Commutative operator algebras and realizations of polynomials on domains in $\mathbb{C}^n$

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January 9, 2011

$p \in M_n \otimes \mathbb{C}[z_1, \dots z_d]$

Let $\mathcal{T}$ be a family of d-tuples of commuting operators, e.g.

$\mathcal{T} =$ all commuting contractions, or $\mathcal{T} =$ all commuting row contractions, or column contractions, etc.

For reasonable families $\mathcal{T}$

$$\|p\|_{\mathcal{T}} := \sup_{T \in \mathcal{T}} \|p(T)\|$$

defines an operator algebra norm.

(Here $p(z) = \sum C_\alpha z^\alpha, \quad p(T) := \sum C_\alpha \otimes T^\alpha$)

Ambrozie/Timotin, Ball/Bolotnikov, Mittal/Paulsen:
Let $Q(z)$ be an analytic $r \times s$ matrix valued polynomial. Define

$$\mathcal{D}_Q := \{z = (z_1 \dots z_d) : \|Q(z)\| < 1\} \subset \mathbb{C}^d$$

$$\mathcal{T}_Q := \{\text{commuting } T = (T_1, \dots T_d) : I - Q(T)Q(T)^* \geq 0\}$$

$$\|p\|_Q := \sup_{T \in \mathcal{T}_Q} \|p(T)\|$$

Say p belongs to the Schur-Agler class $\mathcal{SA}_Q$ if $\|p\|_Q \leq 1$.

Immediate observation—different Q's can give the SAME scalar domain $\mathcal{D}_Q$ but DIFFERENT operator domains $\mathcal{T}_Q$....

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(row space, column space, row $\cap$ column space)

This talk: replace single polynomial Q with family of *linear* maps

$$\sigma : \mathbb{C}^d \to B(H)$$

Each σ has the form

$$\sigma_S(z) = \sum_{j=1}^{d} z_j S_j$$

In particular:

fix E a d-dimensional *operator space*

Consider all completely contractive maps $\sigma : E \to B(H)$.

So, for each n we are given a norm $\|\cdot\|_n$ on

$$M_n \otimes \mathbb{C}^d = \{(A_1 \ A_2 \ \dots \ A_d) : A_j \in M_n\}$$

The map σ_S is completely contractive for E iff

$$\left\| \sum_{j=1}^{d} A_j \otimes S_j \right\|_{M_n \otimes B(H)} \leq \|(A_1 \ \dots \ A_d)\|_n$$

Duality

Concretely, the *dual operator space* E^* is the operator space whose norm at the n^{th} matrix level is

$$\|(B_1 \ \dots \ B_d)\|_{M_n(E^*)} := \sup \left\| \sum_{j=1}^{n} B_j \otimes A_j \right\|_{M_n \otimes M_m},$$

sup over all A's in unit ball of $M_m(E)$, all sizes m

Duality via cc maps:

If T is any d-tuple, we have a linear map

$$\sigma_T(z_1, \dots z_d) = z_1 T_1 + \cdots + z_d T_d$$

For any d-tuples S on $B(H)$ and T on $B(K)$, define

$$\langle S, T \rangle := \sum_{j=1}^{d} S_j \otimes T_j$$

on $B(H \otimes K)$

Lemma

Let E be a d-dim. operator space and T a d-tuple. The map σ_T is completely contractive for E^* if and only if

$$\| \langle S, T \rangle \| \leq 1$$

for all S such that σ_S is completely contractive for E.

Theorem

Let E be a finite dimensional operator space, with underlying Banach space V, and let $\Omega \subset \mathbb{C}^n$ denote the open unit ball of V. For each analytic M_N-valued polynomial p, the following are equivalent:

1) **Agler-Nevanlinna factorization.** There is a cc map $\sigma : E \to B(K)$, and an analytic $F : \Omega \to B(K, \mathbb{C}^N)$ such that

$$1 - p(z)p(w)^* = F(z)\left[I_K - \sigma(z)\sigma(w)^*\right]F(w)^*$$

for all $z, w \in \Omega$.

2) **Transfer function realization.**...

3) **von Neumann inequality.** If S is a commuting d-tuple in $B(K)$ and σ_S is completely contractive for E^*, then

$$\|p(S)\|_{M_N \otimes B(K)} \leq 1.$$

Define

$$\|p\|_{UC(E^*)} := \sup_S \|p(S)\|,$$

sup over all S such that σ_S is completely contractive for E^*.

