DILATIONS AND CONSTRAINED ALGEBRAS

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ABSTRACT. It is well known that contractive representations of the disk algebra are completely contractive. Let $\mathscr{A}$ denote the subalgebra of the disk algebra consisting of those functions f for which $f'(0) = 0$. We prove that there are contractive representations of $\mathscr{A}$ which are not completely contractive, and furthermore characterize those contractive representations which are completely contractive.

1. INTRODUCTION

Let $\mathbb{D}$ denote the unit disk in the complex plane and $\overline{\mathbb{D}}$ its closure. The disk algebra, $\mathbb{A}(\mathbb{D})$, is the closure of analytic polynomials in $C(\overline{\mathbb{D}})$, the space of continuous functions on $\overline{\mathbb{D}}$ with the supremum norm. The *Neil algebra* is the subalgebra of the disk algebra given by

$$\mathscr{A} = \{f \in \mathbb{A}(\mathbb{D}) : f'(0) = 0\}.$$

Constrained algebras, of which $\mathscr{A}$ is perhaps the simplest example, are of current interest as a venue for function theoretic operator theory, such as Pick interpolation. See for instance [DPRS, Ra, JKM, BH] and the references therein.

Let H denote a complex Hilbert space and $B(H)$ the bounded linear operators on H. A unital representation $\pi : \mathscr{A} \to B(H)$ on H is *contractive* if $\|\pi(f)\| \leq \|f\|$ for all $a \in \mathscr{A}$, where $\|f\|$ represents the norm of f as an element of $C(\overline{\mathbb{D}})$ and $\|\pi(f)\|$ is the operator norm of $\pi(f)$. Unless otherwise indicated, in this article representation will mean unital contractive representation.

Let $M_n(\mathscr{A})$ denote the $n \times n$ matrices with entries from $\mathscr{A}$. The norm $\|F\|$ of an element $F = (f_{j,\ell})$ in $M_n(\mathscr{A})$ is the supremum of the set $\{\|F(z)\| : z \in \mathbb{D}\}$, where $\|F(z)\|$ is the operator norm of the $n \times n$ matrix $F(z)$. Applying π to each entry of F,

$$\pi^{(n)}(F) = 1_n \otimes \pi(F) = \big(\pi(f_{j,\ell})\big)$$

produces an operator on the Hilbert space $\oplus_1^n H$ and $\|\pi^{(n)}(F)\|$ is then its operator norm. The mapping π is *completely contractive* if for each n and $F \in M_n(\mathscr{A})$,

$$\|\pi^{(n)}(F)\| \leq \|F\|.$$

The following theorem is the first main result of this article.

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Theorem 1.1. *There exists a finite dimensional Hilbert space and a unital contractive representation $\pi : \mathscr{A} \to B(H)$ which is not completely contractive. In fact, there exists a 2×2 matrix rational inner function F (with poles outside of the closed disk) such that $\|F\| \leq 1$, but $\|\pi(F)\| > 1$.*

Theorem 2.1 gives a necessary and sufficient condition for a unital representation of $\mathscr{A}$ to be completely contractive. An operator $T \in B(H)$ is a *contraction* if it has operator norm less than or equal to one. Since the algebra $\mathscr{A}$ is generated by z^2 and z^3, a contractive representation π of $\mathscr{A}$ is determined by the pair of contractions $X = \pi(z^2)$ and $Y = \pi(z^3)$. In the spirit of the examples of Kaiser and Varopoulos [Va] for the poldisk $\mathbb{D}^d$ $(d > 2)$, Corollary 3.2 asserts the existence of commuting contractions X and Y such that $X^3 = Y^2$, but for which there does not exists a unital contractive representation τ of $\mathscr{A}$ such that $X = \tau(z^2)$ and $\tau(z^3)$.

The remainder of this introduction places Theorems 1.1 and 2.1 as well as Corollary 3.2 in the larger context of rational dilation.

1.1. **Rational dilation.** The Sz.-Nagy dilation theorem states that every contraction operator dilates to a unitary operator. Unitary operators can be characterized in various ways, and in particular, they are normal operators with spectrum contained in the boundary of $\mathbb{D}$; that is, $\mathbb{T}$. A corollary of the Sz.-Nagy dilation theorem is the von Neumann inequality, which implies that T is a contraction if and only if $\|p(T)\| \leq \|p\|$ for every polynomial p, where $\|p\|$ is the again the norm of p in $C(\overline{\mathbb{D}})$.

More generally, and following Arveson [Ar], given a compact subset X of $\mathbb{C}$, let $R(X)$ denote the algebra of rational functions with poles off X with the norm $\|r\|_X$ equal to the supremum of the values of $|r(x)|$ for $x \in X$. The set X is a *spectral set* for the commuting d-tuple T of operators on the Hilbert space H if the spectrum of T lies in X and $\|r(T)\| \leq \|r\|_X$ for each $r \in R(X)$. If N is also a d-tuple of commuting operators with spectrum in X and acting on the Hilbert space K, then T *dilates* to N provided there is an isometry $V : H \to K$ such that $r(T) = V^* r(N)V$ for all $r \in R(X)$. The rational dilation problem asks: if X is a spectral set for T does T dilate to a tuple N of commuting normal operators with spectrum in the Shilov boundary of X relative to the algebra $R(X)$?

Choosing X to be the closure of a finitely connected domain D in $\mathbb{C}$ with analytic boundary, it turns out the Shilov boundary is the topological boundary and the problem has a positive answer when D is an annulus [Ag]. On the other hand, for planar domains of higher connectivity, rational dilation fails (at least when the Schottky double is hyperelliptic, though this is probably an artefact of the proofs and rational dilation will also fail without this extra condition) [DM05, AHR, Pic].

With the choice of $X = \overline{\mathbb{D}}^d$, the question becomes, if $T = (T_1, \ldots, T_d)$ is a tuple of commuting operators acting on a Hilbert space H and if

$$\|p(T_1, \ldots, T_d)\| \leq \|p\|_X$$

for every analytic polynomial $p = p(z_1, \ldots, z_d)$ in d-variables, does there exist a Hilbert space K, an isometry $V : H \to K$, and a commuting tuple $N = (N_1, \ldots, N_d)$ of normal operators on K with spectrum in $\mathbb{T}^d$ (the Shilov boundary of X) such that $p(T) = V^* p(N)V$ for every polynomial p? Andô's theorem implies the result is true for the bidisk $\mathbb{D}^2$. An example due to

Parrott implies that rational dilation fails for the polydisk $\mathbb{D}^d$, $d > 2$. Thus as things stand, the rational dilation problem has been settled for the disk, the annulus, hyperelleptic planar domains, and for polydisks.

Arveson [Ar] profoundly reformulated the rational dilation problem in terms of contractive and completely contractive representations. A tuple T acting on the Hilbert space H with spectrum in X determines a unital representation of π_T of $R(X)$ on H via $\pi_T(r) = r(T)$ and the condition that X is a spectral set for T is equivalent to the condition that this representation is bounded.

Recall that a representation π of $R(X)$ is *completely contractive* if for all n and all $F \in M_n(R(X))$, $\pi^{(n)}(F) := (\pi(F_{i,j}))$ is contractive, the norm of F being given by $\|F\|_\infty = \sup\{\|F(x)\| : x \in X\}$ with $\|F(x)\|$ the operator norm of $F(x)$. Arveson showed that T dilates to a tuple N with spectrum in the (Shilov) boundary of X (with respect to $R(X)$) if and only if π_T is completely contractive. Thus the rational dilation problem can be reformulated. Namely, is every contractive representation of $R(X)$ completely contractive?

Returning to the Neil algebra, recall that a *distinguished variety* [Ra, AM05, AM06, AKM, K10, Ve, JKM] is a variety V in $\mathbb{C}^d$ which intersects the boundary of $\overline{\mathbb{D}}^d$ at its distinguished boundary, $\mathbb{T}^d$. In looking for interesting examples which help us to delineate the border between those domains where rational dilation holds and where it fails, it is therefore particularly interesting to consider $X = V \cap \overline{\mathbb{D}}^2$, where V is a distinguished variety.

The subset V of $\mathbb{C}^2$ with $z_1 = z^2$ and $z_2 = z^3$, $z \in \mathbb{D}$, is a particularly simple but interesting example of a distinguished variety, called the *Neil parabola*. With $X = V \cap \overline{\mathbb{D}}^2$, the mapping from $R(X)$ to $\mathscr{A}$ sending $p(z, w)$ to $p(t^2, t^3)$ is a (complete) isometry. Note that excluding a cusp at $(0, 0)$, V is a manifold, and this cusp makes things just different enough to make $R(X)$ a tractable though nontrivial algebra on which to study the rational dilation problem. In the rest of the paper, we concentrate on studying the connection between contractive and completely contractive representations of $\mathscr{A}$, though the results are readily translated to $R(X)$.

