THE FREDHOLM INDEX FOR ELEMENTS OF TOEPLITZ-COMPOSITION C*-ALGEBRAS

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1. Introduction

The C*-algebras generated by Toeplitz operators have been much studied, and have been shown to have a rich and interesting structure. For example, the well-known theorem of Coburn shows that the C*-algebra $\mathcal{T}$ generated by the Toeplitz operators with continuous symbol, acting on the Hardy space of the unit disk $H^2(\mathbb{D})$, contains the C*-algebra of compact operators $\mathcal{K}$ and there is an exact sequence of C*-algebras

$$0 \to \mathcal{K} \to \mathcal{T} \to C(\mathbb{T}) \to 0.$$

The quotient map takes a Toeplitz operator T_f to its symbol f. From this can be deduced the Toeplitz index theorem, which says that T_f is Fredholm if and only if f is nonvanishing on the unit circle $\mathbb{T}$, in which case the Fredholm index of T_f is equal to the winding number of f about the origin. For these and other basic results concerning Toeplitz operators we refer to the book of R. Douglas [6, Chapter 7].

In this paper we study the C*-algebras generated by $\mathcal{T}$ and a linear-fractional composition operator on the Hardy space H^2 of the unit disk. In particular we are interested in the C*-algebras $\mathcal{TC}_\varphi = C^*(S, C_\varphi)$ where S denotes multiplication by z on H^2 (the unilateral shift) and C_φ is a composition operator with $\varphi : \mathbb{D} \to \mathbb{D}$ a linear fractional map, either an automorphism or non-parabolic non-automorphism. In all of these cases the quotient C*-algebra $\mathcal{TC}_\varphi/\mathcal{K}$ has a discernible structure, and this structure can be used to attack the problem of deciding when an element of $\mathcal{TC}_\varphi$ is Fredholm, and in such cases computing its index.

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This study is motivated by several results: first, earlier work on algebras related to $\mathcal{TC}_\varphi$ by the author in [7] (which considers the C*-algebras generated by composition operators with symbols in a Fuchsian group) and by Kriete, MacCluer and Moorhouse [8] on the algebra $\mathcal{TC}_\varphi$ for certain non-automorphic φ. Second, the work of M.D. Choi and F. Latrémolière [3] on the C*-algebras $C(\overline{\mathbb{D}}) \rtimes_\varphi \mathbb{Z}$ (for disk automorphisms φ) describes the representation theory of these algebras and is closely related to the C*-algebras $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ which we obtain as quotients of $\mathcal{TC}_\varphi$. Finally, a theorem of E. Park [9] describes the Fredholm index of operators in a Toeplitz-like extension of irrational rotation C*-algebras. This extension turns out to be a special case of the extensions of $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ given by $\mathcal{TC}_\varphi$, and this index result generalizes readily to our situation. We thus have tools available to study the Fredholm theory in $\mathcal{TC}_\varphi$.

The paper is divided as follows: Section 2 treats results concerning $\mathcal{TC}_\varphi$ common to all automorphisms φ. We prove that for any automorphism φ there is an exact sequence of C*-algebras

$$0 \to \mathcal{K} \to \mathcal{TC}_\varphi \to C(\mathbb{T}) \rtimes_\varphi \mathbb{Z} \to 0.$$

Using this exact sequence and a computation of the K-theory of $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$, we obtain an integral formula for the Fredholm index in $\mathcal{TC}_\varphi$ which generalizes a result of E. Park [9] for the irrational rotation algebras. We also prove that for many automorphisms φ, the inclusion of the shift S as a generator of $\mathcal{TC}_\varphi$ is unnecessary, that is, we give a sufficient condition on φ for S to be contained in $C^*(C_\varphi)$. Finally we review some basic facts about analytic automorphisms of $\mathbb{D}$ needed in the subsequent sections.

In Sections 3 through 5 we find conditions under which elements of $\mathcal{TC}_\varphi$ are Fredholm, when φ is elliptic (of finite order), hyperbolic, or parabolic respectively. By virtue of the exact sequence obtained in Section 2, this amounts to a study of invertibility in $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$. The finite-order elliptic case, Section 3, can be handled by fairly elementary methods, and we also obtain here a more topological form of the index formula of Section 2. Section 4, the hyperbolic case, is the longest of the three sections, and requires an analysis of representations of $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ which draws on recent related work of M.D. Choi and F. Latrémolière [3]. Since parabolic automorphisms can be viewed as degenerate hyperbolics, the results of Section 5 are obtained by slight modification of the arguments of Section 4.

