Nevanlinna-Pick interpolation in hypo-Dirichlet and related algebras

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Setting

- $\mathcal{A}$ - uniform algebra
- $\mathfrak{M}$ - maximal ideal space of $\mathcal{A}$
- X - Shilov boundary of $\mathcal{A}$
- $b \in \mathfrak{M}$ - fixed base point
- m - representing measure for b on X...so

$$\int_X f\,dm = f(b) \quad \text{for all } f \in \mathcal{A}$$

- H^2 - closure of $\mathcal{A}$ in $L^2(m)$; $\quad H^2_b = \{f \in H^2 : \int f\,dm = 0\}$
- H^∞ - wk-* clousre of $\mathcal{A}$ in $L^\infty(m)$, or $H^2 \cap L^\infty(m)$.

Additional assumptions

- N - space of real annihilating measures for $\mathcal{A}$ is finite dimensional
- our rep. measure m lies in relative interior of rep. measures at b
- Gleason part containing b is non-trivial

Consequences [60's: Ahern-Sarason, Arens-Singer, Gamelin, Hoffman, Wermer,...]:

- each annihilating measure $\mu \in N$ has $d\mu = hdm$ with $h \in L^\infty(m)$
- orthogonal decomposition:

$$L^2(m) = H^2 \oplus (H^2_b)^* \oplus \mathcal{N}$$

where $\mathcal{N}$ is complexification of N.

Final assumptions

- $\mathfrak{M}_b :=$ set of bpe's for H^2, is determining for $\mathcal{A}$

Definition

Say $\mathcal{A}$ has a divisor at $\lambda \in \mathfrak{M}_b$ if there exists $\varphi_\lambda \in H^\infty$ such that:

- $\log|\varphi_\lambda| \in L^\infty$, and
- $\varphi_\lambda H^2 = H^2_\lambda$.

$\Omega :=$ set of divisors for $\mathcal{A}$.

Divisors are far from unique, however if $\varphi_\lambda, \psi_\lambda$ are both divisors at λ then

$$\frac{\varphi_\lambda}{\psi_\lambda} \in (H^\infty)^{-1}.$$

Examples

hypo-Dirichlet algebras: $\log|\mathcal{A}^{-1}|$ is dense in $C_{\mathbb{R}}(X)$ and $\mathrm{Re}\mathcal{A}$ has finite codimension in $C_{\mathbb{R}}(X)$

- E.g. for R a g-holed planar domain, $\mathcal{A} =$ functions holo. on R, cns. on $\overline{R}$
- In this case fix $b \in R$, take $m =$ harmonic measure at b. The annihilating measures N are spanned by

$$Q_j = \frac{\partial w_j}{\partial \nu} \frac{ds}{dm}$$

  where w_j is harmonic measure on j^{th} boundary component, $ds =$ arc length.
- Here $\mathfrak{M}_b = R$ and for each $\lambda \in R$,

$$\varphi_\lambda(z) = z - \lambda$$

  is a divisor.

Examples

Finite codimension subalgebras of hypo-Dirichlets:

- Fix nonzero points $a \neq b$ in $\mathbb{D}$, put $\mathcal{A} = \{f \in A(\mathbb{D}) : f(a) = f(b)\}$
- Pick 0 for base point, $m =$ Lebesgue measure on $\mathbb{T}$
- N is spanned by real and imaginary parts of

$$Q = (1 - \overline{a}e^{i\theta})^{-1} - (1 - \overline{b}e^{i\theta})^{-1}$$

- $\mathfrak{M} = \mathfrak{M}_0$ is quotient of $\mathbb{D}$ with a, b identified
- If $\lambda \neq a, b$ put

$$\rho(\lambda) = \frac{1}{b-a} \log(\frac{a-\lambda}{b-\lambda}), \quad \varphi_\lambda(z) = (z - \lambda)\exp(\rho(\lambda)z)$$

Then φ_λ is a divisor...but NO divisor at identified point $a \sim b$.