While rational dilation fails for the Neil parabola, in Theorem 2.1 we also provide a characterization of the completely contractive representations of $\mathscr{A}$. However, this positive result is not used to establish Theorem 1.1. Rather the proof of Theorem 1.1 essentially comes down to a cone separation argument. The mechanics of this argument appear in Section 3. The construction of the counterexample and preliminary results are in Section 4. The proof of Theorem 1.1 concludes in Section 5, while the statement and proof of Theorem 2.1 and general facts about representations of $\mathscr{A}$ are the subject of Section 2.

2. Representations of $\mathscr{A}$

In this section we characterize the completely contractive representations of $\mathscr{A}$ and consider some examples. The characterization of contractive representations is essentially contained in the paper [DP] on test functions for $\mathscr{A}$, and this is described in the next section.

As a (unital) Banach algebra, $\mathscr{A}$ is generated by the functions z^2 and z^3. It follows that any bounded unital representation is determined by its values on these two functions. If $\pi : \mathscr{A} \to B(H)$ is a bounded representation, $X = \pi(z^2)$ and $Y = \pi(z^3)$, then X, Y are commuting operators which satisfy $X^3 = Y^2$. If we further insist that π is contractive, then X and Y are contractions. In summary, every contractive representation $\pi : \mathscr{A} \to B(H)$ determines a pair of commuting contractions X, Y such that $X^3 = Y^2$. However, as we see in Corollary 3.2, not every such pair gives rise to a contractive representation.

The following theorem characterizes the completely contractive representations of $\mathscr{A}$. For Hilbert spaces $H \subseteq K$, let P_H denote the orthogonal projection of K onto H and $|_H$ the inclusion of H into K.

Theorem 2.1 ([B]). *A representation $\pi : \mathscr{A} \to B(H)$ is completely contractive if and only if there is a Hilbert space $K \supset H$ and a unitary operator $U \in B(K)$ such that for all $n \geq 0$, $n \neq 1$,*

$$\pi(z^n) = P_H U^n|_H. \tag{1}$$

This is a consequence of the Sz.-Nagy dilation theorem together with applications of the Arveson extension and Stinespring dilation theorems. In the case of $\mathbb{A}(\mathbb{D})$, by the Sz.-Nagy dilation theorem every completely contractive representation $\pi : \mathbb{A}(\mathbb{D}) \to B(H)$ is determined by a contraction T, with $\pi(z^n) = T^n$, and $T^n = P_H U^n|_H$ for some unitary U and all $n \geq 0$. Thus a simple way to construct completely contractive representations of $\mathscr{A}$ is to fix a contraction T and restrict: put $\pi(z^2) = T^2$ and $\pi(z^3) = T^3$. However, in spite of Theorem 2.1 it is *not* the case that every completely contractive representation of $\mathscr{A}$ arises in this way, as we see in Example 2.3 below.

Proof of Theorem 2.1. Let $\pi : \mathscr{A} \to B(H)$ be a unital, completely contractive representation. Let $\mathscr{A}^* \subseteq C(\mathbb{T})$ denote the set of complex conjugates of functions in $\mathscr{A}$. Then $\mathscr{A} + \mathscr{A}^*$ is an operator system and $\rho : \mathscr{A} + \mathscr{A}^* \to B(H)$ given by

$$\rho(f + g^*) = \pi(f) + \pi(g)^*$$

is well defined. Since π is unital and $\mathscr{A} \cap \mathscr{A}^* = \mathbb{C}1$, ρ is completely positive. By the Arveson extension theorem, ρ extends to a unital, completely positive (ucp) map $\sigma : C(\mathbb{T}) \to B(H)$. By the Stinespring theorem there is a larger Hilbert space $K \supset H$, and a unitary $U \in B(K)$ such that for all $n \geq 0$,

$$\sigma(z^n) = P_H U^n|_H.$$

Since $\pi(z^n) = \sigma(z^n)$ for all nonnegative $n \neq 1$, one direction follows.

Conversely, suppose that there is a unitary operator $U \in B(K)$ such that for all $n \geq 0$, $n \neq 1$, $\pi(z^n) = P_H U^n|_H$. Then $\tilde{\pi}$ defined as $\tilde{\pi}(z^n) = U^n$, $n \in \mathbb{Z}$ defines a completely contractive representation of $C(\mathbb{T})$. So $\tilde{\pi}$ restricted the operator system $\mathscr{A} \cap \mathscr{A}^*$ is completely positive, as is its compression to H by the Stinespring dilation theorem. Since unital completely positive maps are completely contractive, $\pi = \rho|\mathscr{A}$ is completely contractive. ∎

Remark 2.2. In the above proof, obviously $T = P_H U|_H$ is a contraction. However since the restriction of σ to $\mathbb{A}(\mathbb{D})$ is not necessarily multiplicative, we cannot conclude that $\pi(z^2) = T^2$. Indeed the following example illustrates this concretely:

Example 2.3. Let K be a separable Hilbert space with orthonormal basis $\{e_j\}_{j \in \mathbb{Z}}$, and let U be the bilateral shift. Let $H \subseteq K$ be defined as $H = e_0 \vee \bigvee_{n=2}^{\infty} e_n$. Then H is invariant for U^2 and U^3, and so by Theorem 2.1, π given by $\pi(z^n) = P_H U^n|_H = U^n|_H$, $n \geq 0$, $n \neq 1$, is a completely contractive representation of $\mathscr{A}$.

If it were the case that for some $T \in B(H)$, $T^2 = \pi(z^2)$ and $T^3 = \pi(z^3)$, we would require that

$$e_3 = U^3 e_0 = \pi(z^3) = \pi(z^2)Te_0.$$

However, $\left\langle \pi(z^2)e_n, e_3 \right\rangle = \left\langle U^2 e_n, e_3 \right\rangle = 0$ for $n \geq 0$, $n \neq 1$, and hence e_3 is orthogonal to the range of $\pi(z^2)$. Thus there is no way to define Te_0 so that $e_3 = \pi(z^2)Te_0$, and so there can be no such T.

3. The set of test functions and its cone

Given $\lambda \in \mathbb{D}$, let

$$\varphi_\lambda(z) = \frac{z - \lambda}{1 - \lambda^* z}, \tag{2}$$

and let

$$\psi_\lambda(z) = z^2 \varphi_\lambda(z) \tag{3}$$

the (up to a unimodular constant) *Blaschke factor* with zero at λ, times z^2. It will be convenient to let

$$\psi_\infty = z^2$$

and at the same time let ∞ denote the point at infinity in the one point compactification $\mathbb{D}_\infty$ of the unit disk $\mathbb{D}$. Let

$$\Psi = \{\psi_\lambda : \lambda \in \mathbb{D}_\infty\},$$

with the topology and Borel structure inherited from $\mathbb{D}_\infty$. We refer to this as a set of *test functions*. It has the properties that it separates the points of $\mathbb{D}$ and for all $z \in \mathbb{D}$, $\sup_{\psi \in \Psi} |\psi(z)| < 1$.

Recall that for a set X and C^*-algebra $\mathcal{A}$, a function $k : X \times X \to \mathcal{A}$ is called a *kernel*. It is a *positive kernel* if for every finite subset $\{x_1, \ldots, x_n\}$ of X, $(k(x_i, x_j)) \in M_n(\mathcal{A})$ is positive semidefinite.

Let $M(\Psi)$ be the space of finite Borel measures on the set of test functions. Given a subset S of $\mathbb{D}$, denote by $M^+(S) = \{\mu : S \times S \to M(\Psi)\}$ the collection of positive kernels on $S \times S$ into $M(\Psi)$. Write μ_{xy} for the value of μ at the pair (x, y). By μ being positive, we mean that for all finite sets $\mathcal{G} \subseteq S$ and all Borel sets $\omega \subseteq \Psi$, the matrix

$$(\mu_{x,y}(\omega))_{x,y \in \mathcal{G}} \tag{4}$$

is positive semidefinite. For example, if μ is identically equal to a fixed positive measure ν, or more generally is of the form $\mu_{xy} = f(x)f(y)^*\nu$ for a fixed positive measure ν and bounded measurable function $f : \mathbb{C} \to \mathbb{D}$, or more generally still is a finite sum of such terms, then it is positive.

