C*-algebras, composition operators and dynamics

Michael Jury

University of Florida

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The setting:

- \( \mathbb{D} = \{ |z| < 1 \} \subset \mathbb{C} \)
- \( \mathbb{T} = \partial \mathbb{D} = \{ |z| = 1 \} \)
- \( dm = \text{normalized Lebesgue measure on } \mathbb{T} \)
- \( \varphi : \mathbb{D} \to \mathbb{D} \text{ holomorphic (} \varphi \neq \text{const.} \) \)
- \( H^2 = \text{Hardy space, } P : L^2 \to H^2 \text{ Riesz proj.} \)
- Some operators on \( H^2 \):
  - Toeplitz operators: For \( f \in L^\infty(\mathbb{T}), g \in H^2 \),
    \[
    T_f g := P(fg)
    \]
  - Composition operators: For \( g \in H^2 \),
    \[
    C_{\varphi} g = g \circ \varphi
    \]
**General question:** What can be said about the C*-algebra generated by a set of composition operators? (or composition and Toeplitz operators)?

We seek C*-algebraic relations that obtain between Toeplitz and composition operators (and their adjoints).
One easy relation:

**Lemma**

For any \( \varphi \in \text{Hol}(\mathbb{D}) \) and \( f \in H^\infty \),

\[
C_\varphi T_f = T_{f \circ \varphi} C_\varphi
\]

**Proof:** From definitions—for any \( h \in H^2 \),

\[
(C_\varphi T_f)h = C_\varphi(fh) = (f \circ \varphi)(h \circ \varphi) = (T_{f \circ \varphi} C_\varphi)h
\]
Theorem (Bourdon-MacCluer ’07)

Let \( \varphi \) be inner. Then

\[ C_\varphi^* C_\varphi \]

is Toeplitz, with symbol

\[ F(z) = \frac{1 - |\varphi(0)|^2}{|z - \varphi(0)|^2} \]

**Proof:** Recall the Brown-Halmos criterion: \( T \) is Toeplitz if and only if \( S^* TS = T \).

Now

\[ S^* C_\varphi^* C_\varphi S = C_\varphi^* T_\varphi^* T_\varphi C_\varphi \]

\[ = C_\varphi^* C_\varphi \]

To find symbol, apply to \( C_\varphi^* C_\varphi \) to scalars... \( \square \)
Let now $\Gamma$ be a group of automorphisms of $\mathbb{D}$. What can we say about

$$\mathcal{C}_\Gamma = C^*\{C_\varphi : \varphi \in \Gamma\}$$

E.g., how is it related to C*-algebras associated to $\Gamma$?

We will need the notion of a **covariance algebra**:

Consider:

- $X$ a compact Hausdorff space
- $G$ a group of homeomorphisms of $X$

In the category of C*-algebras, these correspond to

- $C(X)$, a commutative unital C*-algebra
- $C^*(u(G))$, where $u : G \to U(\mathcal{H})$ is a unitary representation
Definition

Given:

- $X$ = compact Hausdorff space
- $G$ = group of homeomorphisms of $X$

$G$ acts on $C(X)$ via $(g \cdot f)(x) = f(g^{-1} \cdot x)$.

A **covariant representation** of $X$, $G$ is a triple $(\pi, u, \mathcal{H})$ with

- $\pi : C(X) \to B(\mathcal{H})$ a $\ast$-homomorphism
- $u : G \to U(\mathcal{H})$ a unitary representation

satisfying

$$u(g)^* \pi(f) u(g) = \pi(g \cdot f)$$

A **covariance algebra** is a $C^*$-subalgebra of $B(\mathcal{H})$ generated by a covariant representation.
Consider now $\varphi \in \Gamma \subset \text{Aut}(\mathbb{D})$. Define

$$U^*_\varphi = C_\varphi (C^*_\varphi C_\varphi)^{-1/2}$$

Suppose also that $\Gamma$ is discrete and \textit{non-elementary} (any orbit accumulates at at least three points of $\mathbb{T}$).

Using the two relations proved so far, it can be shown that

- $U_\varphi U_\psi = U_{\varphi \psi} + \text{compact}$
- $U^*_\varphi T_f U_\varphi = T_{f \circ \varphi^{-1}} + \text{compact}$
- $S \in C^*\{C_\varphi : \varphi \in \Gamma\}$

for all $\varphi, \psi \in \Gamma$ and all $f \in C(\mathbb{T})$.