The $UC(E^*)$ norm is an operator algebra norm. Identify the linear polynomials with $\mathbb{C}^d$:

$$(a_1, \dots a_d) \to \sum a_j z_j$$

Restrict $UC(E^*)$ norms to this space gives an operator space over $\mathbb{C}^d$; by definition and duality lemma this is E^* back. This shows...

The Universal Property

Let E be a finite-dimensional operator space.

1. There exists a canonical completely isometric embedding

$$\iota : E \to UC(E).$$

2. If $\sigma : E \to B(H)$ is a completely contractive map with commutative range, then there exists a unique completely contractive unital homomorphism $\hat{\sigma} : UC(E) \to B(H)$ extending σ.

This is the commutative analog of Pisier's *universal operator algebra* $OA(E)$...

...in fact $UC(E) \cong OA(E)/\mathcal{C}$ completely isometrically.

Examples

$E =$ row space R_d $\quad [Q(z) = (z_1\ z_2 \ldots\ z_d)]$

- σ_S completely contractive for E^* (column space) if and only if S is a row contraction
- norm is multiplier norm on Drury-Arveson space

$MIN(V)$ and $MAX(V)$

Let $V = (\mathbb{C}^d, \|\cdot\|_V)$ be a d-dimensional Banach space. $MIN(V)$ is the operator space with norm

$$\|(A_1 \ \ldots \ A_d)\| = \sup_{\lambda} \|\sum_{j=1}^{d} \lambda_j A_j\|$$

over all $\lambda = (\lambda_1, \ldots \lambda_d) \in \text{ball}(V^*)$.

(Smallest operator space norm whose 1st level is V)

Simlarly $MAX(V)$ is largest operator space norm whose 1st level is V (equivlantly, every contractive map $\sigma : V \to B(H)$ is completely contractive for $MAX(V)$).

Fact: $MIN(V) \cong MAX(V^*)$ completely isometrically

$E = MIN(\ell_d^\infty)$, $E^* = MAX(\ell_d^1)$

- σ_S is completely contractive for $MAX(\ell_d^1)$ iff it is contractive for ℓ_d^1, i.e.

$$\left\| \sum_{j=1}^{d} z_j S_j \right\| \leq \sum_{j=1}^{d} |z_j|,$$

  so iff each S_j is a contraction.

- Norm is Schur-Agler norm

$E = MAX(\ell_d^\infty)$, $E^* = MIN(\ell_D^1)$

Lemma

Let S be a d-tuple in $B(H)$. σ_S is completely contractive for $MIN(\ell_d^1)$ iff there exist commuting unitaries $U_j \in B(K)$ and an isometry $V : H \to K$ such that

$$S_j = V_j^* U_j V_j$$

Fact: the commuting contractions (S_1, S_2, S_3) in the Kaijser-Varopoulos counterexample to von Neumann's inequality on $\mathbb{D}^3$ have this form...

...in other words, there exist

- Hilbert spaces $H \subset K$,
- commuting unitaries $U_1, U_2, U_3 \in B(K)$
- an isometry $V : H \to K$

such that the triple $S_j = V^* U_j V$ commutes but fails von Neumann's inequality.

Corollary: The tridisk algebra $A(\mathbb{D}^3$ (with sup norm) fails the universal property (so sup norm is NOT a $UC(E^*)$-norm)

(Heuristic) corollary: The set of triples of commuting contractions which DO admit a unitary dilation is "non-convex."

Other domains

$V = (\mathbb{C}^d, \|\cdot\|)$, $\Omega = \text{ball}(V)$.

Question For any such Ω, is sup norm a UC norm?

If so, E must be $MAX(V)$ (so $E^* = MIN(V^*)$).
Answer is "yes" for $\mathbb{D}^2$ (Ando), "no" for $\mathbb{D}^n$, $n \geq 3$ (last example)

$\mathbb{B}^n$??? "Easy" counterexamples fail.....

Let $p(z_1, z_2) = 2z_1z_2$. Then $\|p\|_{MIN(\mathbb{B}^2)} = \|p\|_\infty = 1$

$$\sigma(z_1, z_2) = \begin{pmatrix} 0 & z_1 & z_2 & 0 & 0 & 0 \\ z_2 & 0 & 0 & 0 & 0 & 0 \\ z_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_1 & z_2 \\ 0 & 0 & 0 & z_2 & 0 & 0 \\ 0 & 0 & 0 & z_1 & 0 & 0 \end{pmatrix}$$

and

$$F(z_1, z_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & z_1 & z_2 \end{pmatrix}$$

Then

$$1 - 4z_1z_2\overline{w_1w_2} = F(z)(1 - \sigma(z)\sigma(w)^*)F(w)^*.$$