Our starting point is the following result from [DP] (stated there for functions of positive real part):

Proposition 3.1. *An analytic function f in the disk belongs to $\mathscr{A}$ and satisfies $\|f\|_\infty \leq 1$ if and only if there is a positive kernel $\mu \in M^+(\mathbb{D})$ such that*

$$1 - f(x)f(y)^* = \int_\Psi (1 - \psi(x)\psi(y)^*)\, d\mu_{xy}(\psi). \tag{5}$$

for all $x, y \in \mathbb{D}$. Furthermore, Ψ is minimal, in the sense that there is no proper closed subset of $E \subseteq \Psi$ such that for each such f, there exists a μ such that

$$1 - f(x)f(y)^* = \int_E (1 - \psi(x)\psi(y)^*)\, d\mu_{xy}(\psi). \tag{6}$$

For $E \subseteq \Psi$ a closed subset, let $C_{1,E}$ denote the cone consisting of the kernels

$$\left(\int_E (1 - \psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi) \right)_{x,y\in\mathbb{D}}. \tag{7}$$

(Equivalently, we could consider only those μ such that μ_{xy} is supported in E for all x, y.) In particular, if we choose $E = \{z^2, z^3\}$, it follows from [DP, Theorem 3.8] that there exists a function $f \in \mathscr{A}$ with $\|f\|_\infty \leq 1$ such that $1 - f(x)f(y)^* \notin C_{1,E}$. This yields in our context an analogue of the Kaiser and Varopoulos example for the tridisk:

Corollary 3.2. *There exists a pair of commuting contractive matrices X, Y with $X^3 = Y^2$, but such that the representation of $\mathscr{A}$ determined by $\pi(z^2) = X$, $\pi(z^3) = Y$ is not contractive.*

Proof. By a cone separation argument as in the proof of Proposition 3.5, there is a bounded representation π of $\mathscr{A}$ (determined by a pair of matrices X, Y with spectrum in $\mathbb{D}$) such that $\|\pi(\psi)\| \leq 1$ for each $\psi \in E$ but $\|\pi(f)\| > 1$. In particular, if we take E to be the closed set $\{z^2, z^3\}$, we see that $X = \pi(z^2)$ and $Y = \pi(z^3)$ satisfy the conditions of the corollary. ∎

3.1. **The matrix cone.** To study the action of representations on $M_2(\mathscr{A})$, consider a finite subset $\mathscr{F} \subseteq \mathbb{D}$. As usual, $M_2(\mathbb{C})$ stands for the 2×2 matrices with entries from $\mathbb{C}$. Let $\mathcal{X}_{2,\mathscr{F}}$ denote the set of all kernels $G : \mathscr{F} \times \mathscr{F} \to M_2(\mathbb{C})$ and $\mathcal{L}_{2,\mathscr{F}} \subseteq \mathcal{X}_{\mathscr{F}}$ denote the selfadjoint kernels $F : \mathscr{F} \times \mathscr{F} \mapsto M_2(\mathbb{C})$, in the sense that $F(x, y)^* = F(y, x)$. Finally, write $C_{2,\mathscr{F}}$ for the cone in $\mathcal{L}_{2,\mathscr{F}}$ of elements of the form

$$\left(\int_\Psi (1 - \psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi) \right)_{x,y\in\mathscr{F}} \tag{8}$$

where $\mu = (\mu_{x,y}) \in M_2^+(\mathscr{F})$ is a kernel taking its values $\mu_{x,y}$ in the 2×2 matrix valued measure on Ψ such that the measure

$$M(\omega) = \left(\mu_{x,y}(\omega) \right)_{x,y} \tag{9}$$

takes positive semidefinite values (in $M_N(M_2(\mathbb{C}))$). Given $f : \mathscr{F} \to \mathbb{C}^2$, the kernel $(f(x)f(y)^*)_{x,y\in\mathscr{F}}$ is called a *square*.

Lemma 3.3. *The cone $C_{2,\mathscr{F}}$ is closed and contains all squares.*

Proof. For $x \in \mathscr{F}$,

$$\sup_{\psi\in\Psi} |\psi(x)| < |x|.$$

Hence as $\mathscr{F}$ is finite, there exists a there exists $0 < \kappa \leq 1$ such that for all $x \in \mathscr{F}$ and $\psi \in \Psi$

$$1 - \psi(x)\psi(x)^* \geq \kappa.$$

Consequently, if Γ defined by

$$\Gamma(x, y) = \int_{\Psi} (1 - \psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi)$$

is in $C_{2,\mathscr{F}}$, then

$$\frac{1}{\kappa}\Gamma(x, x) \geq \mu_{x,x}(\Psi),$$

where the inequality is in the sense of the positive semidefinite order on 2×2 matrices.

Now suppose (Γ_n) is a sequence from $C_{2,\mathscr{F}}$ converging to some Γ. For each n there is a measure μ^n such that Γ_n given by

$$\Gamma_n(x, y) = \int_{\Psi} (1 - \psi(x)\psi(y)^*)\, d\mu^n_{x,y}(\psi)$$

forms a sequence from $C_{2,\mathscr{F}}$ which converges to some Γ. Hence there exists a $\tilde{\kappa} > 0$ such that for all n and all $x \in \mathscr{F}$, $\tilde{\kappa} \geq \Gamma_n(x, x)$. Consequently, for all n and all $x \in \mathscr{F}$,

$$\tfrac{\tilde{\kappa}}{\kappa} I \geq \mu^n_{x,x}.$$

By positivity of the μ^n's, it now follows that the measures $\mu^n_{x,y}$ are uniformly bounded. Hence there exists a subsequence μ^{n_j} and a measure μ such that μ^{n_j} converges weak-$*$ to μ, which therefore is positive. We conclude that

$$\Gamma = \int_{\Psi} (1 - \psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi) \in C_{2,\mathscr{F}},$$

establishing the fact that $C_{2,\mathscr{F}}$ is closed.

Now let $f : \mathscr{F} \to \mathbb{C}^2$ be given. Let δ denote the unit scalar point mass at z^3 ($\lambda = 0$). Then for $\omega \subseteq \Psi$ a Borel subset,

$$\mu_{x,y}(\omega) = f(x)\frac{1}{1 - x^3 y^{*3}}\delta(\omega) f(y)^*$$

defines a positive measure and

$$\int_{\Psi} (1 - \psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi) = f(x) f(y)^*,$$

showing that $C_{2,\mathscr{F}}$ contains the squares. ■

Elaborating on the construction at the end of the last proof, if

$$\nu(\omega) = \big(\nu_{x,y}(\omega)\big)_{x,y \in \mathscr{F}}$$

is positive semidefinite for every Borel subset ω of Ψ, each ν_{xy} a scalar valued measure, and if $f : \mathscr{F} \to \mathbb{C}^2$, then

$$\mu_{x,y}(\omega) = f(x)\nu_{x,y}(\omega) f(y)^*,$$

defines an $M_2(\mathbb{C})$ valued positive measure μ and

$$\int_{\Psi} (1 - \psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi) \in C_{2,\mathscr{F}}.$$

We therefore have the following from [DP] (see also [BBtH]).

Proposition 3.4. *If $g \in \mathscr{A}$ is analytic in a neighborhood of the closure of the disk and if $\|g\|_\infty \le 1$, then $1 - g(x)g(y)^* \in C_{2,\mathscr{F}}(1)$. Thus, if $f : \mathscr{F} \to \mathbb{C}^2$, then*

$$f(x)(1 - g(x)g(y)^*)f(y)^* \in C_{2,\mathscr{F}}.$$

3.2. **The cone separation argument.** Continue to let $\mathscr{F}$ denote a finite subset of $\mathbb{D}$. Given $F \in M_2(\mathscr{A})$, let $\Sigma_{F,\mathscr{F}}$ denote the kernel

$$\Sigma_{F,\mathscr{F}} = (1 - F(x)F(y)^*)_{x,y\in\mathscr{F}}. \tag{10}$$

Let I denote the ideal of functions in $\mathscr{A}$ which vanish on $\mathscr{F}$. Write $q : \mathscr{A} \to \mathscr{A}/I$ for the canonical projection, which is completely contractive. We use the standard notation $\sigma(T)$ for the spectrum of an operator T on Hilbert space, as well as F^t for the transpose of the matrix function F. Thus, $F^t(z) = F(z)^t$. Obviously, when $F \in M_2(\mathscr{A})$, F^t is as well, and $\|F\|_\infty = \|F^t\|_\infty$.

Proposition 3.5. *If $F \in M_2(\mathscr{A})$, but $\Sigma_{F,\mathscr{F}} \notin C_{2,\mathscr{F}}$, then there exists a a Hilbert space H and representation $\tau : \mathscr{A}/I \to B(H)$ such that*

(i) $\sigma(\tau(a)) \subseteq a(\mathscr{F})$ *for* $a \in \mathscr{A}$;
(ii) $\|\tau(q(a))\| \le 1$ *for all* $a \in \mathscr{A}$*; but*
(iii) $\|\tau(q(F^t))\| > 1$.

Moreover $\|F\| \le 1$, then the representation $\tau \circ q$ is contractive, but not completely so.

Proof. The proof proceeds by a cone separation argument: the representation is obtained by applying the GNS construction to a linear functional that separates $\Sigma_{F,\mathscr{F}}$ from $C_{2,\mathscr{F}}$.

