring's representation theorem, there exists a Hilbert space \mathcal{L} , an isometry $V : \mathcal{H} \rightarrow \mathcal{L}$, and a representation $\pi : \mathcal{B}(H^2(k)) \rightarrow \mathcal{B}(\mathcal{L})$ such that

$$\Phi(T) = V^* \pi(T) V.$$

In particular, for $f \in H^\infty(k)$, we have $\tau(f) = V^* \pi(M_k(f)) V$.

Since $H^2(k)$ is finite dimensional (as X is finite), π is a multiple of the identity representation; i.e., up to unitary equivalence, $\pi(T) = I \otimes T$, and under this identification there is a Hilbert space \mathcal{E} such that $\mathcal{L} = \mathcal{E} \otimes H^2(k)$. \square

3. THE PROOF FOR FINITE X

In this section we prove Theorem 1.4 first under the added hypothesis that X is a finite set. Accordingly, until Section 4, assume that X is finite.

3.1. Representations of quotients. Given $f : X \rightarrow \mathbb{C}$, let $\mathcal{Z}(f)$ denote the zero set of f . The statement of the following lemma also uses the notation $\mathcal{K}|_Y$ from Remark 1.3

Lemma 3.1. *Suppose*

- (i) \mathcal{K} is an Agler interpolation family on the finite set X ;
- (ii) $k \in \mathcal{K}_n$;
- (iii) \mathcal{H} and \mathcal{E} are Hilbert spaces, and $V : \mathcal{H} \rightarrow \mathcal{E} \otimes H^2(k)$ is an isometry;
- (iv) $\sigma : H^\infty(k) \rightarrow \mathcal{B}(\mathcal{H})$ given by

$$H^\infty(k) \ni f \mapsto V^*(I \otimes M_k(f))V$$

is a (unital) representation; and

- (v) $Y \subset X$.

If $\sigma(g) = 0$ whenever $Y \subset \mathcal{Z}(g)$, then, for each $\psi \in H^\infty(k)$,

$$\|\sigma(\psi)^*\| \leq \sup\{\|M_\kappa(\psi|_Y)\| : \kappa \in \mathcal{K}|_Y\}.$$

Remark 3.2. Note $\sigma(\psi)^*$ depends only upon $\psi|_Y$. In fact, σ induces a representation $\tilde{\sigma} : H^\infty(k)/I \rightarrow \mathcal{B}(\mathcal{H})$, where I is the ideal of functions in $H^\infty(k)$ which vanish on the complement, \tilde{Y} , of Y in X .

Proof. Fix $\psi \in H^\infty(k)$ and $\epsilon > 0$. Choose unit vectors h, γ in \mathcal{H} such that

$$(3.1) \quad \|\sigma(\psi)^*\| \leq \langle \sigma(\psi)^* h, \gamma \rangle + \epsilon.$$

Because X is a finite set, there exists a finite dimensional subspace \mathcal{E}_0 of \mathcal{E} such that $V\gamma \in \mathcal{E}_0 \otimes H^2(k)$. Let K denote the kernel $K : X \times X \rightarrow \mathcal{B}(\mathcal{E}_0) \otimes H^2(k)$ defined by

$$K(x, y)e \otimes v = e \otimes k(x, y)v.$$

Since \mathcal{K} is closed with respect to direct sums, $K \in \mathcal{K}$. Indeed, K is the direct sum of k with itself m times, where m is the finite dimension of

\mathcal{E}_0 . Let $N = mn$ and view $K : X \times X \rightarrow \mathbb{C}^N$. Summarizing, $H^\infty(k) = H^\infty(K)$ (as operator algebras), $\mathcal{E} \otimes H^2(k)$ is canonically identified with $H^2(K) \oplus (\mathcal{E}_0^\perp \otimes H^2(k))$, and $V\gamma \in H^2(K)$.