It should be noted here that we do not consider invertibility in $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ when φ is elliptic of infinite order. This is the case of the irrational rotation algebras $\mathcal{A}_\theta$, and because these are known to be simple C*-algebras, there are no accessible "local" criteria for invertibility as in the other cases. Indeed the problem of determining spectra of elements of $\mathcal{A}_\theta$ is extremely difficult even for very simple symbols. For example the elements of the form $z + \overline{z} + u_\varphi + u_\varphi^*$ are the so-called "almost Mathieu operators," and the spectral theory even for these self-adjoint operators is quite difficult and is the subject of a recent book by F.-P. Boca [1].

Finally, Section 6 treats the non-automorphic linear fractional maps considered by Kriete, MacCluer, and Moorhouse [8]. They obtain an exact sequence

$$0 \to \mathcal{K} \to \mathcal{TC}_\varphi \to \mathcal{D} \to 0,$$

where φ is a non-automorphic linear fractional map of $\mathbb{D}$ into itself without a boundary fixed point (so that C_φ^2 is compact) and $\mathcal{D}$ is an explicitly described type I C*-algebra. They also provide an explicit expression for the essential spectrum of elements of $\mathcal{TC}_\varphi$ (which provides a characterization of the Fredholm operators in $\mathcal{TC}_\varphi$) and from that expression we prove an index theorem. It turns out that the situation here is rather simpler than in the case of automorphisms; in fact we prove that the C*-algebra $\mathcal{D}$ is homotopic to $C(\mathbb{T})$, from which the index results follow readily.

2. General considerations

The Hardy space H^2 is defined to be the Hilbert space of functions analytic in the open unit disk $\mathbb{D}$ such that

$$\|f\|^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta$$

is finite; the square root of this quantity defines the norm on H^2. By Fatou's theorem, every $f \in H^2$ has non-tangential limits $\tilde{f}(e^{i\theta}) := \lim_{r\to 1} f(re^{i\theta})$ for almost every $\theta \in [0, 2\pi]$, and the assignment $f \to \tilde{f}$ defines a linear isometry between H^2 and a closed subspace of $L^2(\mathbb{T})$. We let $P : L^2 \to H^2$ denote the corresponding orthogonal projection. For any $f \in L^\infty(\mathbb{T})$, the Toeplitz operator with symbol f is defined on each $g \in H^2$ by

$$T_f g = P(fg).$$

Such an operator is always bounded, and $\|T_f\| = \|f\|_\infty$. For an analytic map $\varphi : \mathbb{D} \to \mathbb{D}$, the composition operator with symbol φ is defined by

$$C_\varphi g = g \circ \varphi.$$

By the Littlewood subordination principle, C_φ is always bounded. In this paper we are concerned with the C*-subalgebra of $\mathcal{B}(H^2)$ generated by the unilateral shift $S = T_z$ and C_φ, for φ either an automorphism of $\mathbb{D}$ or a non-parabolic non-automorphism. We let $\mathcal{TC}_\varphi$ denote this C*-algebra. We observe that since T_z is included as a generator, $\mathcal{TC}_\varphi$ always contains the C*-algebra of compact operators $\mathcal{K}$.

In the remainder of this section we assume φ is an automorphism of $\mathbb{D}$. We will write φ^n for the n^{th} iterate of φ and φ^{-n} for the n^{th} iterate of the inverse automorphism φ^{-1}.

The basic result concerning the structure of the C*-algebras $\mathcal{TC}_\varphi$ is the following:

Theorem 2.1. *There is an exact sequence of C*-algebras*

$$0 \to \mathcal{K} \to \mathcal{TC}_\varphi \to C(\mathbb{T}) \rtimes_\varphi \mathbb{Z} \to 0. \tag{2.1}$$

Proof. The proof is essentially the same as the proof for discrete groups of automorphisms given in [7]. Let U_φ denote the unitary operator $(C_\varphi C_\varphi^*)^{-1/2} C_\varphi$. The calculations of [7] show that the following relations hold:

- For all $f \in C(\mathbb{T})$, $U_\varphi^* T_f U_\varphi - T_{f\circ\varphi^{-1}}$ is compact.
- For all integers m, n, $U_\varphi^m U_\varphi^n - U_\varphi^{m+n}$ is compact.