The cone $C_{2,\mathscr{F}}$ is closed and by assumption $\Sigma_{F,\mathscr{F}}$ is not in the cone. Hence there is an $\mathbb{R}$-linear functional $\Lambda : \mathcal{L}_{\mathscr{F}} \to \mathbb{R}$ such that $\Lambda(C_{2,\mathscr{F}}) \ge 0$, but $\Lambda(\Sigma_{F,\mathscr{F}}) < 0$. Given $f : \mathscr{F} \to \mathbb{C}^2$ (that is, $f \in (\mathbb{C}^2)^{\mathscr{F}}$), recall that the square $ff^* := (f(x)f(y)^*)_{x,y\in\mathscr{F}}$ is in the cone and hence $\Lambda(ff^*) \ge 0$. Since every element of $\mathcal{X}_{\mathscr{F}}$ can be expressed uniquely in the form $G = U + iV$ where $U, V \in \mathcal{L}_{\mathscr{F}}$, there is a unique extension of Λ to a $\mathbb{C}$-linear functional $\Lambda : \mathcal{X}_{\mathscr{F}} \to \mathbb{C}$. With this extended Λ, let H denote the Hilbert space obtained by giving $(\mathbb{C}^2)^{\mathscr{F}}$ the (pre)-inner product

$$\langle f, g\rangle = \Lambda(fg^*)$$

and passing to the quotient by the space of null vectors (those f for which $\Lambda(ff^*) = 0$ — since $\mathscr{F}$ is finite, the quotient will be complete).

Define a representation ρ of $\mathscr{A}$ on H by

$$\rho(g)f(x) = g(x)f(x),$$

where the scalar valued g multiplies the vector valued f entrywise.

If $g \in \mathscr{A}$, is analytic in a neighborhood of the closure of the disk and $\|g\|_\infty \le 1$, then, by Proposition 3.4, $f(x)(1 - g(x)g(y)^*)f(y) \in C_{2,\mathscr{F}}$. Thus,

$$\langle f, f\rangle - \langle \rho(g)f, \rho(g)f\rangle = \Lambda\left((f(x)(1 - g(x)g(y)^*)f(y)^*)_{x,y\in\mathscr{F}}\right) \ge 0. \tag{11}$$

Hence, if $\|g\|_\infty \le 1$, then $\|\rho(g)\| \le 1$ and ρ is a contractive representation of $\mathscr{A}$. Moreover, since the definition of ρ depends only on the values of g on $\mathcal{F}$, it passes to a contractive representation $\tau : \mathscr{A}/I \to B(H)$. The restriction of $\mathscr{A}$ to $\mathcal{F}$ separates points of $\mathcal{F}$ (indeed, the elements of Ψ do so), and so it follows that for each $a \in \mathscr{A}$ the eigenvalues of the matrix representing $\tau(a)$ constitute the set $a(\mathcal{F})$. This proves (*i*) and (*ii*).

To prove (iii), let $\{e_1, e_2\}$ denote the standard basis for $\mathbb{C}^2$ and let $[e_j] : \mathscr{F} \to \mathbb{C}^2$ be the constant function $[e_j](x) = e_j$. Note that $\{e_i e_j^*\}_{i,j=1}^2$ are a system of 2×2 matrix units. We find

$$\rho^{(2)}(F^t)([e_1] \oplus [e_2]) = \begin{pmatrix} F_{1,1}e_1 + F_{2,1}e_2 \\ F_{1,2}e_1 + F_{2,2}e_2 \end{pmatrix}.$$

Since

$$\begin{aligned}(F_{1,1}e_1 + F_{2,1}e_2)(F_{1,1}e_1 + F_{2,1}e_2)^* &= F_{1,1}F_{1,1}^*e_1e_1^* + F_{2,1}F_{1,1}^*e_2e_1^* + F_{1,1}F_{2,1}^*e_1e_2^* + F_{2,1}F_{2,1}^*e_2e_2^* \\ &= \begin{pmatrix} F_{1,1}F_{1,1}^* & F_{1,1}F_{2,1}^* \\ F_{2,1}F_{1,1}^* & F_{2,1}F_{2,1}^* \end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}(F_{1,2}e_1 + F_{2,2}e_2)(F_{1,2}e_1 + F_{2,2}e_2)^* &= F_{1,2}F_{1,2}^*e_1e_1^* + F_{2,2}F_{1,2}^*e_2e_1^* + F_{1,2}F_{2,2}^*e_1e_2^* + F_{2,2}F_{2,2}^*e_2e_2^* \\ &= \begin{pmatrix} F_{1,1}F_{1,1}^* & F_{1,1}F_{2,1}^* \\ F_{2,1}F_{1,1}^* & F_{2,1}F_{2,1}^* \end{pmatrix},\end{aligned}$$

it follows that

$$\begin{aligned}\left\langle \rho^{(2)}(F^t)([e_1] \oplus [e_2]), \rho^{(2)}(F^t)([e_1] \oplus [e_2]) \right\rangle &= \Lambda\left(\begin{pmatrix} F_{1,1}F_{1,1}^* + F_{1,2}F_{1,2}^* & F_{1,1}F_{2,1}^* + F_{1,2}F_{2,2}^* \\ F_{2,1}F_{1,1}^* + F_{2,2}F_{1,2}^* & F_{2,1}F_{2,1}^* + F_{2,2}F_{2,2}^* \end{pmatrix}\right) \\ &= \Lambda(FF^*),\end{aligned}$$

and so

$$\left\langle (I - \rho^{(2)}(F^t)^*\rho^{(2)}(F^t))[e_1] \oplus [e_2], [e_1] \oplus [e_2] \right\rangle < 0.$$

We conclude that $\|\rho(F^t)\| > 1$, and in particular, if it happens to be the case that $\|F\|_\infty \le 1$, then ρ is not 2-contractive, and thus not completely contractive. ∎

Remark 3.6. Though it is not needed in what follows, observe that the converse of the first part of Proposition 3.5 is true: If T is an operator on Hilbert space with spectrum in $\mathscr{F}$, if $\Sigma_{F,\mathscr{F}} \in C_{2,\mathscr{F}}$ and if $\psi(T)$ is contractive for all $\psi \in \Psi$, then $F(T)$ is also contractive.

A proof follows along now standard lines (see, for instance, [DM], where the needed theorems are proved for scalar valued functions, though the proofs remain valid in the matrix case). The assumption that $\Sigma_{F,\mathscr{F}} \in C_{2,\mathscr{F}}$ means that F has a Ψ-unitary colligation transfer function representation. Since the operator T has spectrum in the finite set $\mathscr{F}$, it determines a representation of $\mathscr{A}$ which sends bounded pointwise convergent sequences in $M_2(\mathscr{A})$ to weak operator topology convergent sequences. Representations of $M_2(\mathscr{A})$ with this property and for which $\psi(T)$ is contractive for all $\psi \in \Psi$, are contractive.

4. Construction of the counterexample preliminaries

For $\lambda \in \mathbb{D}\backslash\{0\}$, let

$$\varphi_\lambda = \frac{z - \lambda}{1 - \lambda^* z}.$$

Fix distinct points $\lambda_1, \lambda_2 \in \mathbb{D}$. As a shorthand notation, write $\varphi_j = \varphi_{\lambda_j}$. Set

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} \varphi_1 & 0 \\ 0 & 1 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & \varphi_2 \end{pmatrix}, \tag{12}$$

where U is a 2×2 unitary matrix with no non-zero entries. To be concrete, choose

$$U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

In particular Φ is a 2×2 matrix inner function with $\det \Phi(\lambda) = 0$ at precisely the two nonzero points λ_1 and λ_2. The function

$$F = z^2\Phi \tag{13}$$

is in $M_2(\mathscr{A})$ and is a rational inner function, so $\|F\|_\infty = 1$.

Ultimately we will identify a finite set $\mathcal{F}$ and show that $\Sigma_{F,\mathcal{F}} \notin C_{2,\mathcal{F}}$ and thus, in view of Proposition 3.5 establish Theorem 1.1. In the remainder of this section we collect some needed preliminary lemmas.

Lemma 4.1. *Given distinct points $\lambda_1, \lambda_2 \in \mathbb{D}\backslash\{0\}$ and a 2×2 unitary matrix U, let*

$$\Theta = \begin{pmatrix} \varphi_1 & 0 \\ 0 & 1 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & \varphi_2 \end{pmatrix}, \tag{14}$$

where $\varphi_j = \varphi_{\lambda_j}$. The matrix U is diagonal; that is, there exist unimodular constants s and t such that

$$\Theta = \begin{pmatrix} s\varphi_1 & 0 \\ 0 & t\varphi_2 \end{pmatrix},$$

if and only if there exists points $a, b \in \mathbb{D}$ and 2×2 unitaries V and W such that

$$\Theta = V^* \begin{pmatrix} \varphi_a & 0 \\ 0 & \varphi_b \end{pmatrix} W.$$

Proof. The forward implication is trivial. For the converse, let $\{e_1, e_2\}$ denote the standard basis for $\mathbb{R}^2$. By taking determinants, it follows that $\{a, b\} = \{\lambda_1, \lambda_2\}$. Changing V and W if necessary, without loss of generality it can be assumed that $a = \lambda_1$ and $b = \lambda_2$. Evaluating at λ_2 it follows that $We_2 = \alpha e_2$. Because W is unitary, it now follows that W is diagonal. A similar argument shows that V is diagonal, and the result follows. ∎

Lemma 4.2. *Suppose $\mu_{i,j}$ are 2×2 matrix-valued measures on a measure space (X, Σ) for $i, j = 0, 1$. If $\mu_{i,j}(X) = I$ for all i, j and if, for each $\omega \in \Sigma$ the 4×4 (block 2×2 matrix with 2×2 matrix entries)*

$$\left(\mu_{i,j}(\omega)\right)_{i,j=1}^2$$

is positive semidefinite, then $\mu_{i,j} = \mu_{0,0}$ for each $i, j = 0, 1$.

