Nevanlinna-Pick interpolation in hypo-Dirichlet and related algebras

Michael Jury and Scott McCullough

University of Florida

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Setting

- $\mathcal{A}$ - uniform algebra
- $\mathcal{M}$ - maximal ideal space of $\mathcal{A}$
- $\mathcal{X}$ - Shilov boundary of $\mathcal{A}$
- $b \in \mathcal{M}$ - fixed base point
- $m$ - representing measure for $b$ on $\mathcal{X}$...so

$$\int_{\mathcal{X}} f \, dm = f(b) \quad \text{for all } f \in \mathcal{A}$$

- $\mathcal{H}^2$ - closure of $\mathcal{A}$ in $L^2(m)$; \hspace{1em} $\mathcal{H}_b^2 = \{ f \in \mathcal{H}^2 : \int f \, dm = 0 \}$
- $\mathcal{H}^\infty$ - wk-* closure of $\mathcal{A}$ in $L^\infty(m)$, or $\mathcal{H}^2 \cap L^\infty(m)$. 
**Additional assumptions**

- $N$ - space of real annihilating measures for $\mathcal{A}$ is finite dimensional
- our rep. measure $m$ lies in relative interior of rep. measures at $b$
- Gleason part containing $b$ is non-trivial

**Consequences** [60’s: Ahern-Sarason, Arens-Singer, Gamelin, Hoffman, Wermer,...]:

- each annihilating measure $\mu \in N$ has $d\mu = hdm$ with $h \in L^\infty(m)$
- orthogonal decomposition:

\[ L^2(m) = H^2 \oplus (H^2_b)^* \oplus N \]

where $N$ is complexification of $N$. 
Final assumptions

- $\mathcal{M}_b := \text{set of bpe's for } H^2$, is \textit{determining for } $\mathcal{A}$

**Definition**

Say $\mathcal{A}$ has a \underline{divisor} at $\lambda \in \mathcal{M}_b$ if there exists $\varphi_\lambda \in H^\infty$ such that:
- $\log |\varphi_\lambda| \in L^\infty$, and
- $\varphi_\lambda H^2 = H^2_\lambda$.

$\Omega := \text{set of divisors for } \mathcal{A}$.

Divisors are far from unique, however if $\varphi_\lambda, \psi_\lambda$ are both divisors at $\lambda$ then

$$\frac{\varphi_\lambda}{\psi_\lambda} \in (H^\infty)^{-1}.$$
Examples

hypo-Dirichlet algebras: $\log |A^{-1}|$ is dense in $C_\mathbb{R}(X)$ and $\text{Re} A$ has finite codimension in $C_\mathbb{R}(X)$

- E.g. for $R$ a $g$-holed planar domain, $A =$ functions holo. on $R$, cns. on $\overline{R}$
- In this case fix $b \in R$, take $m =$ harmonic measure at $b$. The annihilating measures $N$ are spanned by

$$Q_j = \frac{\partial w_j}{\partial \nu} \frac{ds}{dm}$$

where $w_j$ is harmonic measure on $j^{th}$ boundary component, $ds =$ arc length.

- Here $M_b = R$ and for each $\lambda \in R$,

$$\varphi_\lambda(z) = z - \lambda$$

is a divisor.
Examples

Finite codimension subalgebras of hypo-Dirichlets:

- Fix nonzero points $a \neq b$ in $\mathbb{D}$, put
  $$\mathcal{A} = \{ f \in A(\mathbb{D}) : f(a) = f(b) \}$$
- Pick 0 for base point, $m = \text{Lebesgue measure on } \mathbb{T}$
- $N$ is spanned by real and imaginary parts of
  $$Q = (1 - \overline{a}e^{i\theta})^{-1} - (1 - \overline{b}e^{i\theta})^{-1}$$
- $\mathcal{M} = \mathcal{M}_0$ is quotient of $\mathbb{D}$ with $a$, $b$ identified
- If $\lambda \neq a$, $b$ put
  $$\rho(\lambda) = \frac{1}{b - a} \log \left( \frac{a - \lambda}{b - \lambda} \right), \quad \varphi_{\lambda}(z) = (z - \lambda) \exp(\rho(\lambda)z)$$

Then $\varphi_{\lambda}$ is a divisor...but NO divisor at identified point $a \sim b$. 
A family of $H^2$ spaces

For each $n \in N$ form the inner product on $A$

$$\langle f, g \rangle_n = \int_X fg^* e^{-2n} dm$$

Closure is called $H^2(n)$.

- Each bpe for $H^2$ is also a bpe for $H^2(n)$, so each $H^2(n)$ is a RKHS on $\mathcal{M}_b$; write $k(s, t; n)$ for its kernel
- some redundancy: let $L \subset N$ be those $n$ for which

$$\exp(2n) = ff^*$$

for some $f \in (H^\infty)^{-1}$. If $n_1 - n_2 \in L$ then

$$f(s)k(s, t; n_1)f(t)^* = k(s, t; n_2)$$
The Pick interpolation theorem

Let $S \subset \mathcal{M}_b \cap \Omega$ be a finite set and suppose $f : S \to \mathbb{C}$. If the kernel

$$S \times S \ni (s, t) \mapsto (1 - f(s)f(t)^*)k(s, t; n)$$

is positive semi-definite for every $n \in \mathbb{N}/L$, then there exists a function $a : \Omega \to \mathbb{C}$ such that the kernel

$$S \times S \ni (s, t) \mapsto (1 - a(s)a(t)^*)k(s, t; n)$$

is positive semi-definite for every $n \in \mathbb{N}$. Further, if $\Omega$ is dense in $\mathcal{M}_b$, then there is an $a \in H^\infty$ such that $\|a\| \leq 1$ and $a|_S = f$. 
Idea of proof

- Introduce a much bigger family of (vector-valued) $H_P^2$ spaces—take $P \in M_d(L^\infty)$ and form $\mathbb{C}^d$-valued spaces

$$\|F\|_P^2 = \int_X F^*PF \, dm, \quad F \in \mathcal{A} \otimes \mathbb{C}^d$$

- The family of kernels $K_P$ of these spaces is an Agler family (closed under direct sums and one-point compressions) –existence of divisors is used here

- By the Pick theorem of [JKM ’09], can interpolate if data are positive against all $K_P$’s

- “Cyclic vector trick” - Look at repn. of $H^\infty$ on $H_P^2$, restrict to cyclic invariant subspace, show each restricted representation is equivalent to the repn. from some $k(\cdot, \cdot; n)$. . . . key is

$$\log F^*PF = g + g^* + n$$
Suppose $\lambda \in \Omega$ with divisor $\varphi_\lambda$, and let $n \in \mathbb{N}$. Let

$$\log |\varphi_\lambda| = f + f^* + 2n_\lambda$$

be the decomposition with respect to $H^2 \oplus (H^2_b)^* \oplus \mathcal{N}$. Then $n_\lambda \in \mathbb{N}$, $e^{-f} \in (H^\infty)^{-1}$, and the operator

$$U_\lambda : h \mapsto e^{-f} \varphi_\lambda h$$

is a unitary transformation from $H^2(n - n_\lambda)$ onto $H^2_\lambda(n)$. 
Clearly $e^{-f} \varphi_\lambda h$ vanishes at $\lambda$. From

$$\log |\varphi_\lambda| = f + f^* + 2n_\lambda,$$

we get

$$|\varphi_\lambda|^2 = e^{2Re f} e^{2n_\lambda}.$$

Thus,

$$\int |e^{-f} \varphi_\lambda h|^2 e^{-2n} \, dm = \int |h|^2 e^{-2(n-n_\lambda)} \, dm.$$

This says $U_\lambda$ is isometric. Surjectivity follows from invertibility of $e^{-f}$ and divisor property of $\varphi_\lambda$. 
We get a map $\lambda \rightarrow n_\lambda$ which measures how representations move when restricted to codim 1 subspaces...BUT this depends on choices of $\varphi_\lambda$...

...however, from def. of divisor we had

$$\frac{\varphi_\lambda}{\psi_\lambda} \in (H^\infty)^{-1}$$

if $\varphi_\lambda$, $\psi_\lambda$ are both divisors. From this we get that the map

$$\Omega \ni \lambda \rightarrow [n_\lambda] \in N/L$$

is independent of the choice of divisors, and is (the real part of) an Abel-Jacobi map...