It follows that, letting f and u_φ denote the images in the Calkin algebra of T_f and U_φ respectively, the quotient $\mathcal{TC}_\varphi/\mathcal{K}$ is generated by a copy of $C(\mathbb{T})$ and a unitary representation of $\mathbb{Z}$ satisfying the covariance relation $u_\varphi^* f u_\varphi = f\circ\varphi^{-1}$. Since the crossed product is the universal C*-algebra generated by these relations, the quotient $\mathcal{TC}_\varphi/\mathcal{K}$ is therefore isomorphic to a quotient of the crossed product $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$. By the same argument as in [7], since the action of φ on $\mathbb{T}$ is amenable and topologically free, in fact $\mathcal{TC}_\varphi/\mathcal{K} \cong C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$. □

Lemma 2.2. *The group $K_1(C(\mathbb{T}) \rtimes_\varphi \mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, generated by $[z]_1$ and $[u_\phi]_1$.*

Proof. This result is proved using the Pimsner-Voiculescu six-term exact sequence, and is a simple generalization of the well-known result for the irrational rotation algebras $\mathcal{A}_\theta$ [5, Example VII.5.2]. The irrational rotation algebras appear in our setting when $\varphi(z) = \lambda z$ with $\lambda = e^{2\pi i\theta}$. The Pimsner-Voiculescu exact sequence is

$$\begin{array}{ccccc} K_1(C(\mathbb{T})) & \xrightarrow{1-\alpha_*} & K_1(C(\mathbb{T})) & \xrightarrow{i_*} & K_1(C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}) \\ \delta_0 \uparrow & & & & \downarrow \delta_1 \\ K_0(C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}) & \xleftarrow{i_*} & K_0(C(\mathbb{T})) & \xleftarrow{1-\alpha_*} & K_0(C(\mathbb{T})) \end{array}$$

The group $K_1(C(\mathbb{T}))$ is generated by $[z]_1$, and by definition $\alpha_*([z]_1) = [\varphi]_1$. Since φ has winding number 1 about the origin, we have $[\varphi]_1 =$

$[z]_1$. Moreover, φ fixes the unit which generates $K_0(C(\mathbb{T}))$ and hence α_* induces the identity map on $K_*(C(\mathbb{T}))$. Thus, the maps $1-\alpha_*$ are 0 and $K_1(C(\mathbb{T})\rtimes_\varphi \mathbb{Z})$ fits into the short exact sequence

$$0 \longrightarrow K_1(C(\mathbb{T})) \xrightarrow{i_*} K_1(C(\mathbb{T})\rtimes_\varphi \mathbb{Z}) \xrightarrow{\delta_1} K_0(C(\mathbb{T})) \longrightarrow 0.$$

Since the K-groups of $C(\mathbb{T})$ are both isomorphic to $\mathbb{Z}$, the above sequence splits and we obtain $K_1(C(\mathbb{T})\rtimes_\varphi \mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$. To find the generators, we observe that the first map is induced by inclusion, so $[z]_1$ is a generator. The second map is the connecting map in the P-V exact sequence, and by construction this map takes the class $[u_\varphi]_1$ to the class of the unit in $K_0(C(\mathbb{T}))$, which is its generator. □

With the exact sequence of Theorem 2.1 and the above description of the K_1 group of $C(\mathbb{T})\rtimes_\varphi \mathbb{Z}$, we can now obtain an integral formula for the index of Fredholm operators in $\mathcal{TC}_\varphi$ which is a straightforward generalization of the formula obtained by E. Park [9] in the case of the irrational rotation algebras. In fact given Theorem 2.1 and Lemma 2.2 the arguments used in [9] go through almost verbatim, so we only sketch the details.

To begin with, we define a "smooth subalgebra" $\mathcal{A}_\varphi^\infty$ of $C(\mathbb{T})\rtimes_\varphi \mathbb{Z}$ by

$$\mathcal{A}_\varphi^\infty = \left\{ \sum_{k\in\mathbb{Z}} f_k u_\varphi^k : f_k \in C^\infty(\mathbb{T}), \{\|f_k\|\}_{k\in\mathbb{Z}} \text{ rapidly decreasing} \right\}$$

and a space of "smooth 1-forms"

$$\Omega^1(\mathcal{A}_\varphi^\infty) = \left\{ \sum_{k\in\mathbb{Z}} \omega_k u_\varphi^k : \omega_k \in \Omega^1(\mathbb{T}), \{\|\omega_k\|\}_{k\in\mathbb{Z}} \text{ rapidly decreasing} \right\}.$$

The left module action of $\mathcal{A}_\varphi^\infty$ on $\Omega^1(\mathcal{A}_\varphi^\infty)$ and the exterior derivative $d : \mathcal{A}_\varphi^\infty \to \Omega^1(\mathcal{A}_\varphi^\infty)$ are defined exactly as in [9], using the diffeomorphism φ. We then obtain a map $\nu : M_n(\Omega^1(\mathcal{A}_\varphi^\infty)) \to \Omega^1(\mathbb{T})$

$$\nu\left(\sum_{k\in\mathbb{Z}} \omega_k u_\varphi^k\right) = \mathrm{Tr}\ \omega_0$$

where Tr is the usual trace on M_n. Finally we define $\widetilde{\mathrm{Ch}} : GL(n, \mathcal{A}_\varphi^\infty) \to \Omega^1(\mathbb{T})$ by