A family of H^2 spaces

For each $n \in N$ form the inner product on $\mathcal{A}$

$$\langle f, g \rangle_n = \int_X f g^* e^{-2n} \, dm$$

Closure is called $H^2(n)$.

- Each bpe for H^2 is also a bpe for $H^2(n)$, so each $H^2(n)$ is a RKHS on $\mathfrak{M}_b$; write $k(s, t; n)$ for its kernel
- some redundancy: let $L \subset N$ be those n for which

  $$\exp(2n) = ff^*$$

  for some $f \in (H^\infty)^{-1}$. If $n_1 - n_2 \in L$ then

  $$f(s)k(s, t; n_1)f(t)^* = k(s, t; n_2)$$

The Pick interpolation theorem

Let $S \subset \mathfrak{M}_b \cap \Omega$ be a finite set and suppose $f : S \to \mathbb{C}$. If the kernel

$$S \times S \ni (s,t) \mapsto (1 - f(s)f(t)^*)k(s,t;n)$$

is positive semi-definite for every $n \in N/L$, then there exists a function $a : \Omega \to \mathbb{C}$ such that the kernel

$$S \times S \ni (s,t) \mapsto (1 - a(s)a(t)^*)k(s,t;n)$$

is positive semi-definite for every $n \in N$.
Further, if Ω is dense in $\mathfrak{M}_b$, then there is an $a \in H^\infty$ such that $\|a\| \leq 1$ and $a|_S = f$.

Idea of proof

- Introduce a much bigger family of (vector-valued) H^2_P spaces–take $P \in M_d(L^\infty)$ and form $\mathbb{C}^d$-valued spaces

$$\|F\|_P^2 = \int_X F^* PF \, dm, \quad F \in \mathcal{A} \otimes \mathbb{C}^d$$

- The family of kernels K_P of these spaces is an Agler family (closed under direct sums and one-point compressions) –existence of divisors is used here
- By the Pick theorem of [JKM '09], can interpolate if data are positive against all K_P's
- "Cyclic vector trick" - Look at repn. of H^∞ on H^2_P, restrict to cyclic invariant subspace, show each restricted representation is equivalent to the repn. from some $k(\cdot, \cdot; n)$....key is

$$\log F^* PF = g + g^* + n$$

Key observation

Suppose $\lambda \in \Omega$ with divisor φ_λ, and let $n \in N$. Let

$$\log|\varphi_\lambda| = f + f^* + 2n_\lambda$$

be the decomposition with respect to $H^2 \oplus (H^2_b)^* \oplus \mathcal{N}$. Then $n_\lambda \in N$, $e^{-f} \in (H^\infty)^{-1}$, and the operator

$$U_\lambda : h \to e^{-f}\varphi_\lambda h$$

is a unitary transformation from $H^2(n - n_\lambda)$ onto $H^2_\lambda(n)$.

Clearly $e^{-f}\varphi_\lambda h$ vanishes at λ. From

$$\log|\varphi_\lambda| = f + f^* + 2n_\lambda,$$

we get

$$|\varphi_\lambda|^2 = e^{2\Re f} e^{2n_\lambda}$$

Thus,

$$\int |e^{-f}\varphi_\lambda h|^2 e^{-2n}\, dm = \int |h|^2 e^{-2(n-n_\lambda)}\, dm.$$

This says U_λ is isometric. Surjectivity follows from invertibility of e^{-f} and divisor property of φ_λ.

We get a map $\lambda \to n_\lambda$ which measures how representations move when restricted to codim 1 subspaces...BUT this depends on choices of φ_λ...

...however, from def. of divisor we had

$$\frac{\varphi_\lambda}{\psi_\lambda} \in (H^\infty)^{-1}$$

if φ_λ, ψ_λ are both divisors. From this we get that the map

$$\Omega \ni \lambda \to [n_\lambda] \in N/L$$

is independent of the choice of divisors, and is (the real part of) an Abel-Jacobi map...