It follows that

$$C^*\{C_\varphi : \varphi \in \Gamma\} = C^*\{S, U_\varphi : \varphi \in \Gamma\}$$
Theorem (J., to appear)

Let $\Gamma$ be a non-elementary Fuchsian group $\Gamma \subset \text{Aut}(\mathbb{D})$. Let

$$\mathcal{C}_\Gamma = C^*\{C_\varphi : \varphi \in \Gamma\}$$

There is an exact sequence of $C^*$-algebras

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{C}_\Gamma \longrightarrow C(\mathbb{T}) \times \Gamma \longrightarrow 0$$

The quotient $\mathcal{C}_\Gamma/\mathcal{K}$ is generated by the images in the Calkin algebra of $T_f$ and $U_\varphi^* = C_\varphi(C_\varphi^*C_\varphi)^{-1/2}$. By previous slide, the quotient is thus a covariance algebra for $(\mathbb{T}, \Gamma)$. With more work it can be shown that it is isomorphic to the full crossed product $C^*$-algebra

$$C(\mathbb{T}) \times \Gamma$$

which is “universal” among covariance algebras (any other is a quotient of this).
What are the “next” examples?

Analogy between dynamics of groups and rational functions on the Riemann sphere ("Sullivan’s dictionary") suggests replacing

\[ \Gamma = \text{Fuchsian group} = \text{group leaving } \mathbb{T} \text{ invariant} \]

with

\[ \{ \phi^n \} = \text{iterates of rational function leaving } \mathbb{T} \text{ invariant} \]

That is: replace \( C_\Gamma \) with the C*-algebra generated by \( C_\varphi \) with \( \varphi \) a finite Blaschke product. Then look for

\[ 0 \longrightarrow \mathcal{K} \longrightarrow C^*(C_\varphi) \longrightarrow ??? \longrightarrow 0 \]

where ??? is some kind of “covariance algebra” for the dynamical system \((\mathbb{T}, \varphi)\).
Consider

- $X = \text{compact Hausdorff space}$
- $\varphi : X \to X$ continuous, surjective

Suppose $\mathcal{L} : C(X) \to C(X)$ is a positive linear map satisfying

$$\mathcal{L}(f_1 \cdot (f_2 \circ \varphi)) = \mathcal{L}(f_1) \cdot f_2$$

for all $f_1, f_2 \in C(X)$ (examples later).

The **Exel crossed product** for the triple $(X, \varphi, \mathcal{L})$ is the universal C*-algebra generated by $C(X)$ and an operator $V$ satisfying

- $(f \circ \varphi)V = Vf$
- $V^*fV = \mathcal{L}(f)$
- “redundancies”...

Thinking $f = T_f, V = C_\varphi$ (mod $\mathcal{K}$) leads us to consider the operators

$$C_\varphi^* T_f C_\varphi$$
Theorem

Suppose $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ is inner and $f \in L^\infty(\mathbb{T})$. Then

$$C_\varphi^* T_f C_\varphi$$

is a Toeplitz operator.

Proof: Brown-Halmos again:

$$S^* C_\varphi^* T_f C_\varphi S = C_\varphi^* T_\varphi^* T_f T_\varphi C_\varphi$$

$$= C_\varphi^* T_f |\varphi|^2 C_\varphi \quad (\varphi \text{ analytic})$$

$$= C_\varphi^* T_f C_\varphi \quad (\varphi \text{ inner})$$

Which Toeplitz operator is it?
Given \( \varphi : \mathbb{D} \rightarrow \mathbb{D}, \alpha \in \mathbb{T}, \)

\[
\text{Re} \left( \frac{\alpha + \varphi(z)}{\alpha - \varphi(z)} \right) = \frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} = \frac{1}{|z|^2} \int \frac{1 - |z|^2}{|\zeta - z|^2} d\mu_\alpha(\zeta)
\]

The measures \( \mu_\alpha \) are the \textit{Aleksandrov-Clark measures} for \( \varphi \).

**Definition (The Aleksandrov operator)**

Let \( f \) be a Borel function on \( \mathbb{T} \). Define

\[
A_\varphi(f)(\alpha) = \int f(\zeta) \, d\mu_\alpha(\zeta)
\]
Theorem (Aleksandrov ’87)

The Aleksandrov operator

$$A_\varphi(f)(\alpha) = \int f \, d\mu_\alpha$$

extends to a well-defined, bounded operator on

- $C(\mathbb{T})$
- $L^p(\mathbb{T}), 1 \leq p \leq \infty$; also $H^p$ if $\varphi(0) = 0$
- $BMO$, $VMO$, Besov spaces...
The Cauchy transform of $\mu_\alpha$:

Assume now $\varphi(0) = 0$. Then for $|\lambda| < 1, |\alpha| = 1$,

$$A_\varphi(k_\lambda)(\alpha) = \int \frac{1}{1 - \lambda \zeta} d\mu_\alpha(\zeta) = \frac{1}{1 - \varphi(\lambda)\alpha} = C_\varphi^*(k_\lambda)(\alpha)$$

Since $A_\varphi$ is bounded on $H^2$, this proves

**Theorem (inner case: Lotto-McCarthy '93; general case–?)**

*If $\varphi(0) = 0$, then*

$$C_\varphi^* = A_\varphi$$

**Application:** we can now compute the symbol of $C_\varphi^* T_f C_\varphi ...
Theorem (J., in progress)

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be inner and $f \in L^\infty(\mathbb{T})$. Then

$$C^*_\varphi T_f C_\varphi$$

is Toeplitz, with symbol $A_\varphi(f)$.

There is an “asymptotic” version for general $\varphi$:

Definition

Say $A \in B(H^2)$ is \textbf{(weakly) asymptotically Toeplitz} if

$$\lim_{n \to \infty} S^*n AS^n$$

exists (WOT). If so, limit is a Toeplitz operator $T_g$; $g$ is called the \textit{asymptotic symbol} of $A$. 
Theorem (J., in progress)

If $\varphi : \mathbb{D} \to \mathbb{D}$, $f \in L^\infty$, then

$$C^*_\varphi T_f C_\varphi$$

is asymptotically Toeplitz, with asymptotic symbol

$$A^s_\varphi(f)(\alpha) := \int f \, d\sigma_\alpha$$

(recall $\mu_\alpha = h_\alpha m + \sigma_\alpha$).
Moreover if

$$E = \{\zeta : |\varphi(\zeta)| = 1\}$$

and $f = 0$ a.e. on $E^c$, then $C^*_\varphi T_f C_\varphi$ is Toeplitz. (Converse holds if $f \geq 0$.)
Returning to the case of finite Blaschke products, it is not hard to show that the Clark measures are given by

\[ \sigma_\alpha = \sum_{\varphi(\zeta) = \alpha} \frac{1}{|\varphi'(\zeta)|} \delta_\zeta \]

So \( A_\varphi(f)(\alpha) \) is a certain weighted average of \( f \) over the preimages of \( \alpha \).

For general inner \( \varphi \), the set

\[ \{ \zeta \in \mathbb{T} : \varphi(\zeta) = \alpha \} \]

is a carrier for \( \sigma_\alpha \) but \( \sigma_\alpha \) has atoms only at points where \( \varphi \) has an angular derivative.
Definition (the transfer operator)

- $\varphi : X \to X$ continuous, surjective, finitely valent
- $g : X \to \mathbb{R}$, continuous

Define for $f \in C(X)$

$$\mathcal{L}_g(f)(x) = \sum_{\varphi(y) = x} \exp(g(y))f(y)$$

$\mathcal{L}_g$ is called the **transfer operator** or **Perron-Frobenius-Ruelle** operator; it is bounded and completely positive on $C(X)$.

For all $f_1, f_2 \in C(X)$ we have

$$\mathcal{L}_g(f_1 \cdot (f_2 \circ \varphi)) = \mathcal{L}_g(f_1) \cdot f_2$$

In particular if $\mathcal{L}_g(1) \equiv 1$ then $\mathcal{L}_g$ is a left inverse for $C_\varphi$. 
Proto-"Perron-Frobenius"-theorem: (unital version, for simplicity) Suppose $\mathcal{L}_g(1) \equiv 1$. With suitable hypotheses on $(\varphi, g)$,

$$\mathcal{L}_g^n(f) \to c \cdot 1$$

uniformly, where $c$ is a scalar. The assignment $f \to c$ determines a probability measure $\mu$ satisfying

$$\mathcal{L}_g^* \mu = \mu$$

Under good conditions such $\mu$ is unique; it describes the asymptotic distribution of the (weighted) backward orbits of $\varphi$. 
Recall that for a rational map $\psi$ of the Riemann sphere, its \textit{Julia set} $J$ is the complement of the maximal open set on which the iterates of $\psi$ are a normal family. (For a finite Blaschke product $J \subset \mathbb{T}$; either $J = \mathbb{T}$ or $J$ is a Cantor set.)

**Theorem (Denker-Urbański ’91 (simplified version))**

Given:
- $\psi$ rational with Julia set $J$
- $g : J \to \mathbb{R}$ Hölder continuous, $\mathcal{L}_g(1) \equiv 1$

Then there is a unique measure $\mu$ with support equal to $J$ such that $\forall f \in C(J)$,

$$\mathcal{L}_g^n(f) \to \left( \int_J f \, d\mu \right) \cdot 1$$

uniformly.
Now fix a finite Blaschke product $\varphi$ and a Hölder continuous function $g : X \to \mathbb{R}$. We have a transfer operator
\[
\mathcal{L}_g(f)(z) = \sum_{\varphi(\zeta) = z} \exp(g(\zeta)) f(\zeta)
\]
defined on $C(\mathbb{T})$. For simplicity we assume $\mathcal{L}_g(1) \equiv 1$; we call such $g$ an admissible weight.

**Theorem**

Let $g$ be an admissible weight. Then there exists $h \in A(\mathbb{D})$ such that
\[
V := T_h C_\varphi
\]
satisfies
\[
V^* T_f V = T_{\mathcal{L}_g(f)}
\]
for all $f \in C(\mathbb{T})$. 
By suitably modifying Przytycki’s proof of the Denker-Urbański result, we obtain:

**Theorem**

*Given:*

- $\varphi$ a finite Blaschke product
- $g : \mathbb{T} \to \mathbb{R}$ an admissible weight

*Then for all $f \in C(\mathbb{T})$,*

$$
L^n_g(f) \to \left( \int f \, d\mu_g \right) \cdot 1
$$

*uniformly, where $\mu_g$ is the unique $\exp(g)$–conformal measure supported on the Julia set of $\varphi$.***
For a fixed admissible weight $g$, we earlier obtained a weighted composition operator

$$V := T_h C_\varphi$$

such that

$$V^* T_f V = T_{\mathcal{L}_g(f)}$$

for all $f \in C(\mathbb{T})$. Using the Perron-Frobenius result for $\mathcal{L}_g$ we obtain

**Theorem**

*Fix an admissible weight $g$ and $V$ as above. For all $f \in C(\mathbb{T})$,*

$$V^n T_f V^n \to \left( \int f \, d\mu_g \right) \cdot I$$

*in norm.*
In general some weight is needed to obtain convergence; for example let
\[ \varphi(z) = \left( \frac{3z + 1}{3 + z} \right)^2 \]
The orbit of 0 under \( \varphi \) tends radially to 1; using this one can show
\[ \| C_\varphi^n C_\varphi^n \| \to \infty. \]
Since the measure $\mu_g$ always has support equal to the Julia set of $\varphi$, the composition operator $C_\varphi$ can “see” the Julia set via iteration of the completely positive map:

**Corollary**

Let $\varphi$ be a finite Blaschke product, $g$ an admissible weight and $V = T_h C_\varphi$ as above. Let $f \in C(\mathbb{T}), f \geq 0$. Then $f$ vanishes on the Julia set $J \subset \mathbb{T}$ of $\varphi$ if and only if

$$V^* V^n T_f V^n \to 0$$

in norm.
Let \( \varphi \) be inner, with Clark measures \( \{\sigma_\alpha\} \). Consider the normalized Aleksandrov operator

\[
\widetilde{A}_\varphi(f)(\alpha) = \int f \frac{d\sigma_\alpha}{\|\sigma_\alpha\|}
\]

In the rational case this corresponds to the admissible weight

\[
g(\zeta) = -\log \left( |\varphi'(\zeta)| \cdot \|\sigma_\varphi(\zeta)\| \right)
\]
**Conjecture:** For any inner function \( \varphi \) and all \( f \in C(\mathbb{T}) \), the sequence

\[
\widetilde{A}_\varphi^n(f)
\]

converges uniformly to a scalar \( c \).

If so, \( c = \int f \, d\mu \) and \( \mu \) describes the (weighted) “asymptotic distribution” of the backwards orbits of \( \varphi \) on \( \mathbb{T} \). This would be particularly interesting when \( \varphi \) has an attracting fixed point on \( \mathbb{T} \); presumably \( \text{supp}(\mu) \subsetneq \mathbb{T} \) (a “Julia set” for \( \varphi \) on \( \mathbb{T} \)).

Where does Hölder continuity come in? (Possibly via Matheson’s smoothness theorem; which implies that \( A_\varphi \) is bounded on the Hölder classes.)
Theorem

Let

- $\varphi$ be a finite Blaschke product ($\text{deg} \varphi \geq 2$), with fixed point in $D$
- $g : \mathbb{T} \rightarrow \mathbb{R}$ Hölder continuous
- $C(\mathbb{T}) \rtimes_{\varphi, \mathcal{L}} \mathbb{N}$ the Exel crossed product

Then there is an exact sequence of $C^*$-algebras

\[
0 \xrightarrow{} \mathcal{K} \xrightarrow{} C^*(S, C\varphi) \xrightarrow{} C(\mathbb{T}) \rtimes_{\varphi, \mathcal{L}} \mathbb{N} \xrightarrow{} 0
\]