$$\widetilde{\mathrm{Ch}}(X) = -\frac{1}{2\pi i}\nu(X^{-1}dX).$$

With these definitions, the proofs of Lemma 6 and Proposition 7 of [9] go through unchanged, so that the map $\widetilde{\mathrm{Ch}}$ induces a group homomorphism $\mathrm{Ch} : K_1(\mathcal{A}_\varphi^\infty) \to H^1_{dR}(\mathbb{T})$. Finally, for X in $GL(n, \mathcal{A}_\varphi^\infty)$ given by the series

$$X = \sum_{k\in\mathbb{Z}} f_k u_\varphi^k$$

(note that $f_k \in M_n(C^\infty(\mathbb{T}))$ here), define $T_X \in \mathcal{TC}_\varphi$ by

$$T_X = \sum_{k\in\mathbb{Z}} T_{f_k} U_\varphi^k.$$

Using Theorem 2.1 and Lemma 2.2 we obtain the following generalization of Theorem 8 of [9], with the same proof:

Theorem 2.3. *For every X in $GL(n, \mathcal{A}_\varphi^\infty)$,*

$$ind\ T_X = \int_{\mathbb{T}} Ch(X).$$

Proof. We prove that the two homomorphisms $ind, \int_{\mathbb{T}} \mathrm{Ch}(\cdot)$ agree on the generators of the group $K_1(C(\mathbb{T}) \rtimes_\varphi \mathbb{Z})$. The map ind is the index map associated to the extension (2.1) and is computed on the generators easily: $ind([u_\varphi]_1) = 0$, since by construction u_φ lifts to the unitary $U_\varphi \in \mathcal{TC}_\varphi$, and $ind([z]_1) = -1$ since z lifts to the unilateral shift, which has Fredholm index -1. On the other hand, using the definition of Ch we find

$$\mathrm{Ch}(u_\varphi) = -\frac{1}{2\pi i} 1 \cdot d1 \quad \text{and} \quad \mathrm{Ch}(z) = -\frac{1}{2\pi i}\frac{dz}{z}$$

so that $\int_{\mathbb{T}} \mathrm{Ch}(u_\varphi) = 0$ and $\int_{\mathbb{T}} \mathrm{Ch}(z) = -1$. □

For the operators appearing in the later sections of the paper a more concrete expression of the above integral formula can be obtained. If we consider an element of $\mathcal{TC}_\varphi$ of the form

$$T_X = \sum_{k=0}^{N} T_{f_k} U_\varphi^k$$

with each $f_k \in C^\infty(\mathbb{T})$, then we have $X = \sum_{k=0}^N f_k u_\varphi^k \in \mathcal{A}_\varphi^\infty$ and $X^{-1} \in \mathcal{A}_\varphi^\infty$ is expressible as the norm convergent series

$$X^{-1} = \sum_{k\in\mathbb{Z}} g_k u_\varphi^k$$

with each $g_k \in C^\infty(\mathbb{T})$. The index formula then takes the form

$$\text{ind}\ T_X = -\frac{1}{2\pi i}\sum_{k=0}^{N}\int_{\mathbb{T}} g_{-k}(\varphi^k(z))\, df_k(z). \tag{2.2}$$

From this description it is also apparent that the index homomorphism from $K_1(\mathcal{A}_\varphi^\infty) \cong K_1(C(\mathbb{T}) \rtimes_\varphi \mathbb{Z})$ to $\mathbb{C}$ given by $[X]_1 \to \int_{\mathbb{T}} \mathrm{Ch}(X)$ coincides with the homomorphism given by the character of the 1-trace constructed by Connes [4, Theorem 1.5] for (reduced) crossed products of actions of countable groups on $\mathbb{T}$. In particular, the validity of the index formula of Theorem 2.3 extends beyond the smooth subalgebra $\mathcal{A}_\varphi^\infty$ to any symbol $X \in C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ with the property that, in the natural representation π of $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ on $L^2(\mathbb{T})$, the commutator $[P, \pi(X)]$ is compact (here $P : L^2 \to H^2$ is the Riesz projection).

While we will work throughout the next three sections with the C*-algebra $\mathcal{TC}_\varphi = C^*(S, C_\varphi)$, it is worth noting that the inclusion of the unilateral shift S as a generator is often unnecessary; that is, it is often the case that $S \in C^*(C_\varphi)$. Theorem 2.6 below gives a sufficient condition on the dynamics of φ (in fact valid for any inner function φ) for this to occur. This condition is satisfied, for example, by all parabolic automorphisms and by all hyperbolic automorphisms for which the fixed points are not the endpoints of a diameter.