Let \mathbf{P} denote the projection onto $H^2(K)$. Thus $\mathbf{P} = P_{\mathcal{E}_0} \otimes I$, from which it follows that the subspace $H^2(K)$ reduces $(I_{\mathcal{E}} \otimes M_k(\varphi)^*)$ for each $\varphi \in H^\infty(k)$. Thus, for $\mathbf{h} \in \mathcal{H}$,

$$\begin{aligned}
 \langle \sigma(\varphi)^* \mathbf{h}, \gamma \rangle &= \langle V^*(I_{\mathcal{E}} \otimes M_k(\varphi)^*) V \mathbf{h}, \gamma \rangle \\
 &= \langle \mathbf{P}(I_{\mathcal{E}} \otimes M_k(\varphi)^*) V \mathbf{h}, V \gamma \rangle \\
 (3.2) \quad &= \langle (I_{\mathcal{E}_0} \otimes M_k(\varphi)^*) \mathbf{P} V \mathbf{h}, V \gamma \rangle \\
 &= \langle V^* M_K(\varphi)^* \mathbf{P} V \mathbf{h}, \gamma \rangle,
 \end{aligned}$$

where $V\gamma = \mathbf{P}V\gamma$ was used in the second equality.

Because of item (iii) in the definition of interpolation family and Lemma 2.2, for $x \in X$,

$$Q_x K(\cdot, y) v = \begin{cases} K(\cdot, x) v & y = x; \\ 0 & y \neq x \end{cases}$$

determines a bounded operator $Q_x : H^2(K) \rightarrow H^2(K)$.

Next observe $Q_x^2 = Q_x$, the range of Q_x is $[K(\cdot, x)v : v \in \mathbb{C}^N]$, there is the (non-orthogonal) resolution $I = \sum_x Q_x$, and

$$(3.3) \quad M_K(\varphi)^* Q_x = \varphi(x)^* Q_x$$

for $\varphi \in H^\infty(k)$.

For $x \in X$, let

$$\mathcal{G}_x = Q_x \mathbf{P} V \mathcal{H}.$$

Observe \mathcal{G}_x is invariant for $\{M_K(\psi)^* : \psi \in H^\infty(K)\}$ because of equation (3.3). Thus $\mathcal{G}_{\bar{Y}} = \sum_{z \notin Y} \mathcal{G}_z$ is invariant for $\{M_K(\psi)^* : \psi \in H^\infty(k)\}$. Let $\mathcal{M} = H^2(K) \ominus \mathcal{G}_{\bar{Y}}$.

If $g \in H^\infty(k)$ and $Y \subset \mathcal{Z}(g)$, and if $\mathbf{h} \in \mathcal{H}$, then

$$\begin{aligned}
 0 &= \langle \sigma(g)^* \mathbf{h}, \gamma \rangle \\
 &= \langle M_K(g)^* \mathbf{P} V \mathbf{h}, V \gamma \rangle \\
 &= \left\langle \sum_x g(x)^* Q_x \mathbf{P} V \mathbf{h}, V \gamma \right\rangle \\
 &= \left\langle \sum_{z \notin Y} g(z)^* Q_z \mathbf{P} V \mathbf{h}, V \gamma \right\rangle.
 \end{aligned}$$

The first equality follows from the hypothesis on σ which gives $\sigma(g) = 0$; the second uses equation (3.2); the third uses equation (3.3) and $I = \sum Q_x$; and the fourth equality from the fact that $g(y) = 0$ for $y \in Y$. Fix a $z_0 \notin Y$ and use item (iii) in the definition of interpolation family to choose $g \in H^\infty(k)$ such that $g(z_0) = 1$ and $g(x) = 0$ otherwise to obtain

$$0 = \langle Q_{z_0} \mathbf{P} V \mathbf{h}, V \gamma \rangle.$$

Thus, $V\gamma$ is orthogonal to each \mathcal{G}_{z_0} and therefore to $\mathcal{G}_{\bar{Y}}$. Hence $V\gamma \in \mathcal{M}$.

Since $P_{\mathcal{M}}Q_z\mathbf{P}V\mathcal{H} = 0$ for $z \notin Y$, if $h \in \mathcal{H}$ and $\mathbf{P}Vh$ is written as

$$\mathbf{P}Vh = \sum_{y \in X} K(\cdot, y)v_y,$$

then, for $z \notin Y$,

$$(3.4) \quad P_{\mathcal{M}}K(\cdot, z)v_z = 0.$$

In particular,

$$(3.5) \quad P_{\mathcal{M}}\mathbf{P}Vh = \sum_{y \in Y} P_{\mathcal{M}}K(\cdot, y)v_y.$$

Thus, with \mathcal{L} equal to the span of $\{K(\cdot, y)v : y \in Y, v \in \mathbb{C}^N\}$, it follows that $P_{\mathcal{M}}\mathbf{P}V\mathcal{H} \subset \mathcal{L}$.

From Lemma 2.1 there is an M , a kernel $\kappa \in \mathcal{K}_M$, and a function $G : X \rightarrow M_{N,M}$ such that

$$(3.6) \quad \langle P_{\mathcal{M}}K(\cdot, y)v, K(\cdot, x)u \rangle = \langle \kappa(x, y)G(y)^*v, G(x)^*u \rangle.$$

In particular, the map $W : \mathcal{L} \rightarrow H^2(\kappa|_Y)$ defined by $WP_{\mathcal{M}}K(\cdot, y)v = \kappa(\cdot, y)G(y)^*v$ is (well defined and) an isometry.

Returning to the vector $h \in \mathcal{H}$ in equation (3.1), there exists $h_x, \gamma_x \in \mathbb{C}^N$ such that

$$\begin{aligned} \mathbf{P}Vh &= \sum_{x \in X} Q_x\mathbf{P}Vh = \sum_{x \in X} K(\cdot, x)h_x \\ V\gamma &= \sum_{x \in X} Q_xV\gamma = \sum_{x \in X} K(\cdot, x)\gamma_x. \end{aligned}$$

Note that, since h and γ are unit vectors, that $\|\mathbf{P}Vh\| \leq 1$ and $\|V\gamma\| = 1$.

With these notations and for $\varphi \in H^\infty(k)$,

$$\begin{aligned}
 \langle \sigma(\varphi)^* h, \gamma \rangle &= \langle M_K(\varphi)^* \mathbf{P}Vh, V\gamma \rangle \\
 &= \sum_{y \in X} \langle \varphi(y)^* K(\cdot, y) h_y, P_{\mathcal{M}} V\gamma \rangle \\
 &= \sum_{x, y \in X} \langle \varphi(y)^* K(\cdot, y) h_y, P_{\mathcal{M}} K(\cdot, x) \gamma_x \rangle \\
 &= \sum_{x, y \in X} \langle \varphi(y)^* P_{\mathcal{M}} K(\cdot, y) h_y, P_{\mathcal{M}} K(\cdot, x) \gamma_x \rangle \\
 &= \sum_{x, y \in Y} \langle \varphi(y)^* P_{\mathcal{M}} K(\cdot, y) h_y, P_{\mathcal{M}} K(\cdot, x) \gamma_x \rangle \\
 &= \sum_{x, y \in Y} \langle \varphi(y)^* W P_{\mathcal{M}} K(\cdot, y) h_y, W P_{\mathcal{M}} K(\cdot, x) \gamma_x \rangle \\
 &= \sum_{x, y \in Y} \langle \varphi(y)^* \kappa(\cdot, y) G(y)^* h_y, \kappa(\cdot, x) G(x)^* \gamma_x \rangle \\
 &= \langle M_{\kappa|_Y}(\varphi|_Y)^* \sum_{y \in Y} k(\cdot, y) G(y)^* h_y, \sum_{x \in Y} k(\cdot, x) G(x)^* \gamma_x \rangle \\
 &= \langle M_{\kappa|_Y}(\varphi|_Y)^* W P_{\mathcal{M}} \mathbf{P}Vh, W V\gamma \rangle.
 \end{aligned}$$

Here the first equality follows from the definition of σ ; the second uses equation 3.3 and $V\gamma \in \mathcal{M}$; the fifth uses equation (3.4); the sixth that $W : \mathcal{L} \rightarrow H^2(\kappa|_Y)$ is an isometry; the seventh the definition of W ; and finally the last equality uses both the definition of W and equation (3.5).