Proof. Fix a unit vector $f \in \mathbb{C}^2$ and let

$$\nu_{i,j}(\omega) = \left\langle \mu_{i,j}(\omega) f, f \right\rangle.$$

It follows that $\nu_{i,j}(X) = 1$ and for each $\omega \in \Sigma$

$$\gamma(\omega) = \left(\nu_{i,j}(\omega)\right)_{i,j=1}^2$$

is positive semidefinite. On the other hand,

$$\gamma(X) - \gamma(\omega) \geq 0$$

and since $\gamma(X)$ is rank one (with a one in each entry), there is a constant $c = c_\omega$ such that

$$\gamma(\omega) = c\gamma(X).$$

Consequently, $\nu_{i,j}(\omega) = \nu_{1,1}(\omega)$. By polarization it now follows that $\mu_{i,j} = \mu_{1,1}$ for each $i, j = 1, 2$. ∎

Lemma 4.3. *There exist independent vectors $v_1, v_2 \in \mathbb{C}^2$ and, for any finite subset $\mathscr{F}$ of the disc, functions $a, b : \mathscr{F} \to \mathbb{C}^2$ in the span of $\{x^2 k_{\lambda_1}(x)v_1, x^2 k_{\lambda_2}(x)v_2\}$ such that*

$$\frac{I - \Phi(x)\Phi(y)^*}{1 - xy^*} = a(x)a(y)^* + b(x)b(y)^*.$$

Proof. Let M_Φ denote the operator of multiplication by Φ on $H^2_{\mathbb{C}^2}$, the Hardy-Hilbert space of $\mathbb{C}^2$-valued functions on the disk. Because Φ is unitary-valued on the boundary, M_Φ is an isometry. In fact, M_Φ is the product of three isometries in view of Equation (12). The adjoints of the first and third have one dimensional kernels. The middle term is unitary and so its adjoint has no kernel. Thus, the kernel of M_Φ^* has dimension at most two. It is evident that $k_{\lambda_1}e_1$ is in the kernel of M_Φ^*. Choose a unit vector v_2 in $\mathbb{C}^2$ with entries α and $\beta \neq 0$ such that

$$\begin{pmatrix} \alpha\varphi_{\lambda_1}(\lambda_2) \\ \beta \end{pmatrix} = Ue_2,$$

with U the unitary appearing in Equation (12). That such a choice of α and $\beta \neq 0$ is possible follows from the assumption that $\lambda_1 \neq \lambda_2$, which ensures that $\varphi_{\lambda_1}(\lambda_2) \neq 0$, and the assumption that U has no non-zero entries, giving $\beta \neq 0$. Further, with this choice of v_2 a simple calculation shows that $k_{\lambda_2}v_2$ is also in the kernel of M_Φ^*. Hence, the dimension of the kernel of M_Φ^* is two. Since M_Φ is an isometry, $I - M_\Phi M_\Phi^*$ is the projection onto the kernel of M_Φ^*.

Choose an orthonormal basis $\{a, b\}$ for the kernel of M_Φ^* so that $I - M_\Phi M_\Phi^* = aa^* + bb^*$. It now follows that, for vectors $v, w \in \mathbb{C}^2$,

$$\begin{aligned}
\left\langle \frac{I - \Phi(x)\Phi(y)^*}{1 - xy^*} v, w \right\rangle &= \left\langle (I - M_\Phi M_\Phi^*)k_y v, k_x w \right\rangle \\
&= \left\langle (aa^* + bb^*)k_y v, k_x w \right\rangle \\
&= \left\langle k_y v, a \right\rangle \langle a, k_x w\rangle + \left\langle k_y v, b \right\rangle \langle b, k_x w\rangle \\
&= \langle (a(x)a(y)^* + b(x)b(y)^*)v, w\rangle.
\end{aligned}$$

∎

The following is well known.

Lemma 4.4. *Let s be the Szegő kernel,*

$$s(x, y) = \frac{1}{1 - xy^*}.$$

If $x_1, \dots, x_m$ and $y_1, \dots, y_m$ are two m-tuples each of distinct points in the unit disk $\mathbb{D}$, then the matrix

$$M = \Big(s(x_j, y_\ell)\Big)_{j,\ell=1}^n$$

is invertible.

Proof. Suppose $Mc = 0$ where c is the vector with entries $c_1, \dots, c_m$. Let

$$r(x) = \sum c_\ell s(x, y_\ell) = [(1 - xy_1^*) \cdots (1 - xy_m^*)]^{-1} \sum c_\ell p_\ell(x),$$

for polynomials p_ℓ of degree $m-1$. Hence r is a rational function with numerator a polynomial p of degree at most $m-1$ and denominator which does not vanish on $\mathbb{D}$. The hypotheses imply that $p(x_j) = 0$ for $j = 1, 2, \dots, m$. Hence p is identically zero, as then is r. Since the kernel functions $\{s(\cdot, t_\ell) : \ell = 1, 2, \dots, m\}$ form a linearly independent set in $H^2(\mathbb{D})$, it follows that $c = 0$. ∎

Given a 2×2 matrix valued measure and a vector $\gamma \in \mathbb{C}^2$, let ν_γ denote the scalar measure defined by $\nu_\gamma(\omega) = \gamma^* \nu(\omega)\gamma$. Note that if ν is a positive measure (that is, takes positive semidefinite values), then each ν_γ is a positive measure. Let $\Psi_0 = \Psi \backslash \{\psi_\infty\}$.

Lemma 4.5. *Suppose ν is a 2×2 positive matrix-valued measure on Ψ_0. For each γ the measure ν_γ is a nonnegative linear combination of at most two point masses if and only if there exist (possibly not distinct) points $\mathfrak{z}_1, \mathfrak{z}_2$ and positive semidefinite matrices Q_1 and Q_2 such that*

$$\nu = \sum_{j=1}^2 \delta_{\mathfrak{z}_j} Q_j,$$

where $\delta_{\mathfrak{z}_1}, \delta_{\mathfrak{z}_2}$ are scalar unit point measures on Ψ supported at $\psi_{\mathfrak{z}_1}, \psi_{\mathfrak{z}_2}$, respectively.

Proof. If $\nu = \sum_{j=1}^2 \delta_{\mathfrak{z}_j} Q_j$ with $\mathfrak{z}_1, \mathfrak{z}_2$ and Q_1, Q_2 as in the statement of the lemma, then clearly each ν_γ is a nonnegative linear combination of at most two point masses.

For the converse, the M_2-valued measure ν, expressed as a 2×2 matrix of scalar measures with respect to the standard orthonormal basis $\{e_1, e_2\}$ of $\mathbb{C}^2$ has the form

$$\nu = \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix}. \tag{15}$$

Since $\nu(\omega)$ is a positive matrix for every measurable set ω, it follows that ν_{11}, ν_{22} are positive measures. Moreover for the off-diagonal entries we have $\nu_{21} = \nu_{12}^*$. If ω is such that $\nu_{11}(\omega) = 0$, then by positivity $\nu_{12}(\omega) = 0$, and similarly if $\nu_{22}(\omega) = 0$. So it follows that ν_{12} and ν_{21} are absolutely continuous with respect to both ν_{11} and ν_{22}. This argument also shows that ν_{12} and ν_{21} are supported on the intersection of the supports for ν_{11} and ν_{22}.

Choosing $\gamma = e_1$, the hypotheses imply there exist $\alpha_1, \alpha_2 \geq 0$ and points $\mathfrak{z}_1, \mathfrak{z}_2$ such that

$$\nu_{11} = \sum_{j=1}^2 \alpha_j \delta_{\mathfrak{z}_j}.$$

Likewise there exist points $\mathfrak{w}_1, \mathfrak{w}_2$ and scalars $\beta_1, \beta_2 \geq 0$ such that

$$\nu_{22} = \sum_{j=1}^{2} \beta_j \delta_{\mathfrak{w}_j}.$$

There are several cases to consider. First suppose that the $\{\mathfrak{z}_1, \mathfrak{z}_2\}$ and $\{\mathfrak{w}_1, \mathfrak{w}_2\}$ have no points in common. Then $\nu_{12} = 0 = \nu_{21}$. Also, for $\gamma = e_1 + e_2$, by assumption

$$\nu_\gamma = \nu_{11} + \nu_{22}$$

has support at two points, and so $\mathfrak{z}_1 = \mathfrak{z}_2$ and $\mathfrak{w}_1 = \mathfrak{w}_2$. It follows that the union of the supports of ν_{11} and ν_{22} has cardinality at most two and $\nu_{12} = 0$, yielding the desired result.