Lemma 2.4. *Let S denote the unilateral shift on H^2. For any $x \in \mathbb{C}$, let $A = S + xSS^*$. Then $S \in C^*(A, I)$.*

Proof. We may obviously assume $x \neq 0$. We calculate

$$|x|^2 SS^* = I + xA^* + \overline{x}A - A^*A \in C^*(A, I),$$

and hence $S = A - xSS^* \in C^*(A, I)$. □

The forward direction of the following lemma was first proved by Bourdon and MacCluer [2, Proposition 3]; we give here a different proof which also suggests the proof of the converse.

Lemma 2.5. *If φ is any inner function then*

$$C_\varphi^* C_\varphi = T_f^* T_f = T_{|f|^2},$$

where

$$f(z) = \frac{(1 - |\varphi(0)|^2)^{1/2}}{1 - \overline{\varphi(0)}z}.$$

Conversely, if $C_\varphi^* C_\varphi$ is a Toeplitz operator, then φ is inner.

Proof. As before let S denote the unilateral shift, and recall that an operator $T \in \mathcal{B}(H^2)$ is a Toeplitz operator if and only if $S^*TS = T$. Since φ is inner, T_φ is an isometry and thus

$$S^* C_\varphi^* C_\varphi S = C_\varphi^* T_\varphi^* T_\varphi C_\varphi = C_\varphi^* C_\varphi,$$

so $C_\varphi^* C_\varphi$ is Toeplitz. Let g be its symbol. To find g, we first observe that since $C_\varphi^* C_\varphi$ is positive, g must be positive, and the projection of g into H^2 is given by

$$h = T_g 1 = C_\varphi^* C_\varphi 1 = C_\varphi^* 1 = k_{\varphi(0)}(z) = (1 - \overline{\varphi(0)}z)^{-1}.$$

Since g must be real-valued, it follows that $g(z) = h(z) + \overline{h(z)} - h(0) = |f(z)|^2$. The factorization $T_{|f|^2} = T_f^* T_f$ is valid because f is analytic.

To prove the converse, the assumption that $C_\varphi^* C_\varphi$ is Toeplitz implies that $C_\varphi^* C_\varphi = S^* C_\varphi^* C_\varphi S = C_\varphi^* T_\varphi^* T_\varphi C_\varphi$, or $C_\varphi^*(T_\varphi^* T_\varphi - I)C_\varphi = 0$. We therefore have

$$\begin{aligned} 0 &= \langle C_\varphi^*(T_\varphi^* T_\varphi - I)C_\varphi 1, 1\rangle \\ &= \langle (T_\varphi^* T_\varphi - I)1, 1\rangle \\ &= \|\varphi\|_{H^2}^2 - 1. \end{aligned}$$

Thus, the L^2 norm (using normalized arc length measure) of $|\varphi|$ on the circle is 1, but since φ is a self-map of $\mathbb{D}$ its L^∞ norm on the circle is at most 1, which implies that $|\varphi(z)| = 1$ almost everywhere on $\mathbb{T}$; that is, φ is inner. □

Theorem 2.6. *Let φ be an inner function. If the orbit $(\varphi^n(0))_{n=1}^\infty$ does not lie on a diameter of $\mathbb{D}$, then $S \in C^*(C_\varphi)$.*

Proof. By Lemma 2.5 we have, for each n, $C_{\varphi^n}^* C_{\varphi^n} = T_{f_n}^* T_{f_n}$ where $f_n(z) = (1 - |\varphi^n(0)|^2)^{1/2}(1 - \overline{\varphi^n(0)}z)^{-1}$. Now $T_{f_n}^* T_{f_n}$ is invertible, with inverse $T_{\frac{1}{f_n}} T_{\frac{1}{f_n}}^*$ So $C^*(C_\varphi)$ contains

$$(1 - |\varphi^n(0)|^2) T_{\frac{1}{f_n}} T_{\frac{1}{f_n}}^* = 1 - \overline{\varphi^n(0)}S - \varphi^n(0)S + |\varphi^n(0)|^2 SS^*$$

for each n. By the hypothesis on the orbit, there exist n, m such that $a = \varphi^n(0)$ and $b = \varphi^m(0)$ are linearly independent over $\mathbb{R}$. Since $C^*(C_\varphi)$ contains an invertible operator, it contains I, so we have the operators $\overline{a}S + aS^* - |a|^2 SS^*$ and $\overline{b}S + bS^* - |b|^2 SS^*$ in $C^*(C_\varphi)$. Since a and b are linearly independent over $\mathbb{R}$, we may find a linear combination of these operators of the form $S + xSS^*$ for some scalar x. Applying the lemma, we find $S \in C^*(C_\varphi)$. □

The remainder of the paper concerns the Fredholm theory in $\mathcal{TC}_\varphi$ for various linear fractional maps φ. When φ is an automorphism, the structure of the algebras $\mathcal{TC}_\varphi$ and of the quotients $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ is dependent upon the fixed point behavior of φ. We therefore divide our analysis into three sections, for φ elliptic, hyperbolic, and parabolic respectively. This classification of automorphisms and the properties of each class are well known, we briefly recall the facts we require.

