DOES EVERY CONTRACTIVE ANALYTIC FUNCTION IN A POLYDISK HAVE A DISSIPATIVE N-DIMENSIONAL SCATTERING REALIZATION?

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Abstract. No.

The title question was posed by D. Kalyuzhnyi-Verbovetskyi [1, Problem 1.3]. Let $L(\mathcal{H},\mathcal{K})$ denote the set of all bounded linear operators between a pair of Hilbert spaces $\mathcal{H},\mathcal{K}$, and let $\mathbb{D}^n$ and $\mathbb{T}^n$ denote the open unit polydisk, and the unit n-torus, respectively.

Definition 1. An *dissipative nD scattering system* is a tuple

$$\alpha = (n; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{U}, \mathcal{Y}) \tag{1}$$

where:

i) $n \geq 1$ is an integer;
ii) $\mathcal{X}, \mathcal{U}, \mathcal{Y}$ are Hilbert spaces;
iii) $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are n-tuples of operators (so $\mathbf{A} = (A_1, \dots A_n)$, etc.) with

$$A_k \in L(\mathcal{X},\mathcal{X}), \quad B_k \in L(\mathcal{U},\mathcal{X}), \quad C_k \in L(\mathcal{X},\mathcal{Y}), \quad D_k \in L(\mathcal{U},\mathcal{Y}); \tag{2}$$

iv) The operator $\zeta\mathbf{G} \in L(\mathcal{X}\oplus\mathcal{U}, \mathcal{X}\oplus\mathcal{Y})$ is contractive for all ζ in the unit n-torus $\mathbb{T}^n$, where

$$\zeta\mathbf{G} := \sum_{k=1}^{n} \zeta_k G_k \tag{3}$$

and the G_k are the 2×2 block operators

$$G_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \tag{4}$$

Given such a system, its *transfer function* is the $L(\mathcal{U},\mathcal{Y})$-valued function

$$\theta_\alpha(z) = z\mathbf{D} + z\mathbf{C}(I_{\mathcal{X}} - z\mathbf{A})^{-1} z\mathbf{B}. \tag{5}$$

defined for all $z \in \mathbb{D}^n$. It is shown in [3] that the transfer function θ_α is a *contractive operator function*; that is, it is analytic in the unit polydisk $\mathbb{D}^n$ and satisfies

$$\|\theta_\alpha(z)\|_{L(\mathcal{U},\mathcal{Y})} \leq 1 \tag{6}$$

for all $z \in \mathbb{D}^n$. The question is then whether every contractive operator function in $\mathbb{D}^n$, vanishing at the origin, is such a transfer function. The answer is known to be "yes" when $n = 1$ or 2, and in fact a stronger result is true; the $\mathbf{G}$ can be chosen unitary (so the scattering system is *conservative*). It was also known that when $n = 3$, there exist contractive operator functions which do not have conservative realizations; this is due to the failure of von Neumann's inequality in three variables. (See [1, 3] for a discussion.) In this note we show the answer is still "no" in the dissipative case when $n = 3$, and give an explicit counterexample (in the scalar case $\mathcal{U} = \mathcal{Y} = \mathbb{C}$).

We first show that any polynomial with a dissipative realization must satisfy a restricted form of von Neumann's inequality. Let $\mathcal{T}$ denote the set of all n-tuples of commuting operators $\mathbf{T} =$

Date: January 16, 2012.
Research partially supported by NSF grant DMS 1101134.

$(T_1, \dots T_n)$ on Hilbert space satisfying the following condition: whenever $\mathbf{X} = (X_1, \dots X_n)$ is an n-tuple of operators satisfying

$$\|\sum_{k=1}^{n} z_k X_k\| \leq 1 \tag{7}$$

for all $z = (z_1, \dots z_n) \in \mathbb{D}^n$, then

$$\|\sum_{k=1}^{n} T_k \otimes X_k\|_{L(\mathcal{H}\otimes\mathcal{K})} \leq 1 \tag{8}$$

where the T_k act on $\mathcal{H}$ and the X_k act on $\mathcal{K}$.

It is easy to see that the $\mathbf{T}$ satisfying this condition must be commuting contractions, but when $n \geq 3$ it is known that not every n-tuple of commuting contractions belongs to $\mathcal{T}$.

Theorem 2. *If p is a polynomial which can be realized as the transfer function of a dissipative nD scattering system, then*

$$\|p(\mathbf{T})\| \leq 1 \tag{9}$$

for all $\mathbf{T} \in \mathcal{T}$.

We will say such p satisfy the *restricted von Neumann inequality.*

Proof of Theorem 2. Suppose p is a polynomial vanishing at 0 and $p = \theta_\alpha$ for some α as in Definition 1; we work only in the scalar case $\mathcal{U} = \mathcal{Y} = \mathbb{C}$. First note that since analytic functions in the polydisk satisfy a maximum principle relative to $\mathbb{T}^n$, the dissipativity condition (iv) implies

$$\|z\mathbf{G}\| := \|\sum_{k=1}^{n} z_k G_k\| \leq 1 \tag{10}$$

for all $z \in \mathbb{D}^n$. Then by definition, if $\mathbf{T} \in \mathcal{T}$, we have

$$\|\sum_{k=1}^{n} T_k \otimes G_k\| \leq 1. \tag{11}$$

Next, we recall the classical fact that if

$$F = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \tag{12}$$

is a block operator and $\|F\| < 1$, then the linear fractional operator

$$Z + Y(I - W)^{-1}X \tag{13}$$

is contractive. Now apply this to the block operator

$$r\mathbf{T}\cdot\mathbf{G} = \begin{pmatrix} r\mathbf{T}\cdot\mathbf{A} & r\mathbf{T}\cdot\mathbf{B} \\ r\mathbf{T}\cdot\mathbf{C} & r\mathbf{T}\cdot\mathbf{D} \end{pmatrix} \tag{14}$$

where $\mathbf{T}\cdot\mathbf{A} := \sum_{k=1}^{n} T_k \otimes A_k$, etc., and $r < 1$. We conclude that if $\mathbf{T} \in \mathcal{T}$, then the linear fractional operator

$$r\mathbf{T}\cdot\mathbf{D} + r\mathbf{T}\cdot\mathbf{C}(I_{\mathcal{H}\otimes\mathcal{X}} - r\mathbf{T}\cdot\mathbf{A})^{-1} r\mathbf{T}\cdot\mathbf{B} \tag{15}$$

is contractive for all $r < 1$. But it is straightforward to check that, since p is assumed to be given by the transfer function realization

$$p(z) = z\mathbf{D} + z\mathbf{C}(I_{\mathcal{X}} - z\mathbf{A})^{-1} z\mathbf{B}, \tag{16}$$

the expression (15) is equal to $p(r\mathbf{T})$ (This can be done by expanding (16) in a power series, substituting $r\mathbf{T}$ for z, and comparing coefficients with the expansion of (15) in powers of $rT_1, \dots rT_n$.) But then $\|p(r\mathbf{T})\| \leq 1$ for all $\mathbf{T} \in \mathcal{T}$ and $r < 1$, which suffices to establish the theorem. $\square$

It follows that any contractive polynomial which fails the restricted von Neumann inequality will fail to have a dissipative realization. In fact, the counterexample to the classical von Neumann inequality produced by Kaijser and Varopoulos is, it turns out, also a counterexample to the restricted inequality, as we now show. The computations are taken from a closely related example considered in [2].

Let $e_1, \dots e_5$ denote the standard basis of $\mathbb{C}^5$. Consider the unit vectors

$$v_1 = \frac{1}{\sqrt{3}}(-e_2 + e_3 + e_4)$$
$$v_2 = \frac{1}{\sqrt{3}}(e_2 - e_3 + e_4)$$
$$v_3 = \frac{1}{\sqrt{3}}(e_2 + e_3 - e_4)$$

The Kaijser-Varopoulos contractions are the commuting 5×5 matrices T_1, T_2, T_3 defined by

$$T_j = e_{j+1} \otimes e_1 + e_5 \otimes v_j$$

If p is the polynomial

$$p(z_1, z_2, z_3) = \frac{1}{5}(z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_1z_3 - 2z_2z_3) \tag{17}$$

then it is known that $\sup_{\zeta \in \mathbb{T}^3} |p(\zeta)| = 1$ but

$$\|p(\mathbf{T})\| = \frac{3\sqrt{3}}{5} > 1, \tag{18}$$

so p fails the classical von Neumann inequality [5]. To show that this p fails the restricted von Neumann inequality, we show that already this $\mathbf{T}$ belongs to $\mathcal{T}$; that is, if X_1, X_2, X_3 are operators which satisfy

$$\|z_1X_1 + z_2X_2 + z_3X_3\| \leq 1 \tag{19}$$

for all $z \in \mathbb{D}^n$, then $\|\sum_{k=1}^3 T_k \otimes X_k\| \leq 1$. To see this, we compute and find

$$T_1 \otimes X_1 + T_2 \otimes X_2 + T_3 \otimes X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ X_1 & 0 & 0 & 0 & 0 \\ X_2 & 0 & 0 & 0 & 0 \\ X_3 & 0 & 0 & 0 & 0 \\ 0 & Y_1 & Y_2 & Y_3 & 0 \end{pmatrix} \tag{20}$$

where

$$Y_1 = \frac{1}{\sqrt{3}}(-X_1 + X_2 + X_3)$$
$$Y_2 = \frac{1}{\sqrt{3}}(X_1 - X_2 + X_3)$$
$$Y_3 = \frac{1}{\sqrt{3}}(X_1 + X_2 - X_3)$$

The norm of the matrix (20) is equal to the maximum of the norms of the first column and the last row. By (19), we have $\| \pm X_1 \pm X_2 \pm X_3\| \leq 1$ for all choices of signs, so the last row of (20) has norm at most 1. To say that the first column has norm at most 1 amounts to saying that

$$I - \sum_{k=1}^{n} X_k^* X_k \geq 0. \tag{21}$$

This may be seen by averaging: by (19), the matrix valued function

$$I - \sum_{i,j=1}^{n} \zeta_i \overline{\zeta_j} X_j^* X_i$$

is positive semidefinite on $\mathbb{T}^n$. Integrating against normalized Lebesgue measure on $\mathbb{T}^n$ gives (21).

There is a general principle that transfer function realizations should be equivalent to von Neumann-type inequalities. Some recent, general results in this direction may be found in [2, 4].

References

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