C*-algebras, composition operators and dynamics

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SEAM 23, March 9, 2007

The setting:

- $\mathbb{D} = \{|z| < 1\} \subset \mathbb{C}$
- $\mathbb{T} = \partial\mathbb{D} = \{|z| = 1\}$
- dm=normalized Lebesgue measure on $\mathbb{T}$
- $\varphi : \mathbb{D} \to \mathbb{D}$ holomorphic ($\varphi \neq$ const.)
- H^2= Hardy space, $P : L^2 \to H^2$ Riesz proj.
- Some operators on H^2:
    - Toeplitz operators: For $f \in L^\infty(\mathbb{T})$, $g \in H^2$,

$$T_f g := P(fg)$$

    - Composition operators: For $g \in H^2$,

$$C_\varphi g = g \circ \varphi$$

General question: What can be said about the C*-algebra generated by a set of composition operators? (or composition and Toeplitz operators)?

We seek C*-algebraic relations that obtain between Toeplitz and composition operators (and their adjoints).

One easy relation:

Lemma

For any $\varphi \in Hol(\mathbb{D})$ and $f \in H^\infty$,

$$C_\varphi T_f = T_{f\circ\varphi} C_\varphi$$

Proof: From definitions—for any $h \in H^2$,

$$\begin{aligned}(C_\varphi T_f)h &= C_\varphi(fh)\\ &= (f\circ\varphi)(h\circ\varphi)\\ &= (T_{f\circ\varphi}C_\varphi)h \qquad \square\end{aligned}$$

Theorem (Bourdon-MacCluer '07)

Let φ be inner. Then

$$C_\varphi^* C_\varphi$$

is Toeplitz, with symbol

$$F(z) = \frac{1 - |\varphi(0)|^2}{|z - \varphi(0)|^2}$$

Proof: Recall the *Brown-Halmos criterion*: T is Toeplitz if and only if $S^* TS = T$.
Now

$$\begin{aligned} S^* C_\varphi^* C_\varphi S &= C_\varphi^* T_\varphi^* T_\varphi C_\varphi \\ &= C_\varphi^* C_\varphi \end{aligned}$$

To find symbol, apply to $C_\varphi^* C_\varphi$ to scalars... $\square$

Let now Γ be a group of automorphisms of $\mathbb{D}$. What can we say about

$$\mathcal{C}_\Gamma = C^*\{C_\varphi : \varphi \in \Gamma\}$$

E.g., how is it related to C*-algebras associated to Γ?

We will need the notion of a *covariance algebra*:
Consider:

- X a compact Hausdorff space
- G a group of homeomorphisms of X

In the category of C*-algebras, these correspond to

- $C(X)$, a commutative unital C*-algebra
- $C^*(u(G))$, where $u : G \to U(\mathcal{H})$ is a unitary representation

Definition

Given:

- X = compact Hausdorff space
- G = group of homeomorphisms of X

G acts on $C(X)$ via $(g \cdot f)(x) = f(g^{-1} \cdot x)$.
A *covariant representation* of X, G is a triple $(\pi, u, \mathcal{H})$ with

- $\pi : C(X) \to B(\mathcal{H})$ a *-homomorphism
- $u : G \to U(\mathcal{H})$ a unitary representation

satisfying

$$u(g)^* \pi(f) u(g) = \pi(g \cdot f)$$

A *covariance algebra* is a C*-subalgebra of $B(\mathcal{H})$ generated by a covariant representation.

Consider now $\varphi \in \Gamma \subset \mathrm{Aut}(\mathbb{D})$. Define

$$U_\varphi^* = C_\varphi (C_\varphi^* C_\varphi)^{-1/2}$$

Suppose also that Γ is discrete and *non-elementary* (any orbit accumulates at at least three points of $\mathbb{T}$).
Using the two relations proved so far, it can be shown that

- $U_\varphi U_\psi = U_{\varphi\psi} +$ compact
- $U_\varphi^* T_f U_\varphi = T_{f\circ\varphi^{-1}} +$ compact
- $S \in C^*\{C_\varphi : \varphi \in \Gamma\}$

for all $\varphi, \psi \in \Gamma$ and all $f \in C(\mathbb{T})$.
It follows that

$$C^*\{C_\varphi : \varphi \in \Gamma\} = C^*\{S, U_\varphi : \varphi \in \Gamma\}$$

Theorem (J., to appear)

Let Γ be a non-elementary Fuchsian group $\Gamma \subset Aut(\mathbb{D})$. Let

$$\mathcal{C}_\Gamma = C^*\{C_\varphi : \varphi \in \Gamma\}$$

There is an exact sequence of C-algebras*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{C}_\Gamma \longrightarrow C(\mathbb{T}) \times \Gamma \longrightarrow 0$$

The quotient $\mathcal{C}_\Gamma/\mathcal{K}$ is generated by the images in the Calkin algebra of of T_f and $U_\varphi^* = C_\varphi(C_\varphi^* C_\varphi)^{-1/2}$. By previous slide, the quotient is thus a covariance algebra for $(\mathbb{T}, \Gamma)$. With more work it can be shown that it is isomorphic to the *full crossed product* C*-algebra

$$C(\mathbb{T}) \times \Gamma$$

which is "universal" among covariance algebras (any other is a quotient of this).

What are the "next" examples?
Analogy between dynamics of groups and rational functions on the Riemann sphere ("Sullivan's dictionary") suggests replacing

$$\Gamma = \text{ Fuchsian group } = \text{ group leaving } \mathbb{T} \text{ invariant}$$

with

$$\{\varphi^n\} = \text{ iterates of rational function leaving } \mathbb{T} \text{ invariant}$$

That is: replace $\mathcal{C}_\Gamma$ with the C*-algebra generated by C_φ with φ a finite Blaschke product. Then look for

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(C_\varphi) \longrightarrow ?? \longrightarrow 0$$

where ?? is some kind of "covariance algebra" for the dynamical system $(\mathbb{T}, \varphi)$.

Consider

- X = compact Hausdorff space
- $\varphi : X \to X$ continuous, surjective

Suppose $\mathcal{L} : C(X) \to C(X)$ is a positive linear map satisfying

$$\mathcal{L}(f_1 \cdot (f_2 \circ \varphi)) = \mathcal{L}(f_1) \cdot f_2$$

for all $f_1, f_2 \in C(X)$ (examples later).

The *Exel crossed product* for the triple $(X, \varphi, \mathcal{L})$ is the universal C*-algebra generated by $C(X)$ and an operator V satisfying

- $(f \circ \varphi)V = Vf$
- $V^* f V = \mathcal{L}(f)$
- "redundancies"...

Thinking $f = T_f$, $V = C_\varphi$ (mod $\mathcal{K}$) leads us to consider the operators

$$C_\varphi^* T_f C_\varphi$$

Theorem

Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is inner and $f \in L^\infty(\mathbb{T})$. Then

$$C_\varphi^* T_f C_\varphi$$

is a Toeplitz operator.

Proof: Brown-Halmos again:

$$\begin{aligned} S^* C_\varphi^* T_f C_\varphi S &= C_\varphi^* T_\varphi^* T_f T_\varphi C_\varphi \\ &= C_\varphi^* T_{f|\varphi|^2} C_\varphi \qquad (\varphi \text{ analytic}) \\ &= C_\varphi^* T_f C_\varphi \qquad (\varphi \text{ inner}) \end{aligned}$$

Which Toeplitz operator is it?

Given $\varphi : \mathbb{D} \to \mathbb{D}$, $\alpha \in \mathbb{T}$,

$$\mathrm{Re}\left(\frac{\alpha + \varphi(z)}{\alpha - \varphi(z)}\right) = \frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} = \int \frac{1 - |z|^2}{|\zeta - z|^2}\, d\mu_\alpha(\zeta)$$

The measures μ_α are the *Aleksandrov-Clark measures* for φ.

Definition (The Aleksandrov operator)

Let f be a Borel function on $\mathbb{T}$. Define

$$A_\varphi(f)(\alpha) = \int f(\zeta)\, d\mu_\alpha(\zeta)$$

Theorem (Aleksandrov '87)

The Aleksandrov operator

$$A_\varphi(f)(\alpha) = \int f \, d\mu_\alpha$$

extends to a well-defined, bounded operator on

- $C(\mathbb{T})$
- $L^p(\mathbb{T})$, $1 \leq p \leq \infty$; *also* H^p *if* $\varphi(0) = 0$
- *BMO, VMO, Besov spaces...*

The Cauchy transform of μ_α:

Assume now $\varphi(0) = 0$. Then for $|\lambda| < 1, |\alpha| = 1$,

$$A_\varphi(k_\lambda)(\alpha) = \int \frac{1}{1-\overline{\lambda}\zeta}\, d\mu_\alpha(\zeta) = \frac{1}{1-\overline{\varphi(\lambda)}\alpha} = C_\varphi^*(k_\lambda)(\alpha)$$

Since A_φ is bounded on H^2, this proves

Theorem (inner case: Lotto-McCarthy '93; general case–?)

If $\varphi(0) = 0$, then

$$C_\varphi^* = A_\varphi$$

Application: we can now compute the symbol of $C_\varphi^* T_f C_\varphi$...

Theorem (J., in progress)

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be inner and $f \in L^\infty(\mathbb{T})$. Then

$$C_\varphi^* T_f C_\varphi$$

is Toeplitz, with symbol $A_\varphi(f)$.

There is an "asymptotic" version for general φ:

Definition

Say $A \in B(H^2)$ is *(weakly) asymptotically Toeplitz* if

$$\lim_{n\to\infty} S^{*n} A S^n$$

exists (WOT). If so, limit is a Toeplitz operator T_g; g is called the *asymptotic symbol* of A.

Theorem (J., in progress)

If $\varphi : \mathbb{D} \to \mathbb{D}$, $f \in L^\infty$, then

$$C_\varphi^* T_f C_\varphi$$

is asymptotically Toeplitz, with asymptotic symbol

$$A_\varphi^s(f)(\alpha) := \int f \, d\sigma_\alpha$$

(recall $\mu_\alpha = h_\alpha m + \sigma_\alpha$).
Moreover if

$$E = \{\zeta : |\varphi(\zeta)| = 1\}$$

and $f = 0$ a.e. on E^c, then $C_\varphi^* T_f C_\varphi$ is Toeplitz. (Converse holds if $f \geq 0$.)

Returning to the case of finite Blaschke products, it is not hard to show that the Clark measures are given by

$$\sigma_\alpha = \sum_{\varphi(\zeta)=\alpha} \frac{1}{|\varphi'(\zeta)|}\delta_\zeta$$

So $A_\varphi(f)(\alpha)$ is a certain weighted average of f over the preimages of α.

For general inner φ, the set

$$\{\zeta \in \mathbb{T} : \varphi(\zeta) = \alpha\}$$

is a carrier for σ_α but σ_α has atoms only at points where φ has an angular derivative.

Definition (the transfer operator)

- $\varphi : X \to X$ continuous, surjective, finitely valent
- $g : X \to \mathbb{R}$, continuous

Define for $f \in C(X)$

$$\mathcal{L}_g(f)(x) = \sum_{\varphi(y)=x} \exp(g(y))f(y)$$

$\mathcal{L}_g$ is called the *transfer operator* or *Perron-Frobenius-Ruelle* operator; it is bounded and completely positive on $C(X)$.

For all $f_1, f_2 \in C(X)$ we have

$$\mathcal{L}_g(f_1 \cdot (f_2 \circ \varphi)) = \mathcal{L}_g(f_1) \cdot f_2$$

In particular if $\mathcal{L}_g(1) \equiv 1$ then $\mathcal{L}_g$ is a left inverse for C_φ.

Proto-"Perron-Frobenius"-theorem: (unital version, for simplicity) Suppose $\mathcal{L}_g(1) \equiv 1$. With suitable hypotheses on (φ, g),

$$\mathcal{L}_g^n(f) \to c \cdot 1$$

uniformly, where c is a scalar. The assignment $f \to c$ determines a probability measure μ satisfying

$$\mathcal{L}_g^* \mu = \mu$$

Under good conditions such μ is unique; it describes the asymptotic distribution of the (weighted) backward orbits of φ.

Recall that for a rational map ψ of the Riemann sphere, its *Julia set* J is the complement of the maximal open set on which the iterates of ψ are a normal family. (For a finite Blaschke product $J \subset \mathbb{T}$; either $J = \mathbb{T}$ or J is a Cantor set.)

Theorem (Denker-Urbański '91 (simplified version))

Given:

- *ψ rational with Julia set J*
- *$g : J \to \mathbb{R}$ Hölder continuous, $\mathcal{L}_g(1) \equiv 1$*

Then there is a unique measure μ with support equal to J such that $\forall f \in C(J)$,

$$\mathcal{L}_g^n(f) \to \left(\int_J f \, d\mu \right) \cdot 1$$

uniformly.

Now fix a finite Blaschke product φ and a *Hölder continuous* function $g : X \to \mathbb{R}$. We have a transfer operator

$$\mathcal{L}_g(f)(z) = \sum_{\varphi(\zeta)=z} \exp(g(\zeta))f(\zeta)$$

defined on $C(\mathbb{T})$. For simplicity we assume $\mathcal{L}_g(1) \equiv 1$; we call such g an *admissible weight*.

Theorem

Let g be an admissible weight. Then there exists $h \in A(\mathbb{D})$ such that

$$V := T_h C_\varphi$$

satisfies

$$V^* T_f V = T_{\mathcal{L}_g(f)}$$

for all $f \in C(\mathbb{T})$.

By suitably modifying Przytycki's proof of the Denker-Urbański result, we obtain:

Theorem

Given:

- φ *a finite Blaschke product*
- $g : \mathbb{T} \to \mathbb{R}$ *an admissible weight*

Then for all $f \in C(\mathbb{T})$,

$$\mathcal{L}_g^n(f) \to \left(\int f \, d\mu_g \right) \cdot 1$$

uniformly, where μ_g *is the unique* $\exp(g)-$*conformal measure supported on the Julia set of* φ.

For a fixed admissible weight g, we earlier obtained a weighted composition operator

$$V := T_h C_\varphi$$

such that

$$V^* T_f V = T_{\mathcal{L}_g(f)}$$

for all $f \in C(\mathbb{T})$. Using the Perron-Frobenius result for $\mathcal{L}_g$ we obtain

Theorem

Fix an admissible weight g and V as above. For all $f \in C(\mathbb{T})$,

$$V^{*n} T_f V^n \to \left(\int f \, d\mu_g \right) \cdot I$$

in norm.

In general some weight is needed to obtain convergence; for example let

$$\varphi(z) = \left(\frac{3z+1}{3+z}\right)^2$$

The orbit of 0 under φ tends radially to 1; using this one can show

$$\|C_\varphi^{*n} C_\varphi^n\| \to \infty.$$

Since the measure μ_g always has support equal to the Julia set of φ, the composition operator C_φ can "see" the Julia set via iteration of the completely positive map:

Corollary

Let φ be a finite Blaschke product, g an admissible weight and $V = T_h C_\varphi$ as above. Let $f \in C(\mathbb{T}), f \geq 0$. Then f vanishes on the Julia set $J \subset \mathbb{T}$ of φ if and only if

$$V^{*n} T_f V^n \to 0$$

in norm.

Let φ be inner, with Clark measures $\{\sigma_\alpha\}$. Consider the normalized Aleksandrov operator

$$\widetilde{A_\varphi}(f)(\alpha) = \int f \, \frac{d\sigma_\alpha}{\|\sigma_\alpha\|}$$

In the rational case this corresponds to the admissible weight

$$g(\zeta) = -\log\left(|\varphi'(\zeta)| \cdot \|\sigma_{\varphi(\zeta)}\|\right)$$

Conjecture: For any inner function φ and all $f \in C(\mathbb{T})$, the sequence

$$\widetilde{A_\varphi}^n(f)$$

converges uniformly to a scalar c.

If so, $c = \int f \, d\mu$ and μ describes the (weighted) "asymptotic distribution" of the backwards orbits of φ on $\mathbb{T}$. This would be particularly interesting when φ has an attracting fixed point on $\mathbb{T}$; presumably $\mathrm{supp}(\mu) \subsetneq \mathbb{T}$ (a "Julia set" for φ on $\mathbb{T}$).

Where does Hölder continuity come in? (Possibly via Matheson's smoothness theorem; which implies that A_φ is bounded on the Hölder classes.)

Theorem

Let

- φ *be a finite Blaschke product* ($\deg \varphi \geq 2$)*, with fixed point in* $\mathbb{D}$
- $g : \mathbb{T} \to \mathbb{R}$ *Hölder continuous*
- $C(\mathbb{T}) \rtimes_{\varphi,\mathcal{L}} \mathbb{N}$ *the Exel crossed product*

Then there is an exact sequence of C-algebras*

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(S, C_\varphi) \longrightarrow C(\mathbb{T}) \rtimes_{\varphi,\mathcal{L}} \mathbb{N} \longrightarrow 0$$