Every (nontrivial) Möbius transformation of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ fixes exactly two points (counting multiplicity). A Möbius transformation of the disk is called *hyperbolic* if it has two distinct fixed points in the closed disk $\overline{\mathbb{D}}$, these necessarily lie on the boundary $\partial\mathbb{D}$. If φ has only one fixed point in $\overline{\mathbb{D}}$, then φ is called *elliptic* if the fixed point lies in the interior $\mathbb{D}$, and *parabolic* if the fixed point lies on $\partial\mathbb{D}$ (in which case this point is a fixed point of multiplicity two for the automorphism of the Riemann sphere given by φ). Obviously, the automorphism φ^{-1} is of the same type as φ, as are the iterates φ^n of φ. Every elliptic automorphism is conjugate to a rotation of $\mathbb{D}$ about the origin. If we replace the disk by the upper half-plane, every hyperbolic automorphism is conjugate to a dilation $z \to rz$ of the upper half plane (for some nonzero real r) and every parabolic automorphism is conjugate to a translation $z \to z + r$ for some real r.

When φ is hyperbolic, the fixed points can be distinguished by the modulus of the derivative: at one fixed point λ_+, called the *attracting fixed point*, we have $|\varphi'(\lambda_+)| < 1$ and at the *repelling fixed point* λ_- we have $|\varphi'(\lambda_-)| > 1$. Moreover for each $w \in \overline{\mathbb{D}} \setminus \{\lambda_-\}$, we have

$$\lim_{n\to\infty} \varphi^n(w) = \lambda_+.$$

When φ is replaced by φ^{-1}, the roles of the attracting and repelling fixed points are reversed; and in particular we have for all $w \in \overline{\mathbb{D}} \setminus \{\lambda_+\}$

$$\lim_{n\to\infty} \varphi^{-n}(w) = \lambda_-.$$

When φ is parabolic the single fixed point λ is called *indifferent*, since it is neither attracting nor repelling in the topological sense, however we do have for all $w \in \overline{\mathbb{D}}$

$$\lim_{n\to\pm\infty} \varphi^n(w) = \lambda.$$

Finally, we recall that for any automorphism φ of the Riemann sphere $\widehat{\mathbb{C}}$, its Fatou set (that is, the largest subset of $\widehat{\mathbb{C}}$ on which the iterates of φ are a normal family) is the complement of its fixed point set. In particular the set of iterates of φ is equicontinuous on $\mathbb{T} \setminus \{\lambda_+, \lambda_-\}$ (in the hyperbolic case) and $\mathbb{T} \setminus \{\lambda\}$ (in the parabolic case).

3. Elliptic automorphisms of finite order

In this section we assume φ is an elliptic automorphism of finite order, that is, there exists a nonnegative integer q such that $\varphi^q(z) = z$. Conjugating by an automorphism if necessary, we may assume that the fixed point of φ is the origin, so that $\varphi(z) = \lambda z$ where $\lambda = e^{2\pi i(p/q)}$ with p/q in lowest terms. It is well-known that the crossed product

$C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ is isomorphic to the subalgebra of $M_q \otimes C(\mathbb{T})$ consisting of elements of the form

$$\begin{pmatrix} f_0 & f_1 & \cdots & f_{q-1} \\ f_{q-1}\circ\varphi & f_0\circ\varphi & \cdots & f_{q-2}\circ\varphi \\ \cdots & \cdots & \cdots & \cdots \\ f_1\circ\varphi^{(q-1)} & \cdots & \cdots & f_0\circ\varphi^{(q-1)} \end{pmatrix}.$$

In particular, in the regular representation π of $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ on $L^2(\mathbb{T})^{(q)}$, we have $\pi(f) = \operatorname{diag}(M_{f\circ\varphi^j})$ for all $f \in C(\mathbb{T})$ and $\pi(u_\varphi)$ is the permutation matrix with a 1 in the (i,j) entry if $j - i \equiv 1 \mod q$ and zeroes elsewhere. With this description of $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ the Fredholm theory in $\mathcal{TC}_\varphi$ is easily worked out, and we also obtain a more topological form of the index formula of Section 2.