Hence,

$$\begin{aligned}
 \|\sigma(\varphi)^* - \epsilon\| &\leq |\langle \sigma(\varphi)^* h, g \rangle| \\
 &= |\langle M_{\kappa|_Y}(\varphi|_Y)^* W P_{\mathcal{M}} \mathbf{P}Vh, V\gamma \rangle| \\
 &\leq \|M_{\kappa|_Y}(\varphi|_Y)^*\| \|W P_{\mathcal{M}} \mathbf{P}Vh\| \|W V\gamma\| \\
 &\leq \|M_{\kappa|_Y}(\varphi|_Y)^*\| \|h\| \|\gamma\|.
 \end{aligned}$$

and the proof is complete. \square

3.2. The end of the proof for finite X . In this subsection we complete the proof of Theorem 1.4 in the case that X is finite, in which case there exists m and x_1, \dots, x_m such that $Y = X \setminus \{x_1, \dots, x_m\}$. Fix $g : Y \rightarrow \mathbb{C}$. Suppose $\psi : X \rightarrow \mathbb{C}$ extends g so that $g = \psi|_Y$, and define,

$$\rho = \sup\{\|M_{k|_Y}(\psi|_Y)\| : k \in \mathcal{K}\}.$$

Note that ρ depends only upon g .

Let \tilde{k} be a given element of \mathcal{K} . Let $\mathcal{I}_{\tilde{k}}$ denote the ideal of functions in $H^\infty(\tilde{k})$ which vanish Y . The quotient $H^\infty(\tilde{k})/\mathcal{I}_{\tilde{k}}$ is a unital operator algebra and hence (by the BRS theorem) it has a completely contractive unital representation τ on a Hilbert space \mathcal{H} .

The quotient mapping

$$\pi : H^\infty(\tilde{k}) \rightarrow H^\infty(\tilde{k})/I_{\tilde{k}}$$

is completely contractive and unital. Thus, $\sigma = \tau \circ \pi : H^\infty(\tilde{k}) \rightarrow \mathcal{B}(\mathcal{H})$ is a completely contractive unital representation. Further, because τ is a (complete) isometry,

$$\|\pi(\psi)\| = \|\sigma(\psi)\|$$

for $\psi \in H^\infty(\tilde{k})$. Since π is a unital completely contractive representation of $H^\infty(\tilde{k})$, π has the form given in Lemma 2.3. Hence, Lemma 3.1 applies to give

$$\|\pi(\psi)\| \leq \rho.$$

Suppose now that $\rho' > \rho$. Then, by the definition of the quotient norm, there exists a φ such that $\pi(\varphi) = \pi(\psi)$ and so that

$$(3.7) \quad X \times X \ni (x, y) \rightarrow [(\rho')^2 - \varphi(x)\varphi(y)^*]\tilde{k}(x, y)$$

is positive semi-definite.

Consider the set

$$C_{\tilde{k}} = \{(\varphi(x_1), \dots, \varphi(x_m)) : \pi(\varphi) = \pi(\psi) \text{ and equation (3.7) holds}\} \subset \mathbb{C}^m.$$

From above $C_{\tilde{k}}$ is nonempty. It is also closed, and item (iv) in the definition of interpolation family implies it is bounded. Because \mathcal{K} is closed with respect to direct sums, the collection $\{C_{\tilde{k}} : \tilde{k} \in \mathcal{K}\}$ has the finite intersection property. Hence, there exists a φ such that $\varphi|_Y = g$ and, for each $k \in \mathcal{K}$, the kernel

$$(3.8) \quad X \times X \ni (x, y) \rightarrow [(\rho')^2 - \varphi(x)\varphi(y)^*]k(x, y)$$

is positive semi-definite.

To finish the proof, choose a sequence $\rho_\ell > \rho$ converging to ρ . There exists φ_ℓ such that the kernel in equation (3.8), with φ_ℓ in place of φ and ρ_ℓ in place of ρ' , is positive semi-definite. Because φ_ℓ is uniformly bounded (again using item (iv) of the definition of interpolation family) it has a subsequence converging pointwise to some f which then satisfies the conclusion of the Theorem 1.4

4. THE CASE OF ARBITRARY X

The passage from finite X to infinite X involves a Zorn's Lemma argument.

Let \mathcal{K} denote a given interpolation family on a set X . Let Y , a finite subset of X , $g : Y \rightarrow \mathbb{C}$ and $\rho > 0$ such that for each $k \in \mathcal{K}$ the kernel

$$Y \times Y \ni (x, y) \mapsto (\rho^2 - g(x)g(y)^*)k(x, y)$$

is positive semi-definite, be given.

Consider the collection \mathcal{S} of pairs (U, f) where $Y \subset U \subset X$, $f : U \rightarrow \mathbb{C}$, $f|_Y = g$, and for each $k \in \mathcal{K}$ the kernel

$$U \times U \ni (x, y) \mapsto (\rho^2 - f(x)f(y)^*)k(x, y)$$

is positive semi-definite.

Partially order \mathcal{S} as follows. Say $(U, f) \leq (W, h)$ if $U \subset W$ and $h|_U = f$. Suppose $\mathcal{C} = \{(U, f_U)\}$ is a well ordered chain from \mathcal{S} . To see that \mathcal{C} has an upper bound, let $T = \cup U$ and define $h : T \rightarrow \mathbb{C}$ by $h(x) = f_U(x)$, where (U, f_U) is any element of \mathcal{C} for which $x \in U$. The fact that \mathcal{C} is linearly ordered implies that h is well defined. Further, if F is any finite subset of T , then there exists a $(U, f_U) \in \mathcal{C}$ such that $F \subset U$ and hence, for each $k \in \mathcal{K}$, the matrix

$$\begin{aligned} A_{k,F} &= ((\rho^2 - f_U(x)f_U(y)^*)k(x, y))_{x,y \in F} \\ &= ((\rho^2 - h(x)h(x)^*)k(x, y)) \end{aligned}$$

is positive semi-definite. It follows that $(T, h) \in \mathcal{S}$ and is an upper bound for \mathcal{C} .

By Zorn's Lemma, \mathcal{C} has a maximal element (W, h) . Suppose $W \neq X$. In this case, there is a point $z \in X \setminus W$. Given a finite subset $F \subset Y$, let $G = F \cup \{z\}$. For each $u \in \mathbb{C}$, define a function $q : G \rightarrow \mathbb{C}$ by declaring $q|_F = h|_F$ and $q(z) = u$. Now define C_F to be the set of $u \in \mathbb{C}$ for which the kernel

$$G \times G \mapsto (\rho^2 - q(x)q(y)^*)k(x, y)$$

is positive semidefinite for all $k \in \mathcal{K}$. The set C_F is nonempty by the finite case of Theorem 1.4 and is also closed. It is bounded by condition (iv) of Definition 1.2. Thus C_F is compact.

The collection $\{C_F : F \subset X, |F| < \infty\}$ has the finite intersection property and hence there is a u_* such that

$$u_* \in \cap \{C_F : F \subset X, |F| < \infty\}.$$

Define $h_* : Y \cup \{z\} \rightarrow \mathbb{C}$ by $h_*|_Y = h$ and $h_*(z) = u_*$. Then $(W \cup \{z\}, h_*) \in \mathcal{C}$ and is greater than (W, h) , a contradiction which completes the proof.

5. SCALAR INTERPOLATION

Let \mathcal{K} be an Agler interpolation family of kernels on a set X . Let \mathcal{K}_* denote those scalar kernels k on X which have the form

$$k(x, y) = G(y)^* K(x, y) G(x)$$

for some N , kernel $K \in \mathcal{K}_N$, and function $G : X \rightarrow \mathbb{C}^N$. The following lemma says, under the the conditions of equations 1.1 and 1.2 in the statement of Theorem 1.4, that \mathcal{K} can be replaced by \mathcal{K}_* .

Lemma 5.1. *If Y is a subset of X , $\rho > 0$, and $f : Y \rightarrow \mathbb{C}$, then the kernel*

$$Y \times Y \ni (x, y) \mapsto (\rho^2 - f(x)f(y)^*)k(x, y)$$

is positive semi-definite for every $k \in \mathcal{K}$ if and only if the kernel

$$Y \times Y \ni (x, y) \mapsto (\rho^2 - f(x)f(y)^*)k_*(x, y)$$

is positive for every $k_ \in \mathcal{K}_*$.*

Proof. Fix $k \in \mathcal{K}$ and a finite subset $F \subset X$ and consider the block matrix

$$A = ((\rho^2 - f(x)f(y)^*)k(x, y))_{x, y \in F}.$$

Thus A is a matrix with $n \times n$ matrix entries. Given a function $H : F \rightarrow \mathbb{C}^n$ viewed as a vector,

$$\begin{aligned} \langle AH, H \rangle &= \sum \langle A_{x, y} H(y), H(x) \rangle \\ &= \langle [(\rho^2 - f(x)f(y)^*)H(y)^*k(x, y)H(y)]_{x, y \in F} o(y), o(x) \rangle, \end{aligned}$$

where $o : F \rightarrow \mathbb{C}$ is the constant function $o(x) = 1$. Hence, if

$$F \times F \ni (x, y) \mapsto (\rho^2 - f(x)f(y)^*)H(y)^*k(x, y)H(y)$$

is positive semi-definite for each H , then A is positive semi-definite. \square

6. EXAMPLES: THE DISC AND THE ANNULUS

For the case of the disc, let $\mathcal{K}_n = \{s_n = I_n \otimes s\}$, where I_n is the identity $n \times n$ matrix and s is Szegő's kernel. Given a unit vector $\gamma \in \mathbb{C}^n$ and $\lambda \in \mathbb{D}$ let $Q = I - \gamma\gamma^*$, and let φ_λ denote a Möbius map of the disc sending λ to 0, and $G = \varphi_\lambda \gamma \gamma^* + Q$. It is readily verified that

$$\begin{aligned} k'(z, w) &= s_n(z, w) - \frac{s_n(z, \lambda)\gamma\gamma^*s_n(\lambda, w)}{\gamma^*s_n(\lambda, \lambda)\gamma} \\ &= G(w)^*s_n(z, w)G(z). \end{aligned}$$

Hence \mathcal{K} is an Agler interpolation family.

Let \mathbb{A} denote an annulus, $\{r < |z| < \frac{1}{r}\}$. There is a family $k_t(z, w)$ of scalar kernels parameterized by T in the unit circle \mathbb{T} which collectively play a role on the annulus similar to that played by Szegő's kernel on the disc [S65]. These are the kernels appearing in Abrahamse's interpolation theorem on \mathbb{A} [Ab]. It turns out that given $t \in \mathbb{T}$ and $\lambda \in \mathbb{A}$ there is an $s \in \mathbb{T}$ (which can be explicitly described in terms of the Abel-Jacobi map) and an analytic function φ_λ such that

$$k_t(z, w) - \frac{k_t(z, \lambda)k_t(\lambda, w)}{k_t(\lambda, \lambda)} = \varphi_\lambda(w)^*k_s(z, w)\varphi_\lambda(z).$$

Moreover, to each t and s there is a λ such that the above identity holds, explaining, at least heuristically, the need to consider the whole Sarason collection of kernels when interpolating on \mathbb{A} .

Let \mathcal{K}_n denote the collection of kernels of the form $k_{t_1} \oplus \cdots \oplus k_{t_n}$. The results in [AD] show that $\mathcal{K} = (\mathcal{K}_n)$ is an Agler interpolation family on \mathbb{A} . Moreover, interpolation with respect to this family is interpolation in $H^\infty(\mathbb{A})$ as in [Ab].

As a final remark, note that in the proof of Lemma 3.1 and using the notations there if k is a direct sum of kernels and if $\mathcal{G}_{\tilde{\gamma}} = \mathcal{L}$, then κ is also the direct sum of scalar kernels. If this were always the case, then there would be no need to consider direct sums in the definition of interpolation family. Thus, the fact that, for scalar interpolation on a multiply connected domain

it suffices to consider scalar kernels only represents additional structure not modeled by Theorem 1.4.

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