Next suppose that the sets $\{\mathfrak{z}_1, \mathfrak{z}_2\}$ and $\{\mathfrak{w}_1, \mathfrak{w}_2\}$ have one point in common, say $\mathfrak{z}_1 = \mathfrak{w}_1$. In this case ν_{12} is supported at $\mathfrak{z}_1$ and there is a complex number s so that

$$\nu_{12} = s\delta_{\mathfrak{z}_1}.$$

If $s = 0$, choose $\gamma = e_1 + e_2$, so that $\nu_\gamma = \nu_{11} + \nu_{22}$. Otherwise set $\gamma = e_1 + s^* e_2$, in which case,

$$\nu_\gamma = \nu_{11} + 2|s|^2 \delta_{\mathfrak{z}_1} + |s|^2 \nu_{22}.$$

In either case, ν_γ has support at $\{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{w}_2\}$ and only two of these can be distinct.

The remaining case has the sets $\{\mathfrak{z}_1, \mathfrak{z}_2\}$ and $\{\mathfrak{w}_1, \mathfrak{w}_2\}$ equal, and the result is immediate.

Positivity of ν implies positivity of Q_1 and Q_2. ■

5. The proof of Theorem 1.1

Fix a finite set $\mathcal{F}$ containing $0, \lambda_1, \lambda_2$ and consisting of at least six distinct points. This choice of $\mathcal{F}$ along with the prior choices of Φ and F as in Equations (12) and (13) remain in effect for the rest of the paper. Accordingly, let $\Sigma_F = \Sigma_{F,\mathcal{F}}$.

We next prove the following diagonalization result.

Theorem 5.1. *If Σ_F lies in the cone $C_{2,\mathcal{F}}$, that is there exists an $M_2(\mathbb{C})$ valued μ such that*

$$I - F(x)F(y)^* = \int_\Psi (1 - \psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi) \qquad x, y \in \mathcal{F}, \tag{16}$$

then there exists rank one orthogonal projections Q_1, Q_2 summing to I, such that, for $x, y \in \mathcal{F}$,

$$I - F(x)F(y)^* = (1 - x^2 y^{*2} \varphi_1(x)\varphi_1(y)^*)Q_1 + (1 - x^2 y^{*2} \varphi_2(x)\varphi_2(y)^*)Q_2. \tag{17}$$

The proof proceeds by a sequence of lemmas which increasingly restrict the measures $\mu_{x,y}$ in (16).

Assume that $\Sigma_F \in C_{2,\mathcal{F}}$. Multiplying (16) by the Szegő kernel $s(x,y) = (1 - xy^*)^{-1}$ obtains

$$\left(\frac{I - F(x)F(y)^*}{1 - xy^*} \right)_{x,y \in \mathcal{F}} = \left(\int_\Psi \left(\frac{1 - \psi(x)\psi(y)^*}{1 - xy^*} \right) d\mu_{x,y}(\psi) \right)_{x,y \in \mathcal{F}}. \tag{18}$$

Next, since F has the form $x^2\Phi(x)$,

$$\begin{aligned}\frac{I - F(x)F(y)^*}{1 - xy^*} &= \frac{I_2 - x^2y^{*2}I_2 + x^2y^{*2}I_2 - x^2y^{*2}\Phi(x)\Phi(y)^*}{1 - xy^*}\\ &= (1 + xy^*)I_2 + x^2y^{*2}\left(\frac{I - \Phi(x)\Phi(y)^*}{1 - xy^*}\right).\end{aligned}$$

Similarly, for the test functions $\psi_\lambda(x) = x^2\varphi_\lambda(x)$ at points $\lambda \in \mathbb{D}$,

$$\frac{1 - \psi_\lambda(x)\psi_\lambda(y)^*}{1 - xy^*} = (1 + xy^*) + x^2y^{*2}\left(\frac{1 - \varphi_\lambda(x)\varphi_\lambda(y)^*}{1 - xy^*}\right). \tag{19}$$

(Here we take $\varphi_\infty = 1$.) Letting

$$k_\lambda(x) = \frac{\sqrt{1 - |\lambda|^2}}{1 - \lambda^* x}$$

denote the normalized Szegő kernel at λ and using the identity

$$\frac{1 - \varphi_\lambda(x)\varphi_\lambda(y)^*}{1 - xy^*} = k_\lambda(x)k_\lambda(y)^*, \tag{20}$$

for $\lambda \neq \infty$, equation (19) gives,

$$\frac{1 - \psi_\lambda(x)\psi_\lambda(y)^*}{1 - xy^*} = (1 + xy^*) + x^2y^{*2}k_\lambda(x)k_\lambda(y)^*,$$

while for $\lambda = \infty$ (correspondingly, $\psi_\infty(z) = z^2$ and $k_\infty(x) = 0$),

$$\frac{1 - \psi_\infty(x)\psi_\infty(y)^*}{1 - xy^*} = 1 + xy^*.$$

Putting these computations together, we rewrite (18) as

$$\begin{aligned}\frac{I - F(x)F(y)^*}{1 - xy^*} &= (1 + xy^*)I_2 + x^2y^{*2}\left(\frac{I - \Phi(x)\Phi(y)^*}{1 - xy^*}\right)\\ &= (1 + xy^*)\int_\Psi d\mu_{x,y}(\psi) + x^2y^{*2}\int_{\Psi_0} k_\lambda(x)k_\lambda(y)^*\, d\mu_{x,y}(\psi).\end{aligned} \tag{21}$$

Note that the first integral is over Ψ while the second is just over $\Psi_0 = \Psi\backslash\{z^2\}$ since $k_\infty(x) = 0$.

Combining Lemma 4.3 with Equation (21) gives

$$\begin{aligned}&(1 + xy^*)I + x^2y^{*2}\left(a(x)a(y)^* + b(x)b(y)^*\right)\\ &\qquad = \int_\Psi (1 + xy^*)\, d\mu_{x,y}(\psi) + \int_{\Psi_0} x^2y^{*2}\, k_\lambda(x)k_\lambda(y)^*\, d\mu_{x,y}(\psi).\end{aligned} \tag{22}$$

The next step will be to remove the x, y dependence in μ. Introducing some notation, let

$$\begin{aligned}\tilde{A}(x, y) &= \int_\Psi d\mu_{x,y}(\psi);\\ R(x, y) &= x^2y^{*2}\left(a(x)a(y)^* + b(x)b(y)^*\right); \quad \text{and}\\ \tilde{R}(x, y) &= (xy^*)^2\int_{\Psi_0} k_\lambda(x)k_\lambda(y)^*\, d\mu_{x,y}(\psi).\end{aligned}$$

Thus, $\tilde{A}$, R, and $\tilde{R}$ are all positive kernels on $\mathcal{F}$. With this notation and some rearranging of Equation (22), for $x, y \in \mathcal{F}$,

$$(1 + xy^*)(\tilde{A}(x,y) - I) = R(x,y) - \tilde{R}(x,y). \tag{23}$$

Let

$$\mathbb{K} = \{x^2 k_{\lambda_1}(x)v_1, x^2 k_{\lambda_2}(x)v_2\}, \tag{24}$$

the set of vectors spanning the kernel of $I - M_\Phi M_\Phi^*$ appearing in Lemma 4.3.

Lemma 5.2. *With the above notations, the assumption that $\Sigma_F \in C_{2,\mathcal{F}}$ and for $x, y \in \mathcal{F}$,*

(i) The $M_2(\mathbb{C})$ valued kernel $(\tilde{A} - I)(x,y) = \tilde{A}(x,y) - I$ is positive semidefinite;
(ii) The $M_2(\mathbb{C})$ valued kernel $R(x,y) - \tilde{R}(x,y)$ is positive semidefinite with rank at most two;
(iii) The range of $\tilde{R}$ lies in the range of R, which is in the span of $\mathbb{K}$; and
(iv) Either
  (a) The kernel $\tilde{A} - I$ has rank at most one; i.e., there is a function $r : \mathcal{F} \to \mathbb{C}^2$ such that

$$\tilde{A}(x,y) = I + r(x)r(y)^*, \quad \text{or;} \tag{25}$$

  (b) there exist functions $r, s : \mathcal{F} \to \mathbb{C}^2$ such

$$\tilde{A}(x,y) = I + r(x)r(y)^* + s(x)s(y)^*,$$

  and a point $\mathfrak{z} \in \mathcal{F} \setminus \{0\}$ such that $r(\mathfrak{z}) = 0 = s(\mathfrak{z})$.