Theorem 3.1. *Let $T = \sum_{j=1}^q T_{f_j} C_\varphi^j + K$ and suppose each $f_j \in C^1(\mathbb{T})$. Then T is Fredholm if and only if the $C(\mathbb{T})$-valued determinant*

$$h_T = \begin{vmatrix} f_0 & f_1 & \cdots & f_{q-1} \\ f_{q-1}\circ\varphi & f_0\circ\varphi & \cdots & f_{q-2}\circ\varphi \\ \cdots & \cdots & \cdots & \cdots \\ f_1\circ\varphi^{(q-1)} & \cdots & \cdots & f_0\circ\varphi^{(q-1)} \end{vmatrix}$$

is nonvanishing on $\mathbb{T}$, in which case

$$ind(T) = \frac{-1}{2\pi i q}\int_{\mathbb{T}} \frac{dh_T}{h_T}, \tag{3.1}$$

that is, $\frac{-1}{q}$ times the winding number of h_T about the origin.

Proof. By the exact sequence 2.1 the operator T is Fredholm if and only if the element $f = \sum_{j=0}^q f_j u_\varphi^j$ is invertible in $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$, and by the above description of the regular representation this is the case if and only if h_T is nonvanishing. To prove the index formula, we observe that it follows from the matrix-valued Toeplitz index theorem that if $H(z)$ is invertible in $M_q \otimes C(\mathbb{T})$ and $h(z) = \det H(z)$, the (integer) quantity

$$\chi(h) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{dh}{h} \tag{3.2}$$

is homotopy invariant. By restricting to the subalgebra $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z} \subset M_q \otimes C(\mathbb{T})$ (identified with its image in the regular representation), we obtain a homomorphism from $K_1(C(\mathbb{T}) \rtimes_\varphi \mathbb{Z})$ to $\mathbb{Z}$. To prove the index formula we must show that this map agrees with the index map of the extension 2.1. Since this K_1 group is generated by the elements $w(z) = z$ and u_φ, it suffices to check that the two maps agree on these generators. On $w(z) = z$, we have $\det \pi(w)(z) = \prod_{j=1}^q \varphi^j(z)$ so the formula (3.1) gives $(-1/q) \cdot q = -1$, which is equal to the index of the unilateral shift T_z, which is the lifting of $w(z) = z$ to $\mathcal{TC}_\varphi$. On the other hand, applied to u_φ the formula (3.1) gives 0, which agrees with the index map since u_φ lifts to the unitary U_φ. □

4. Hyperbolic automorphisms

In this section, φ is a hyperbolic automorphism with attracting fixed point λ_+ and repelling fixed point λ_-. While the problem of determining the spectrum of an arbitrary element $f \in C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ is likely intractable, we can obtain some sufficient conditions for invertibility (and hence for Fredholmness in $\mathcal{TC}_\varphi$).

4.1. **Localization in $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$.** Fix an orthonormal basis $\{\xi_n\}_{n\in\mathbb{Z}}$ for $\ell^2(\mathbb{Z})$. We let u denote the bilateral shift on this basis, that is, $u\xi_n = \xi_{n+1}$. For each $x \in \mathbb{T}$ we define a representation

$$\pi_x : C(\mathbb{T}) \rtimes_\varphi \mathbb{Z} \to \mathcal{B}(\ell^2(\mathbb{Z}))$$

as follows: let $\pi_x(u_\varphi) = u$ and for $g \in C(\mathbb{T})$ let $\pi_x(g)(\xi_n) = g(\varphi^n(x))\xi_n$.

In this subsection we collect some results about the representations π_x which we will require for our main theorems. The main result of this section is Theorem 4.4, which reduces the question of invertibility in $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ to invertibility in the "local" representations π_x (by [3, Lemma 3.9], π_x is irreducible for $x \neq \lambda^\pm$).

Lemma 4.1. *For each $f \in C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$, the function $x \to \pi_x(f)$ is continuous in the point-norm topology from $\mathbb{T} \setminus \{\lambda_+, \lambda_-\}$ to $\mathcal{B}(\ell^2(\mathbb{Z}))$.*

Proof. We prove the lemma for the "polynomial" expressions

$$f = \sum_{k=m}^{n} f_k u_\varphi^k.$$

Since these are norm dense in $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$, the general case follows by an "$\epsilon/3$" argument. Let $f \in C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ be as above, fix $x \in \mathbb{T} \setminus \{\lambda_+, \lambda_-\}$, and let $\epsilon > 0$ be given. Since each of the finitely many functions f_k is uniformly continuous, there exists $\delta_1 > 0$ such that for all $k = m, \dots n$ and all $y, z \in \mathbb{T}$ such that $|y - z| < \delta_1$,

$$|f_k(y) - f_k(z)| < \frac{\epsilon}{n - m + 1}.$$

Since the iterates of φ are equicontinuous on $\mathbb{T} \setminus \{\lambda_+, \lambda_-\}$, there exists $\delta > 0$ such that for all $|y - x| < \delta$, we have $\sup_j |\varphi^j(y) - \varphi^j(x)| < \delta_1$. Using the fact that $\pi_x(u_\varphi) = \pi_y(u_\varphi) = u$, we then have for all $|y - x| < \delta$

