PARTIALLY ISOMETRIC DILATIONS OF NONCOMMUTING N-TUPLES OF OPERATORS

MICHAEL T. JURY[1] AND DAVID W. KRIBS[2]

Abstract. Given a row contraction of operators on Hilbert space and a family of projections on the space which stabilize the operators, we show there is a unique minimal joint dilation to a row contraction of partial isometries which satisfy natural relations. For a fixed row contraction the set of all dilations forms a partially ordered set with a largest and smallest element. A key technical device in our analysis is a connection with directed graphs. We use a Wold Decomposition for partial isometries to describe the models for these dilations, and discuss how the basic properties of a dilation depend on the row contraction.

1. Introduction

Dilation theory has played a central role in operator theory since Sz.-Nagy [**8**] proved in 1953 that every contraction operator on Hilbert space has a unique minimal dilation to an isometry on a larger space. This result was extended to the noncommutative multivariable setting by Frazho [**9**] (for $n = 2$), Bunce [**4**] (for $2 \leq n < \infty$), and Popescu [**23**] (for $n = \infty$ and uniqueness in general). Specifically, every row contraction of n operators on Hilbert space was shown to have a joint minimal dilation to n isometries on a larger space with mutually orthogonal ranges. The study of isometries with orthogonal ranges has provided the technical underpinning for a number of far reaching enquiries (see [**2, 3, 5, 6, 13, 17, 24**] for examples from different perspectives). While the Frazho-Bunce-Popescu (FBP) dilation has played a role in many of these instances, there are deep reasons from the representation theory of infinite dimensional operator algebras which suggest it may have limited utility.

In this paper, we present a dilation theory for n-tuples of operators on Hilbert space which is, in general, more in tune with properties of the

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[2]second author partially supported by an NSERC research grant.

n-tuple as compared to the FBP dilation. Given a row contraction $T = (T_1, \ldots, T_n)$ and a family of projections which stabilize the operators in a certain sense (such families always exist), we show there is a unique minimal joint dilation of T to an n-tuple of partial isometries $S = (S_1, \ldots, S_n)$ which satisfy natural relations. This dilation theorem may be regarded as a refinement of a special case of the recent Muhly-Solel **[22]** theorem for the more abstract setting of tensor algebras over C*-correspondences.

For fixed T, the set of all dilations forms a partially ordered set of directed graphs with a largest and smallest element. The smallest element is the Sz.-Nagy dilation (for $n = 1$) or the FBP dilation (for $n \geq 2$), which corresponds to the directed graph with a single vertex and n loop edges. There is a 'finest' dilation which is the largest element in the ordering. This dilation is maximal amongst the set of all minimal dilations of T in the sense that if we are given a minimal dilation of T, the corresponding directed graph is a 'deformation' of the graph for the finest dilation.

The main technical drawback of the FBP dilation is that the analogue of the unitary part in the Wold Decomposition **[23]** determines a representation of the Cuntz algebra $\mathcal{O}_n$, which is an 'NGCR' algebra **[11]**, and hence its representations cannot be classified up to unitary equivalence. While this has been accomplished for special classes of row contractions **[6]**, in general it is not possible. On the other hand, for many row contractions, the dilation theory developed here avoids this problem. Indeed, the analogue of the unitary part here determines a representation of a Cuntz-Krieger graph C*-algebra **[18, 19]**, and there are many graphs for which the representation theory of the algebra is type I. In fact, Ephrem **[7]** has recently obtained a complete graph-theoretic characterization of when this happens.

In the first section we discuss the models for this dilation theory and recall the Wold Decomposition from **[14]** for families of partial isometries. We prove the dilation theorem in the second section, and show how basic properties of a dilation depend on the dilated row contraction. We also describe the class of row contractions for which this dilation theory gives an improvement on the FBP dilation theory. In the final section we discuss the partially ordered set of minimal dilations generated by a given row contraction.

Throughout the paper n is a positive integer or $n = \infty$, but we behave as though n is finite.

2. Wold Decomposition

The models for the dilation theory presented here are n-tuples $S = (S_1, \dots, S_n)$ of (non-zero) operators acting on a Hilbert space $\mathcal{K}$ which satisfy the following relations:

$$(\dagger) \left\{ \begin{array}{ll} (1) & \forall\, 1 \leq i \leq n,\ (S_i^* S_i)^2 = S_i^* S_i \\ (2) & \sum_{i=1}^n S_i S_i^* \leq I \\ (3) & \forall\, 1 \leq i, j \leq n,\ (S_i^* S_i)(S_j^* S_j) = 0 \text{ or } S_i^* S_i = S_j^* S_j \\ (4) & \forall\, i\, \exists\, j \text{ such that } S_i S_i^* \leq S_j^* S_j \\ (5) & \text{If } \{Q_k\} \text{ are the distinct elements from } \{S_i^* S_i\}, \\ & \text{then } \sum_k Q_k = I. \end{array} \right.$$

Such an n-tuple consists of partial isometries with mutually orthogonal ranges, with initial projections equal or orthogonal, with each final projection supported by some initial projection, and distinct initial projections summing to the identity operator. Observe there is a natural directed graph G (with no sinks) associated with each n-tuple which satisfies ($\dagger$). The vertex set $V(G)$ for G is identified with the index set for $\{Q_k\}_k$, and the edge set $E(G)$ includes a directed edge for each S_i; specifically, S_i determines an edge in G from vertex k to vertex l where $S_i^* S_i = Q_k$ and $S_i S_i^* \leq Q_l$. We will use the orderings of $S = (S_1, \dots, S_n)$ induced by G, and write $S = (S_e)_{e \in E(G)}$ when an ordering has been chosen.

If $S = (S_1, \dots, S_n)$ satisfies ($\dagger$) with $SS^* = \sum_{i=1}^n S_i S_i^* = I$, then we say S is *fully coisometric*. From the operator algebra perspective, fully coisometric n-tuples generate what are sometimes called Cuntz-Krieger directed graph C*-algebras (see **[7, 18, 19]** for instance).

At the other extreme, we say $S = (S_e)_{e \in E(G)}$ satisfying ($\dagger$) is *pure* if

$$(1) \qquad \lim_{d \to \infty} \Big(\sum_{w \in \mathbb{F}^+(G);\ |w| \geq d} ||w(S)^* \xi||^2 \Big) = 0 \quad \text{for all} \quad \xi \in \mathcal{K}.$$

Here we denote the *semigroupoid* of G by $\mathbb{F}^+(G)$. This is the set of all vertices in G and all finite paths w in the edges e of $E(G)$, with the natural operations of concatenation of allowable paths. We write $|w|$ for the number of edges which make up the path w, and put $w = k_2 w k_1$ when the initial and final vertices of w are, respectively, k_1 and k_2. The notation $w(S)$ stands for the partial isometry given by the product $w(S) = S_{e_{i_1}} \cdots S_{e_{i_m}}$ when $w = e_{i_1} \cdots e_{i_m}$ belongs to $\mathbb{F}^+(G)$.

We now discuss the fundamental examples for the pure case. Let G be a countable directed graph and let $\mathcal{K}_G = \ell^2(G)$ be the Hilbert space with orthonormal basis $\{\xi_w : w \in \mathbb{F}^+(G)\}$. Define partial isometries on

$\mathcal{K}_G$ by

$$\mathbf{L}_e \xi_w = \begin{cases} \xi_{ew} & \text{if } ew \in \mathbb{F}^+(G) \\ 0 & \text{otherwise.} \end{cases}$$

The operators $\mathbf{L}_G = (\mathbf{L}_e)_{e\in E(G)}$ are easily seen to be pure and satisfy (†). This generalized 'Fock space' construction was introduced by Muhly [**20**] and there is now a growing literature for the nonselfadjoint operator algebras generated by such tuples [**12, 14, 15, 16, 21, 22**]. The C*-algebra generated by a tuple $\mathbf{L}_G$ is said to be of 'Cuntz-Krieger-Toeplitz' type since it is the extension of a Cuntz-Krieger algebra by the compact operators.

Every pure tuple $S = (S_e)_{e\in E(G)}$ which satisfies (†) for G is determined by $\mathbf{L}_G$ in the following sense: Let $\mathcal{V}_k$, $k \in V(G)$, be the subspace of $\mathcal{K}_G$ generated by basis vectors from paths which begin at vertex k, that is, $\mathcal{V}_k = \text{span}\{\xi_w : w = wk \in \mathbb{F}^+(G)\}$. Then there is a joint unitary equivalence such that

$$S_e \simeq \sum_{k\in V(G)} \oplus \mathbf{L}_e^{(\alpha_k)}\Big|_{\mathcal{V}_k^{(\alpha_k)}} \quad \text{for} \quad e \in E(G),$$

where $\alpha_k = \dim \left[Q_k\big(I - \sum_e S_e S_e^*\big)\right]$ and recall $\{Q_k\}_{k\in V(G)}$ are the distinct projections amongst $\{S_e^* S_e : e \in E(G)\}$. The basic idea is as follows. Let $\mathcal{W} = \text{Ran}\left(I - \sum_e S_e S_e^*\right)$ be the *wandering subspace* [**14**] for S. A unitary producing the joint equivalence is defined by making a natural identification between orthonormal bases for the non-zero subspaces of the form $w(S)Q_k\mathcal{W}$ and corresponding subspaces of $\mathcal{V}_k^{(\alpha_k)}$. We refer to the α_k as the *vertex multiplicities* in this decomposition.

The following Wold Decomposition was established in [**14**] for n-tuples satisfying (†).

Theorem 2.1. *Let $S = (S_1, \ldots, S_n)$ be operators on $\mathcal{K}$ satisfying (†) and let $S = (S_e)_{e\in E(G)}$ be an induced ordering. Then these operators are jointly unitarily equivalent to the direct sum of a pure n-tuple and a fully coisometric n-tuple which both satisfy (†) for the directed graph G. In other words, there is a unitary U and a fully coisometric n-tuple $(V_e)_{e\in E(G)}$ such that*

$$US_eU^* = V_e \oplus \Big(\sum_{k\in V(G)} \oplus\, \mathbf{L}_e^{(\alpha_k)}\Big|_{\mathcal{V}_k^{(\alpha_k)}} \Big) \quad \textit{for} \quad e \in E(G), \tag{2}$$

and the α_k are determined as above.

Let $\mathcal{K}_p = \sum_{w\in\mathbb{F}^+(G)} \oplus w(S)\mathcal{W}$ where $\mathcal{W} = \text{Ran}\left(I - \sum_e S_e S_e^*\right)$ and let $\mathcal{K}_c = (\mathcal{K}_p)^\perp$. The subspaces $\mathcal{K}_c$ and $\mathcal{K}_p$ reduce $S = (S_e)_{e\in E(G)}$, and the restrictions $S_e|_{\mathcal{K}_c}$ and $S_e|_{\mathcal{K}_p}$ determine the joint unitary equivalence in (2). This decomposition is unique in the sense that if $\mathcal{V}$ is *a subspace of* $\mathcal{K}$ *which reduces* $S = (S_e)_{e\in E(G)}$*, and if the restrictions* $\{S_e|_{\mathcal{V}} : e \in E(G)\}$ *are pure, respectively fully coisometric, then* $\mathcal{V} \subseteq \mathcal{K}_p$*, respectively* $\mathcal{V} \subseteq \mathcal{K}_c$*.*

We finish this section by identifying a large class of pure row contractions which will be used in the sequel. Let $\mathfrak{H} = \{\mathcal{H}_k : k \in \mathcal{J}\}$ be a countable collection of Hilbert spaces. Let G be a countable directed graph with vertex set $V(G) = \mathcal{J}$. We define $\ell^2(G, \mathfrak{H})$ to be the Hilbert space given by the ℓ^2-direct sum $\ell^2(G, \mathfrak{H}) = \sum_{w\in\mathbb{F}^+(G)} \oplus\mathcal{H}_w$ where $\mathcal{H}_w \equiv \mathcal{H}_k$ when $w = wk$, that is, the initial vertex of w is k. For each non-zero $\mathcal{H}_k$ choose an orthonormal basis $\{\xi_j^{(k)}\}$, and for $w = wk \in \mathbb{F}^+(G)$ let $\{\xi_j^{(w)}\}$ be the corresponding orthonormal basis for the wth coordinate space $\mathcal{H}_w$ of $\ell^2(G, \mathfrak{H})$. Then the *canonical (pure) shift* on $\ell^2(G, \mathfrak{H})$ consists of operators $(L_e)_{e\in E(G)}$ defined on $\ell^2(G, \mathfrak{H})$ by

$$L_e\xi_j^{(w)} = \begin{cases} \xi_j^{(ew)} & \text{if } ew \in \mathbb{F}^+(G) \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that every canonical shift $(L_e)_{e\in E(G)}$ is pure and satisfies (†), and hence Theorem 2.1 explicitly gives its form up to joint unitary equivalence. In particular, the fully coisometric part is vacuous and the vertex multiplicities are given by $\alpha_k = \dim(\mathcal{H}_k)$ for $k \in V(G)$.

3. Minimal Partially Isometric Dilations

Let $T = (T_1, \dots, T_n)$ be operators on a Hilbert space $\mathcal{H}$ such that $TT^* = \sum_{i=1}^n T_iT_i^* \leq I_{\mathcal{H}}$. We say an n-tuple $S = (S_1, \dots, S_n)$ of operators on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ is a *minimal partially isometric dilation* of T if the following conditions hold:

(*i*) $S = (S_1, \dots, S_n)$ satisfy the relations (†).
(*ii*) $\mathcal{H}$ reduces each $S_i^*S_i$, $1 \leq i \leq n$, and $\mathcal{H}$ is invariant for each S_i^* with $S_i^*|_{\mathcal{H}} = T_i^*$, $1 \leq i \leq n$.
(*iii*) $\mathcal{K} = \mathcal{H} \vee \Big(\bigvee_{i_1,\dots,i_k;\, k\geq 1} S_{i_1}\cdots S_{i_k}\mathcal{H}\Big)$.

Given $T = (T_1, \dots, T_n)$, consider all countable families $\mathcal{P} = \{P_k : k \in \mathcal{J}\}$ of projections on $\mathcal{H}$ which *stabilize* T in the following sense:

$$P_kT_i,\ T_iP_k \in \{T_i, 0\},\ 1 \leq i \leq n, \quad \text{and} \quad \sum_{k\in\mathcal{J}} P_k = I_{\mathcal{H}}. \tag{3}$$

For each i, it will be convenient to let $k_s = s(T_i)$ and $k_r = r(T_i)$ be the elements of $\mathcal{J}$ such that $T_iP_{k_s} = T_i$ and $P_{k_r}T_i = T_i$. To avoid pathologies we shall assume each T_i is non-zero. (If some $T_i = 0$, there is ambiguity in the choice of k_s, k_r.) Further, we clearly lose no generality in restricting our attention to families $\mathcal{P}$ such that there is no P_k with $T_iP_k = 0$ for all i. We show there is a minimal dilation of T generated by each such family of projections.

Given a family $\mathcal{P} = \{P_k\}_{k\in\mathcal{J}}$ which satisfies (3), we let $I_{\mathcal{P}}$ be the projection on the Hilbert space direct sum $\mathcal{H}^{(n)}$ defined by the $n \times n$ diagonal matrix with (i,i) entry equal to P_{k_i} where $k_i = s(T_i)$. Observe that the relations (3) guarantee that $I_{\mathcal{P}} - T^*T \geq 0$ is a positive operator on $\mathcal{H}^{(n)}$, here regarding T as a row matrix. Thus we may define a *defect operator* for $T = (T_1,\dots,T_n)$ on $\mathcal{H}^{(n)}$ by $D \equiv D_{\mathcal{P},T} = \big(I_{\mathcal{P}} - T^*T\big)^{1/2}$. Let $\mathcal{D} = \overline{D\mathcal{H}^{(n)}}$. Further let $E_i : \mathcal{H} \to \mathcal{H}^{(n)}$ be the injection of $\mathcal{H}$ onto the ith coordinate space of $\mathcal{H}^{(n)}$ for $1 \leq i \leq n$. Consider the operators $D_i = DE_i : \mathcal{H} \to \mathcal{D}$ for $1 \leq i \leq n$.

Lemma 3.1. *If $r(T_i) \neq r(T_j)$, then the range subspaces* $\mathrm{Ran}(D_i)$ *and* $\mathrm{Ran}(D_j)$ *are orthogonal.*

Proof. It suffices to show that $\mathrm{Ran}(D^{2a}E_i)$ and $\mathrm{Ran}(D^{2b}E_j)$ are orthogonal for $a, b \geq 1$; then a standard functional calculus argument can be applied. Recall that $D^2 = I_{\mathcal{P}} - T^*T$ is an $n \times n$ matrix which acts on $\mathcal{H}^{(n)}$. The operator D^2E_i picks out the ith column of D^2. When $r(T_i) \neq r(T_j)$, it follows from the identities (3) that in the ith and jth columns of D^2 there are no rows m such that both the (m,i) and (m,j) entries are non-zero. This property is easily seen to carry over to the self-adjoint powers $(D^2)^a$. Hence $D^{2a}E_i$ and $D^{2b}E_j$ have orthogonal ranges for $a, b \geq 1$ when $r(T_i) \neq r(T_j)$. ∎

We will use these operators to define generalized Schaffer matrices [**10, 22, 23, 25**] in the following proof.

Theorem 3.2. *Let $T = (T_1,\dots,T_n)$ be operators on a Hilbert space $\mathcal{H}$ such that $\sum_{i=1}^n T_iT_i^* \leq I_{\mathcal{H}}$. Let $\mathcal{P} = \{P_k\}_{k\in\mathcal{J}}$ be a family of projections which stabilize T as in* (3)*. Then there is a minimal partially isometric dilation $S = (S_1,\dots,S_n)$ of T on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ with $\mathcal{P} = \{S_i^*S_i|_{\mathcal{H}} : 1 \leq i \leq n\}$. This dilation is unique up to joint unitary equivalence which fixes $\mathcal{H}$.*

Proof. By Lemma 3.1 we may decompose $\mathcal{D}$ into the orthogonal direct sum $\mathcal{D} = \sum_{k\in V(G)} \oplus \mathcal{D}_k$, where $\mathcal{D}_k = \bigvee_{r(T_i)=k} \overline{D_i\mathcal{H}}$ are subspaces of $\mathcal{D}$ and G is the directed graph (with no sinks) determined by $\mathcal{P}$ and the relations (3). Put $\mathfrak{D} = \{\mathcal{D}_k\}_{k\in V(G)}$. Let $\mathcal{K}$ be the Hilbert space $\mathcal{K} = \mathcal{H} \oplus \ell^2(G,\mathfrak{D})$ and let $\mathcal{H}$ and $\ell^2(G,\mathfrak{D})$ be embedded into $\mathcal{K}$ in the natural way. For the rest of this proof it is convenient to re-label $T = (T_1,\dots,T_n)$ as $T = (T_e)_{e\in E(G)}$ by using a natural ordering induced by (3). We shall carry this notation over to the operators D_i, E_i, denoting them by D_e, E_e. For each $e \in E(G)$ define operators $S_e : \mathcal{K} \to \mathcal{K}$ by

$$S_e = \big(T_e + D_e\big) \oplus L_e,$$

where $(L_e)_{e\in E(G)}$ is the canonical shift on $\ell^2(G, \mathfrak{D})$.

We first verify (i) and (ii) for a minimal dilation. Observe that

$$T_e^* T_f + D_e^* D_f = T_e^* T_f + E_e^* \big(I_{\mathcal{P}} - T^* T\big) E_f.$$

When $e = f$, this identity yields $T_e^* T_e + D_e^* D_e = P_{k_0}$ where $k_0 = s(T_e)$. On the other hand, if $e \neq f$, then $T_e^* T_f + D_e^* D_f = T_e^* T_f - T_e^* T_f = 0$. It follows that the operators S_e are partial isometries with $T_e = P_{\mathcal{H}} S_e|_{\mathcal{H}} = \big(S_e^*|_{\mathcal{H}}\big)^*$, and initial projections which satisfy $\{S_e^* S_e|_{\mathcal{H}} : e \in E(G)\} = \{P_k : k \in V(G)\}$. Moreover, the ranges of the S_e are mutually orthogonal, $S_e^* S_f = 0$ for $e \neq f$, and hence $\sum_e S_e S_e^* \leq I_{\mathcal{K}}$. Lastly, by construction each range projection $S_e S_e^*$ is supported by an initial projection, $S_e S_e^* \leq S_f^* S_f$ for some $f \in E(G)$, and the distinct initial projections $\{Q_k\}_{k\in V(G)} = \{S_e^* S_e\}_{e\in E(G)}$ sum to the identity.

To verify minimality, first notice that $\mathcal{H} \bigvee S_e \mathcal{H} = \mathcal{H} \oplus \mathcal{D}$. But

$$\begin{aligned} \mathcal{K} \ominus (\mathcal{H} \oplus \mathcal{D}) &= \ell^2(G, \mathfrak{D}) \ominus \mathcal{D} \\ &= \sum_{e\in E(G)} \oplus\, S_e(\ell^2(G, \mathfrak{D})) = \sum_{w\in \mathbb{F}^+(G);\, |w|\geq 1} \oplus\, w(S)\mathcal{D}, \end{aligned}$$

and thus we have $\mathcal{K} = \mathcal{H} \vee \Big(\bigvee_{w\in\mathbb{F}^+(G);\, |w|\geq 1} w(S)\mathcal{H}\Big)$.

Finally, the uniqueness assertion is that if $S' = (S_1', \dots, S_n')$ on $\mathcal{K}' \supseteq \mathcal{H}$ is another minimal dilation of T with respect to $\mathcal{P}$, then there is a unitary $U : \mathcal{K} \to \mathcal{K}'$ such that $U|_{\mathcal{H}} = I_{\mathcal{H}}$ and $U^* S_i' U = S_i$ for $1 \leq i \leq n$. This proof is a relatively simple adaptation of the single variable case [**8**], hence we omit the details. ■

Remark 3.3. In the case that the family $\mathcal{P} = \{I\}$ is a singleton, Theorem 3.2 collapses to the Sz.-Nagy dilation theorem [**8**] when $n = 1$ and the FBP dilation theorem [**9, 4, 23**] when $2 \leq n \leq \infty$. This is the only case for which the minimal dilation consists entirely of isometries. In its most general form, Theorem 3.2 may be regarded as a refinement of the Muhly-Solel dilation theorem for a subclass of the representations considered in [**22**]. In the language of [**22**], a row contraction T and collection of projections $\mathcal{P}$ satisfying (3) can be seen to induce a *covariant representation* of a C*-correspondence generated by T and $\mathcal{P}$. These representations form a subclass of those considered in [**22**], and the class of all such representations are shown to have minimal dilations. Hence the basic existence of minimal dilations in our setting can be deduced from [**22**]. However, our short spatial proof and the particular details we obtain are not easily seen there. Furthermore, we suggest that the results of the current paper provide a more accessible dilation theory for row contractions, as the abstract machinery of Hilbert modules and C*-correspondences is not required in the formulation here.

We next discuss how properties of T can be used to identify properties of its minimal dilations.

Proposition 3.4. *Every minimal partially isometric dilation of $T = (T_e)_{e\in E(G)}$ is pure if and only if*

$$\lim_{d\to\infty}\Big(\sum_{w\in\mathbb{F}^+(G);\,|w|\geq d}||w(T)^*\xi||^2\Big) = 0 \quad \textit{for all} \quad \xi\in\mathcal{H}. \tag{4}$$

Proof. If $S = (S_e)_{e\in E(G)}$ is a pure minimal dilation of T, then $S_e^*|_{\mathcal{H}} = T_e^*$ for $e \in E(G)$ and (4) follows from the corresponding identity (1) for S. Conversely, when (4) holds we may use the (†) relations to obtain the necessary estimates which show that (1) holds for every minimal dilation S of T. ■

Corollary 3.5. *If $\sum_{i=1}^n T_iT_i^* \leq rI$, with $r < 1$, then every minimal partially isometric dilation of $T = (T_1,\dots,T_n)$ is pure.*

Next we obtain detailed information on the pure part of a dilation.

Proposition 3.6. *Let $S = (S_e)_{e\in E(G)}$ be a minimal partially isometric dilation of $T = (T_e)_{e\in E(G)}$ with respect to the projections $\mathcal{P} = \{P_k\}_{k\in V(G)}$. Then for $k \in V(G)$ we have*

$$\operatorname{rank}\Big(Q_k\big(I_{\mathcal{K}} - \sum_e S_eS_e^*\big)\Big) = \operatorname{rank}\Big(P_k\big(I_{\mathcal{H}} - \sum_e T_eT_e^*\big)\Big). \tag{5}$$

Proof. Recall $\{Q_k\} = \{S_e^*S_e\}$. By the Wold Decomposition and Theorem 3.2 we may assume that $Q_k|_{\mathcal{H}} = P_k$ for $k \in V(G)$. Fix $k \in V(G)$. Observe the (†) relations imply Q_k commutes with $P = I_{\mathcal{K}} - \sum_e S_eS_e^*$. Let R_k be the projection $R_k = Q_kP$ and let $P_{\mathcal{H}}$ be the projection of $\mathcal{K}$ onto $\mathcal{H}$. The minimality of the dilation ensures the subspace $P\mathcal{K}$ does not intersect $\mathcal{H}^\perp$, and hence neither does the subspace $Q_kP\mathcal{K} = PQ_k\mathcal{K}$. Thus $P_{\mathcal{H}}(Q_kP)P_{\mathcal{H}}$ has the same rank as Q_kP (even though $Q_kP\mathcal{K}$ is not contained in $\mathcal{H}$ in general). But notice that

$$P_{\mathcal{H}}Q_kP\Big|_{\mathcal{H}} = P_{\mathcal{H}}\Big(Q_k\big(I_{\mathcal{K}} - \sum_e S_eS_e^*\big)\Big)P_{\mathcal{H}}\Big|_{\mathcal{H}} = P_k\big(I_{\mathcal{H}} - \sum_e T_eT_e^*\big),$$

and the result follows. ■

Corollary 3.7. *Every minimal partially isometric dilation of* $T = (T_1, \ldots, T_n)$ *is fully coisometric if and only if* $\sum_{i=1}^{n} T_i T_i^* = I_{\mathcal{H}}$.

Remark 3.8. The identity (5) shows how to compute the vertex multiplicities for a minimal dilation strictly in terms of the dilated row contraction T and the projection family $\mathcal{P}$. Thus, by Theorem 2.1 this gives a method for explicitly finding the pure part of a dilation.

On the other hand, Corollary 3.7 identifies when the fully coisometric case occurs in terms of T. In complete generality it is not possible to explicitly describe the fully coisometric part of a minimal dilation. As mentioned above, the representation theory of $\mathcal{O}_n$ is the obstacle. However, note that the fully coisometric part of a minimal dilation here will determine a representation of a Cuntz-Krieger directed graph C*-algebra $\mathrm{C}^*(G)$. Ephrem [**7**] characterizes when the representation theory of such algebras is type I strictly in terms of the directed graph G. Interestingly, in the case of finite graphs his graph-theoretic condition can be seen to be precisely the condition obtained by the second author and Power [**15, 16**] as a description of when a nonselfadjoint 'free semigroupoid algebra' is partly free. Specifically, $\mathrm{C}^*(G)$ is type I if and only if the following two conditions hold:

(*i*) G contains no double-cycles; there are no distinct cycles $w_1 = xw_1x$, $w_2 = xw_2x$ at a vertex x in G.

(*ii*) Given a non-overlapping infinite directed path in G, there are only finitely many ways to exit and return to the path.

Thus, whenever the G obtained in a minimal dilation of T satisfies these conditions, the dilation theory here gives an improvement on the dilation theory derived from the FBP dilation. For example, let $\mathcal{H}$ be a Hilbert space and let T_1, T_2, T_3 be operators on $\mathcal{H}$ such that T_1 is a co-isometry and (T_2, T_3) forms a row contraction with $T_2T_2^* + T_3T_3^* = I$. Define a row contraction $V = (V_1, V_2, V_3)$ on $\mathcal{H} \oplus \mathcal{H}$ by

$$V_1 = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad V_2 = \begin{bmatrix} 0 & 0 \\ 0 & T_2 \end{bmatrix}, \qquad V_3 = \begin{bmatrix} 0 & 0 \\ T_3 & 0 \end{bmatrix}.$$

Let $P_i \equiv P_{\mathcal{H}}$, $i = 1, 2$, be the projections of the direct sum $\mathcal{H} \oplus \mathcal{H}$ onto its two coordinate spaces. Observe that $VV^* = \sum_{i=1}^{3} V_i V_i^* = I$. Hence, the minimal partially isometric dilation of V with respect to $\mathcal{P} = \{P_1, P_2\}$ determines a representation of the C*-algebra $\mathrm{C}^*(G)$ where G is the directed graph with two vertices, a loop edge over each vertex, and a directed edge from the first to the second vertex. As G satisfies the above conditions, $\mathrm{C}^*(G)$ is type I and hence GCR [**11**]. That is, every representation of $\mathrm{C}^*(G)$ can be obtained as a direct integral of irreducible subrepresentations [**1**].

4. Partial Orders on Minimal Dilations

Let $T = (T_1, \ldots, T_n)$ be a fixed row contraction on $\mathcal{H}$. As a convenience, in this section we assume that $\mathcal{H} = \vee_i(\mathrm{Ran}(T_i) \vee \mathrm{Ran}(T_i^*))$. (Observe that the restrictions of each T_i and T_i^* to the orthogonal complement of this joint reducing subspace are zero.)

Lemma 4.1. *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are families of projections which stabilize T as in* (3)*, then the families $\mathcal{P}_1$ and $\mathcal{P}_2$ are mutually commuting.*

Proof. Let $P_\alpha \in \mathcal{P}_1$ and $P_\beta \in \mathcal{P}_2$. Then for $1 \leq i \leq n$,

$$P_\alpha P_\beta T_i = P_\beta P_\alpha T_i = \begin{cases} T_i & \text{if } P_\beta T_i = T_i = P_\alpha T_i \\ 0 & \text{if } P_\beta T_i = 0 \text{ or } P_\alpha T_i = 0. \end{cases}$$

Similarly, $T_i P_\alpha P_\beta = T_i P_\beta P_\alpha$, so that $P_\beta P_\alpha T_i^* = P_\alpha P_\beta T_i^*$. By the assumption above, the ranges of $\{T_i, T_i^* : 1 \leq i \leq n\}$ are dense inside $\mathcal{H}$, hence the result follows. ∎

Let $G_\mathcal{P}$ be the directed graph determined by the relations (3) for a family of projections $\mathcal{P}$ which stabilize T. Using the Wold Decomposition and Theorem 3.2, we may identify the set of all graphs $G_\mathcal{P}$ with the set $\mathrm{MinDil}(T)$ of all equivalence classes of minimal partially isometric dilations of T. Define a partial ordering on $\mathrm{MinDil}(T)$ by: $G_{\mathcal{P}_1} \leq G_{\mathcal{P}_2}$ if and only if

$$\forall\, P \in \mathcal{P}_1 \quad \exists\, \mathbb{P} \subseteq \mathcal{P}_2 \quad \text{such that} \quad P = \sum_{P_\alpha \in \mathbb{P}} P_\alpha.$$

This set has a natural join operation defined by $G_{\mathcal{P}_1} \bigvee G_{\mathcal{P}_2} \equiv G_{\mathcal{P}_1 \bigvee \mathcal{P}_2}$ where

$$\mathcal{P}_1 \bigvee \mathcal{P}_2 = \Big\{ P_1 \wedge P_2 : P_i \in \mathcal{P}_i,\ i = 1, 2 \Big\},$$

and $P_1 \wedge P_2 = P_1 P_2 = P_2 P_1$ is the projection onto the intersection of the range subspaces for P_1, P_2 by Lemma 4.1.

In terms of the directed graph structures, the relation $G_{\mathcal{P}_1} \leq G_{\mathcal{P}_2}$ means $G_{\mathcal{P}_2}$ may be *deformed*, by identifying certain vertices in $V(G_{\mathcal{P}_2})$, to obtain $G_{\mathcal{P}_1}$. Conversely, to every deformation of $G_{\mathcal{P}_2}$ there corresponds an element of $\mathrm{MinDil}(T)$.

Proposition 4.2. *The partially ordered set* $\mathrm{MinDil}(T)$ *has a largest element and a smallest element.*

Proof. The smallest element of $\mathrm{MinDil}(T)$ is clearly the minimal *isometric* dilation of T [**8, 9, 4, 23**], corresponding to the directed graph with a single vertex and n distinct loop edges. On the other hand, if we let $\overline{\mathcal{P}} = \{\mathcal{P}_\alpha\}_{\alpha \in \mathbb{A}}$ be the set of all sets of projections which satisfy (3) for T, we may define a largest element of $\overline{\mathcal{P}}$ by $\mathcal{P}_0 = \big\{ \bigwedge_{\alpha \in \mathbb{A}} P_\alpha : P_\alpha \in \mathcal{P}_\alpha \big\}$.

Indeed, the family $\mathcal{P}_0$ is clearly a set of pairwise orthogonal projections which stabilize T. Further, by Lemma 4.1 the P_α from distinct $\mathcal{P}_\alpha$ commute, and hence $\mathcal{P}_0$ determines a partition of the identity which is the supremum of $\overline{\mathcal{P}}$. ■

The unique minimal partially isometric dilation S_0 which corresponds to $\mathcal{P}_0$ may be regarded as the 'finest' of all the minimal partially isometric dilations of $T = (T_1, \dots, T_n)$. It is the minimal dilation which best reflects the joint behaviour of the T_i. Of course, in many instances this will be the minimal isometric dilation of T, when $\mathcal{P} = \{I\}$ is the only projection family which stabilizes T, but in general there may be non-trivial families of such projections. Amongst the set of all directed graphs $G_{\mathcal{P}}$ which come from minimal dilations of T, the directed graph $G_{\mathcal{P}_0}$ will be the largest in the sense that any other directed graph in this set can be obtained from $G_{\mathcal{P}_0}$ by a series of deformations.

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Department of Mathematics, Purdue University, West Lafayette, IN, USA 47907

E-mail address: jury@math.purdue.edu

Department of Mathematics and Statistics, University of Guelph, Guelph, ON, CANADA N1G 2W1

E-mail address: kribs@math.purdue.edu