Proof. Since $\psi(0) = 0$ for all $\psi \in \Psi$, it follows from (16) for all $y \in \mathcal{F}$,

$$I = I - F(0)F(y)^* = \int_\Psi (1 - \psi(0)\psi(y)^*) d\mu_{0,y}(\psi) = \int d\mu_{0,y}(\lambda) = \tilde{A}(0,y)$$

and (i) follows.

That $R - \tilde{R}$ is positive semidefinite follows from item (i) and Equation (23). Since R is rank two it must be the case that the rank of $R - \tilde{R}$ is rank at most two, completing the proof of item (ii).

By item (ii) and Douglas' lemma, the range of $\tilde{R}$ is contained in the range of R. By Lemma 4.3, the range of R is spanned by the set $\mathbb{K}$ and (iii) follows.

To prove item (iv), first note that in any case Equation (23) and item (ii) imply $\tilde{A} - I$ has at most rank two; i.e., there exists $r, s : \mathcal{F} \to \mathbb{C}^2$ such that

$$\tilde{A} - I = r(x)r(y)^* + s(x)s(y)^*.$$

From Equation (23), each of r, xr, s, xs lie in the range of R, which equals the span of $\mathbb{K}$. If r is nonzero at two points in $\mathcal{F}$, then r and xr are linearly independent and hence span the range of R. In this case, as both s and xs are in the range of R there exists α_j and β_j (for $j = 1, 2$) such that

$$\begin{aligned} s =& \alpha_1 r + \alpha_2 xr \\ xs =& \beta_1 r + \beta_2 xr. \end{aligned}$$

It follows that

$$0 = xs - xs = (\beta_1 + (\beta_2 - \alpha_1)x + \alpha_2 x^2)r(x). \tag{26}$$

If $\alpha_2 = 0$, then s is a multiple of r and case $(iv)(a)$ holds. Otherwise, in view of (26), r is zero with the exception of at most two points. Thus r is zero at two points, one of which, say $\mathfrak{z}$, must be different from 0. Since s must be zero when r is, $s(\mathfrak{z}) = 0$ too and $(iv)(b)$ holds.

The remaining possibility is that both r and s are non-zero at at most one point each, and these points may be distinct. In this situation r and s have at least two common zeros, one of which must be different from 0 and again $(iv)(b)$ holds. ∎

Lemma 5.3. *Under the assumption that $\Sigma_F \in C_{2,\mathcal{F}}$, the 2×2 matrix-valued kernel $\tilde{A}$ is constantly equal to I; i.e., $\tilde{A}(x,y) = I_2$ for all $x, y \in \mathcal{F}$.*

Proof. In the case that $(iv)(a)$ holds in Lemma 5.2, it (more than) suffices to prove that the r in Equation (25) is 0. To this end, let $\mathfrak{R}$ denote the range of R which, by Lemma 5.2, is spanned by the set $\mathbb{K}$ appearing in Equation (24). From Equations (23) and (25),

$$\tilde{R} + (1 + xy^*)r(x)r(y)^* = R.$$

Thus, $\mathfrak{R}$ contains both r and xr; that is, both r and xr are in the span of $\mathbb{K}$. Consequently, there exists α_j and β_j $(j = 1, 2)$ such that

$$\begin{aligned} r =& x^2 \sum_{j=1}^{2} \alpha_j k_{\lambda_j}(x) v_j \\ xr =& x^2 \sum_{j=1}^{2} \beta_j k_{\lambda_j}(x) v_j. \end{aligned}$$

Hence,

$$0 = xr - xr = x^2 \sum_{j=1}^{2} (\beta_j - x\alpha_j) k_{\lambda_j}(x) v_j. \tag{27}$$

Since the set $\{v_1, v_2\}$ is a basis for $\mathbb{C}^2$ (see Lemma 4.3), it has a dual basis $\{w_1, w_2\}$. Taking the inner product with w_ℓ in Equation (27) gives,

$$0 = x^2(\beta_\ell - x\alpha_\ell) k_{\lambda_\ell}(x)$$

for $x \in \mathcal{F}$. Choosing $x = \lambda_\ell$ (which is not zero) implies $\beta_\ell - \lambda_\ell \alpha_\ell = 0$. But then choosing any $x \in \mathcal{F}$ different from both 0 and λ_j (and using $k_{\lambda_j}(x) \neq 0$) implies $\beta_\ell - x\alpha_\ell = 0$. Hence $\alpha_\ell = 0 = \beta_\ell$ and consequently $r(x) = 0$ for all x.

Now suppose $(iv)(b)$ in Lemma 5.2 holds. In particular, there exists a point $\mathfrak{z}$ in $\mathcal{F} \setminus \{0\}$ such that $r(\mathfrak{z}) = 0 = s(\mathfrak{z})$. By the same reasoning as in the first part of this proof, there exist α_j and β_j such that

$$\begin{aligned} r =& x^2 \sum_{j=1}^{2} \alpha_j k_{\lambda_j}(x) v_j \\ s =& x^2 \sum_{j=1}^{2} \beta_j k_{\lambda_j}(x) v_j. \end{aligned}$$

Taking the inner product with w_ℓ and evaluating at $\mathfrak{z}$ yields

$$0 = \alpha_\ell k_{\lambda_\ell}(\mathfrak{z}).$$

Thus $\alpha_\ell = 0$. Likewise, $\beta_\ell = 0$. Thus $r = 0 = s$ and the proof is complete. ∎

Remark 5.4. Observe that if it were the case that $v_1 = v_2$ in Equation (27), then it would not be possible to conclude that the α_j and β_j are 0. Indeed, in such a situation, choosing $\beta_j = (-1)^j$ and $\alpha_j = (-1)^j \lambda_j^*$ gives a non-trivial solution. However, the case $v_1 = v_2$ corresponds to a Φ having the form

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & \varphi_{\lambda_1}\varphi_{\lambda_2} \end{pmatrix},$$

which is explicitly ruled out by our choice of Φ and Lemma 4.1.

Lemma 5.5. *There exists a 2×2 matrix valued positive measure μ on Ψ such that $\mu(\Psi) = I_2$ and*

$$K^\Phi(x, y) := \frac{1 - \Phi(x)\Phi(y)^*}{1 - xy^*} = \int_{\Psi_0} k_\lambda(x) k_\lambda(y)^* \, d\mu(\psi) \tag{28}$$

for all $x, y \in \mathcal{F} \setminus \{0\}$.

Proof. By Lemma 5.3, $\tilde{A}(x, y) = I$ for all $x, y \in \mathcal{F}$. An examination of the definition of $\tilde{A}$ and application of Lemma 4.2 implies there is a positive measure μ such that $\mu_{x,y} = \mu$ for all (x, y). Substituting this representation for $\mu_{x,y}$ into and some canceling and rearranging of (21) gives,

$$(xy^*)^2 \left(\frac{I - \Phi(x)\Phi(y)^*}{1 - xy^*} \right) = x^2 y^{*2} \int_{\Psi_0} k_\lambda(x) k_\lambda(y)^* \, d\mu(\psi).$$

Dividing by $(xy^*)^2$ (and of course excluding either $x = 0$ or $y = 0$) gives the result. ∎

Now that μ has no x, y dependence, the next step is to restrict its support. For this we employ Lemma 4.3. Recall that μ is a positive 2×2 matrix-valued measure on Ψ. Let δ_∞ denote point mass at the point $\psi_\infty = z^2$.

Lemma 5.6. *Under the assumption that $\Sigma_F \in C_{2,\mathcal{F}}$, and with notation as above, there are two points $\mathfrak{z}_1, \mathfrak{z}_2$ in $\mathcal{F}$ such that the measure μ has the form $\mu = \delta_{\mathfrak{z}_1} Q_1 + \delta_{\mathfrak{z}_2} Q_2 + \delta_\infty P$, where Q_1, Q_2, P are 2×2 matrices satisfying $0 \leq Q_1, Q_2, P \leq 1$ and $Q_1 + Q_2 + P = I$, and $\delta_{\mathfrak{z}_1}, \delta_{\mathfrak{z}_2}$ are scalar unit point measures on Ψ supported at $\psi_{\mathfrak{z}_1}, \psi_{\mathfrak{z}_2}$, respectively.*

Proof. We first show that the restriction of μ to $\mathbb{D}$ has support at no more than two points. Accordingly, let ν denote the restriction of μ to $\mathbb{D}$.