$$\begin{aligned} \|\pi_y(f) - \pi_x(f)\| &\leq \sum_{k=m}^{n} \|\pi_y(f_k) - \pi_x(f_k)\| \|u\| \\ &= \sum_{k=m}^{n} \|\mathrm{diag}(f_k(\varphi^j(y)) - f_k(\varphi^j(x)))\| \\ &= \sum_{k=m}^{n} \sup_{j \in \mathbb{Z}} |f_k(\varphi^j(y)) - f_k(\varphi^j(x))| \\ &< \epsilon \end{aligned} \tag{4.1}$$

where the last inequality holds because of the choices of δ and δ_1. □

Lemma 4.2. *Let x_1 and x_2 be points in $\mathbb{T} \setminus \{\lambda_+, \lambda_-\}$. The representations π_{x_1} and π_{x_2} are equivalent if and only if x_1 and x_2 lie on the same orbit of φ.*

Proof. This is a special case of [3, Lemma 3.9]. □

Lemma 4.3. *There exist closed arcs $I_1, I_2 \subset \mathbb{T}$ such that for each $x \in \mathbb{T} \setminus \{\lambda_+, \lambda_-\}$ there is a $y \in I_1 \cup I_2$ such that π_x is equivalent to π_y.*

Proof. Choose one point from each of the two open arcs of $\mathbb{T} \setminus \{\lambda_+, \lambda_-\}$, call these points x_1 and x_2. Let I_i be the closed arc with endpoints x_i and $\varphi(x_i)$. Since the translates of I_1 and I_2 cover $\mathbb{T} \setminus \{\lambda_+, \lambda_-\}$, each point of the latter set lies on the orbit of some point of $I_1 \cup I_2$. The lemma now follows from Lemma 4.2. □

Theorem 4.4. *Let $f \in C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$. Then f is invertible if and only if $\pi_x(f)$ is invertible for each $x \in \mathbb{T}$.*

Proof. The "only if" statement is immediate, so we must prove the "if" statement. Let I_1 and I_2 be the closed arcs provided by Lemma 4.3 and let $\{x_n\}$ be a countable dense subset of $I = I_1 \cup I_2$ which includes the endpoints of the intervals. We first claim that the representation

$$\pi := \pi_{\lambda_+} \oplus \pi_{\lambda_-} \oplus \{\oplus_n \pi_{x_n}\}$$

of $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ is faithful. Since the action of φ on $\mathbb{T}$ is topologically free, a nontrivial ideal in $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ must have nontrivial intersection with

$C(\mathbb{T})$. It therefore suffices to show that the restriction of π_x to $C(\mathbb{T})$ is faithful. Let $g \in C(\mathbb{T})$ be a nonzero function. If $\pi_{\lambda_\pm}(g)$ is nonzero, we are done. Otherwise, let $x \in \mathbb{T} \setminus \{\lambda_+, \lambda_-\}$ with $g(x)$ nonzero. Then g is nonvanishing in a neighborhood U of x, and by Lemma 4.3 there exists an integer k such that $\varphi_k(U) \cap I \neq \emptyset$. For any x in this intersection $\pi_x(g) \neq 0$, and by Lemma 4.1 $\pi_{x_n}(g) \neq 0$ for some x_n.

Now let $f \in C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ and suppose $\pi_x(f)$ is invertible for each $x \in \mathbb{T}$. Since the representation π is faithful, it suffices to prove that $\pi(f)$ is invertible, and to prove this it suffices to show that the sequence $\|\pi_{x_n}(f)^{-1}\|$ is bounded. By Lemma 4.1 and the continuity of the holomorphic functional calculus, for each n there is a neighborhood U_n of x_n such that

$$\|\pi_x(f)^{-1} - \pi_{x_n}(f)^{-1}\| < 1$$

for all $x \in U_n$. By compactness, I is covered by finitely many of these U_n, say $U_{n_1}, \ldots U_{n_k}$. Then for all $x \in I$ (and in particular for all x_n)

$$\|\pi_x(f)^{-1}\| \leq \max_k \{\|\pi_{x_{n_k}}(f)^{-1}\| + 1\},$$

so the sequence $\|\pi_{x_n}(f)^{-1}\|$ is bounded. □

4.2. **Fredholm criteria.** From the exact sequence 2.1 we know that an element of $\mathcal{TC}_\varphi$ of the form

$$T = \sum_{n=0}^{N} T_{f_n} U_\varphi^n$$

is Fredholm if and only if

$$f = \sum_{n=0}^{N} f_n u_\varphi^n$$

is invertible in $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$. