“Noncommutative” Aleksandrov-Clark measures

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THE DISK: Fix $b \in \text{Ball}(H^\infty)$, and a scalar $|\alpha| = 1$. Define:

- the *de Branges-Rovnyak space* $\mathcal{H}(b)$: RKHS with kernel

$$k^b(z,w) = \frac{1 - b(z)b(w)^*}{1 - zw^*}$$

- the *Aleksandrov-Clark measure* μ_α:

$$\frac{1 + b(z)\alpha^*}{1 - b(z)\alpha^*} = \int_{\mathbb{T}} \frac{1 + z\zeta^*}{1 - z\zeta^*}\, d\mu_\alpha(\zeta) + it$$

- $$P^2(\mu) = \text{closure of analytic polynomials in } L^2(\mu)$$

Facts:

- $\mathcal{H}(b) \subset H^2$ contractively
- (dB-R) $\mathcal{H}(b)$ is invariant under the backward shift

$$S^* f(z) = \frac{f(z) - f(0)}{z}$$

and if $X = S^*|_{\mathcal{H}(b)}$,

$$\|Xf\|_b^2 \leq \|f\|_b^2 - |f(0)|^2$$

- (Clark, Aleksandrov) The *normalized Cauchy transform*

$$(V_\mu f)(z) := (1 - b(z)\alpha^*) \int_{\mathbb{T}} \frac{f(\zeta)}{1 - z\zeta^*} \, d\mu(\zeta)$$

extends to a unitary operator from $P^2(\mu)$ onto $\mathcal{H}(b)$.

More facts:

- (Clark, ...) V_μ intertwines M_ζ^* on $P^2(\mu)$ with a rank-one perturbation of X:

$$V_\mu M_\zeta^* V_\mu^* = X + (1 - b(0))^{-1}(S^* b) \otimes k_0^b$$

  If $P^2(\mu) = P_0^2(\mu)$, then M_ζ^* is unitary.

- (dB-R, Sarason) TFAE:
  - $\mathcal{H}(b)$ is z-invariant
  - b is NOT an extreme point of Ball(H^∞)
  - $b \in \mathcal{H}(b)$

  (Reason: b IS extreme iff $P^2(\mu) = L^2(\mu)$, iff $P_0^2(\mu) = P^2(\mu)$, iff $\log(1 - |b|^2) \notin L^1$)

KEY IDENTITY: If $|z|, |w| < 1$ and $|\zeta| = 1$,

$$\frac{1 + z\zeta^*}{1 - z\zeta^*} + \frac{1 + w^*\zeta}{1 - w^*\zeta} = 2\frac{1 - zw^*}{(1 - z\zeta^*)(1 - w^*\zeta)}$$

since

$$(z\zeta^*)(w^*\zeta) = zw^*$$

Integrating this $d\mu$ proves

$$\langle k_w, k_z \rangle_\mu = (1 - b(z))^{-1}(1 - b(w)^*)^{-1}\frac{1 - b(z)b(w)^*}{1 - zw^*}$$

Unitarity of V_μ follows.

THE BALL ($\mathbb{B}^d \subset \mathbb{C}^d$)

- *Drury-Arveson space* H^2_d: RKHS with kernel

$$k(z,w) = \frac{1}{1-\langle z,w\rangle}$$

- *de Branges-Rovnyak space* $\mathcal{H}(b)$: RKHS with kernel

$$k^b(z,w) = \frac{1-b(z)b(w)^*}{1-\langle z,w\rangle}$$

(when $\|bf\| \leq \|f\|$ for all $f \in H^2_d$)

- $\mathcal{H}(b) \subset H^2_d$ contractively.

Backward shift?

Look for solution operators for the *Gleason problem*:

$$f(z) - f(0) = \sum z_j f_j(z)$$

Theorem (Ball-Bolotnikov-Fang)

For all b, there exists a contractive solution to the Gleason problem*: operators $X_1, \dots X_d$ on $\mathcal{H}(b)$ such that*

$$f(z) - f(0) = \sum z_j (X_j f)(z)$$

and

$$\sum \|X_j f\|_b^2 \leq \|f\|_b^2 - |f(0)|^2.$$

A structure theorem for the X's:

Theorem (J.)

A tuple $(X_1, \ldots X_d)$ is a contractive solution to the Gleason problem in $\mathcal{H}(b)$ if and only if it acts on kernels by

$$X_j k_w^b = w_j^* k_w^b - b(w)^* b_j$$

where the $b_j \in \mathcal{H}(b)$ satisfy

i) $b(z) - b(0) = \sum z_j b_j(z)$

ii) $\sum \|b_j\|_b^2 \leq 1 - |b(0)|^2$

Equivalently,

$$(X_j^* f)(z) = z_j f(z) - \langle f, b_j \rangle_b b(z)$$

Corollary (J.)

$\mathcal{H}(b)$ is z_j-invariant for all $j = 1, \dots d$ if and only if $b \in \mathcal{H}(b)$.

When is $b \in \mathcal{H}(b)$? In the disk, if and only if b is not an extreme point of Ball(H^∞).

In the ball?...

What is μ?

Fix a Hilbert space H and a system of d isometries with orthogonal ranges:

$$L_i^* L_j = \delta_{ij} I_H$$

Let

$$\mathcal{A} = \overline{\text{alg}\{I, L_1, \dots L_d\}}$$

(the *noncommutative disk algebra*; disk algebra when $d = 1$), and

$$\overline{\mathcal{A} + \mathcal{A}^*}$$

the *Cuntz-Toeplitz operator system* ($C(\mathbb{T})$ when $d = 1$).