From Lemma 4.3, for $x, y \in \mathcal{F} \setminus \{0\}$,

$$\frac{I_2 - \Phi(x)\Phi(y)^*}{1 - xy^*} = a(x)a(y)^* + b(x)b(y)^*$$

where a, b are $\mathbb{C}^2$ valued functions on $\mathfrak{F}$. Fix a vector γ and define a scalar measure ν_γ on Ψ by $\nu_\gamma(\omega) = \gamma^*\nu(\omega)\gamma$. Note that

$$\begin{aligned}\gamma^*\left(a(x)a(y)^* + b(x)b(y)^*\right)\gamma =&\gamma^*\left(\int_\Psi k_\lambda(x)k_\lambda(y)^* d\mu(\psi)\right)\gamma \\ =&\int_{\Psi_0} k_\lambda(x)k_\lambda(y) d\nu_\gamma(\psi)\end{aligned}$$

is a kernel of rank (at most) two.

Choosing a three-point subset $\mathfrak{G} \subseteq \mathcal{F} \setminus \{0\}$ and a nonzero scalar-valued function $c : \mathfrak{G} \to \mathbb{C}$ such that

$$\sum_{x,y\in\mathfrak{G}} c(x)\gamma^*\left(a(x)a(y)^* + b(x)b(y)^*\right)c(y)^* = 0 \tag{29}$$

gives

$$0 = \int_{\Psi_0}\left|\sum_{x\in\mathfrak{G}} k_\lambda(x)c(x)\right|^2 d\nu_\gamma(\psi), \tag{30}$$

which means that the function $f = \sum_{x\in\mathfrak{G}} k_\lambda(x)c(x)$ vanishes for ν_γ-a.e. on Ψ_0. The function f is a linear combination of at most three Szegő kernels, and hence can vanish at at most two points in $\mathbb{D}$. It follows that ν_γ is supported at at most two points in $\mathbb{D}$. An application of Lemma 4.5 now implies that there exist points $\mathfrak{z}_1, \mathfrak{z}_2$ and positive semidefinite matrices Q_1, Q_2 such that

$$\nu = \sum_{j=1}^{2} \delta_{\mathfrak{z}_j} Q_j.$$

Letting $P = \mu(\{\infty\})$, it follows that μ has the promised form,

$$\mu = \delta_{\mathfrak{z}_1}Q_1 + \delta_{\mathfrak{z}_2}Q_2 + \delta_\infty P.$$

Finally, because μ has total mass the identity,

$$I = \mu(\Psi) = Q_1 + Q_2 + P. \qquad \blacksquare$$

To eliminate P and show that the Q_i are orthogonal, rank one projections, return to Equation (28) and rearrange it once again: recalling the identity of Equation (20) and multiplying through by $1 - xy^*$ and using Lemma 5.6, we have by the description of μ from the previous lemma, for $x, y \in \mathcal{F} \setminus \{0\}$,

$$1 - \Phi(x)\Phi(y)^* = (1 - \varphi_{\mathfrak{z}_1}(x)\varphi_{\mathfrak{z}_1}(y)^*)Q_1 + (1 - \varphi_{\mathfrak{z}_2}(x)\varphi_{\mathfrak{z}_2}(y)^*)Q_2,$$

where $\psi_{\mathfrak{z}_1}, \psi_{\mathfrak{z}_2}$ are the support points of the measure μ. Using the fact that $Q_1 + Q_2 + P = I$, we obtain, for $x, y \in \mathcal{F} \setminus \{0\}$,

$$\Phi(x)\Phi(y)^* = \varphi_{\mathfrak{z}_1}(x)\varphi_{\mathfrak{z}_1}(y)^*Q_1 + \varphi_{\mathfrak{z}_2}(x)\varphi_{\mathfrak{z}_2}(y)^*Q_2 + P. \tag{31}$$

Lemma 5.7. *Let Φ be as above. In the representation (31),*

(i) $\{\mathfrak{z}_1, \mathfrak{z}_2\} = \{\lambda_1, \lambda_2\}$;
(ii) $P = 0$; *and*

(iii) Q_1, Q_2 *are rank one projections summing to* I (*and hence mutually orthogonal*).

Proof. Since $\det \Phi(\lambda_1) = 0$, the identity (31) implies

$$\Phi(\lambda_1)\Phi(\lambda_1)^* = |\varphi_{\mathfrak{z}_1}(\lambda_1)|^2 Q_1 + |\varphi_{\mathfrak{z}_2}(\lambda_1)|^2 Q_2 + P,$$

that both sides have rank at most one. It follows that at least one of $\varphi_{\mathfrak{z}_1}, \varphi_{\mathfrak{z}_2}$ (and hence exactly one, since the λ_j are distinct) must have a zero at λ_1 (otherwise the three positive matrices Q_1, Q_2, P would all be scalar multiples of the same rank one matrix, which violates $Q_1 + Q_2 + P = I$). Similarly for λ_2, so (*i*) is proved. Further, without loss of generality, it can be assumed that $\mathfrak{z}_j = \lambda_j$ for $j = 1, 2$.

It follows from evaluating at the λ_j that each of Q_1, Q_2, P has rank at most one. In particular we have for $j = 1, 2$,

$$\Phi(\lambda_j)\Phi(\lambda_j)^* = |\varphi_k(\lambda_j)|^2 Q_k + P, \tag{32}$$

where $k \in \{1, 2\}$ and $k \neq j$. This means that $\operatorname{ran} P \subseteq \operatorname{ran} Q_1 \cap \operatorname{ran} Q_2$. On the other hand, if $\operatorname{ran} Q_1 \cap \operatorname{ran} Q_2 \neq \{0\}$, we have $\operatorname{ran} Q_1 \subseteq \operatorname{ran} Q_2$ or vice versa, which again contradicts $Q_1 + Q_2 + P = 1$. Thus $\operatorname{ran} Q_1 \vee \operatorname{ran} Q_2 = \mathbb{C}^2$, and so $P = 0$, which is (*ii*). Since $Q_2 = 1 - Q_1$, if $f \in \ker Q_1$, then $Q_2 f = f$. However, Q_2 is a rank one contraction, so it must be a projection, and then the same follows for Q_1. Thus we have (*iii*). ∎

Proof of Theorem 5.1. Since $F(x) = x^2\Phi(x)$, Theorem 5.1 is now immediate from Lemma 5.7. ∎

5.1. **The proof of Theorem 1.1.**

The proof of Theorem 1.1 concludes in this subsection. Recall that we are assuming that $F(z) = z^2\Phi(z)$, where Φ is as in (12).

Suppose that $\Sigma_F \in C_{2,\mathcal{F}}$. From Equation (31) and Lemma 5.7,

$$\Phi(x)\Phi(y)^* = \sum_{j=1}^{2} \varphi_j(x)\varphi_j(y)^* Q_j, \tag{33}$$

valid for $x, y \in \mathcal{F} \setminus \{0\}$. Since the Q_j are rank one projections which sum to I, there exists an orthonormal basis $\{\gamma_1, \gamma_2\}$ such that

$$Q_j = \gamma_j\gamma_j^*.$$

Let U be the unitary matrix with columns γ_j, and let

$$G(z) = U\begin{pmatrix} \varphi_1(z) & 0 \\ 0 & \varphi_2(z) \end{pmatrix}.$$

Observe $\Phi(x)\Phi(y)^* = G(x)G(y)^*$ for $x, y \in \mathcal{F} \setminus \{0\}$.

Fix $\zeta \in \mathcal{F} \setminus \{0, \lambda_1, \lambda_2\}$. Then $\Phi(\zeta)$ is invertible and further $\Phi(\zeta)\Phi(\zeta)^* = G(\zeta)G(\zeta)^*$. Hence by Douglas' Lemma, there is a unitary W such that $\Phi(\zeta) = G(\zeta)W^*$. Consequently,

$$0 = \Phi(\zeta)\Phi(y)^* - G(\zeta)G(y)^* = G(\zeta)(\Phi(y)W - G(y))^*,$$

and therefore $\Phi(y)W = G(y)$, for $y \in \mathcal{F} \setminus \{0\}$. Returning to the definition of G, we arrive at the conclusion that, for $x \in \mathcal{F} \setminus \{0\}$,

$$\Phi(x) = U\begin{pmatrix} \varphi_1(x) & 0 \\ 0 & \varphi_2(x) \end{pmatrix} W^*. \tag{34}$$

Now Φ and G are both rational matrix inner functions of degree at most two. Since $\mathcal{F} \setminus \{0\}$ contains at least five points it is a set of uniqueness for rational functions of degree at most two, and hence (34) must hold on all of $\mathbb{D}$. Returning to Φ, it now follows that, on all of $\mathbb{D}$,

$$\Phi = U \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} W^*.$$

By Lemma 4.1,

$$\Phi = \begin{pmatrix} s\varphi_1 & 0 \\ 0 & t\varphi_2 \end{pmatrix}$$

for unimodular constants $s$and t, contrary to our choice of Φ in (12). We conclude that $\Sigma_F \notin C_{2,\mathcal{F}}$, and so by Proposition 3.5, there exists a contractive representation of $\mathscr{A}$ which is contractive, but not completely contractive.

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