The L's give a NC Herglotz formula for b:

Theorem (McCarthy-Putinar; Popescu)

If b is a contractive multiplier of H^2_d and $|\alpha| = 1$, there exists a positive linear functional (state) μ_α on $\mathcal{A} + \mathcal{A}^*$ such that

$$\frac{1 + b(z)\alpha^*}{1 - b(z)\alpha^*} = \mu_\alpha((I + \langle z, L\rangle)(I - \langle z, L\rangle)^{-1}) + it$$

where

$$\langle z, L\rangle = z_1 L_1^* + \cdots + z_d L_d^*.$$

When $d = 1$, the measure μ_α is unique.

NOT unique (in general) for $d > 1$...

...to recover uniqueness, shrink the operator system.

Introduce the *NC Cauchy-Fantappiè kernel*

$$K_z = (I - \langle z, L \rangle)^{-1*}$$

and the *symmetric part* of the NC disk algebra;

$$\mathcal{B} = \overline{\mathrm{span}\{K_z : z \in \mathbb{B}^d\}}$$

(not an algebra!!)

and the *symmetric part* of the Cuntz-Toeplitz operator system:

$$\mathcal{B} + \mathcal{B}^* \subset \mathcal{A} + \mathcal{A}^*$$

Define the μ_α as states on $\mathcal{B} + \mathcal{B}^*$; then uniqueness holds. These are the *NC-AC measures*.

Why "symmetric"? Expand K_z in a power series:

$$(I - \langle z, L \rangle)^{-1} = \sum_{\mathbf{n} \in \mathbb{N}^\mathbf{d}} z^{\mathbf{n}} L^{*(\mathbf{n})}$$

Notation: L term with $z_1^2 z_2$ is

$$L_1^2 L_2 + L_1 L_2 L_1 + L_2 L_1^2.$$

So $\mathcal{B}$ is spanned by symmetric combinations $L^{(\mathbf{n})}$ of the L's.

$P^2(\mu)$:

Lemma

$\mathcal{B}^*\mathcal{B} \subset \mathcal{B} + \mathcal{B}^*$.

Proof by example:

$$L_1^*(L_1^2L_2 + L_1L_2L_1 + L_2L_1^2) = L_1L_2 + L_2L_1$$

We can now define $P^2(\mu)$ as a GNS space: if $F, G \in \mathcal{B}$, define

$$\langle F, G\rangle_\mu = \mu(G^*F)$$

Close up etc. to get $P^2(\mu)$.

NC version of KEY IDENTITY:

$$\langle K_w, K_z \rangle_\mu = (1 - b(z))^{-1}(1 - b(w)^*)^{-1} \frac{1 - b(z)b(w)^*}{1 - \langle z, w \rangle}$$

This works because

$$\langle z, L \rangle \langle w, L \rangle^* = \sum_{i,j=1}^{d} z_i w_j^* L_i^* L_j = \langle z, w \rangle I$$

For the "vacuum state" $m(I) = 1$, $m(L^{(\mathbf{n})}) = 0$, get

$$\langle K_w, K_z \rangle_m = \frac{1}{1 - \langle z, w \rangle}.$$

Immediately:

Theorem (J.)

The (normalized) NC Cauchy transform

$$V_\mu(F)(z) = (1 - b(z))\langle F, K_z\rangle_\mu$$

extends to a unitary from $P^2(\mu)$ *onto* $\mathcal{H}(b)$.

Define

$$P_0^2(\mu) := \text{closed span of } F\text{'s in } \mathcal{B} \text{ with no } I \text{ term}$$

Definition

Say b is *quasi-extreme* if $P^2(\mu) = P_0^2(\mu)$.

Theorem (J.)

TFAE:

- $b \in \mathcal{H}(b)$
- $\mathcal{H}(b)$ *is* z_j*-invariant for each* $j = 1, \dots d$
- b *is NOT quasi-extreme.*

CONJECTURE: b is quasi-extreme if and only if it is an extreme point of the unit ball of multipliers of H^2_d. (True when $d = 1$.)

Theorem (J.)

If b is quasi-extreme, there exists a unique solution of the Gleason problem in $\mathcal{H}(b)$ satisfying

$$\sum X_j^* X_j = I - k_0^b \otimes k_0^b$$

For this choice of X's there is a Clark theorem...

Theorem (J.)

Let b be quasi-extreme; X the canonical solution to the Gleason problem. Then:

i) *the NC-AC measure μ has a unique positive extension ν to the full Cuntz-Toeplitz operator system $\mathcal{A} + \mathcal{A}^*$,*

ii) *the rank one perturbation*

$$U_j := X_j + (1 - b(0))^{-1} b_j \otimes k_0^b$$

is a row-coisometry–NOT unitary, but

iii) *the minimal isometric dilation $(V_1, \ldots V_d)$ of the tuple $(U_1, \ldots U_d)$ is unitarily equivalent to the generators $\pi_\nu(L_i)$ of the GNS representation from ν.*

(iii) says ν is the "spectral measure" of the V's.

Summary:

- In the disk,

$$k(z, w) = \frac{1}{1 - zw^*} \xrightarrow{w \to \partial\mathbb{D}} \frac{1}{1 - z\zeta^*} \in C(\mathbb{T})$$

  In the ball, used "quantized" kernels instead:

$$K_z = (1 - \langle z, L \rangle)^{-1} \in \mathcal{B} + \mathcal{B}^*$$

- Conjecture: b is an extreme point of the contractive multipliers of H^2_d if and only if $P^2(\mu) = P^2_0(\mu)$.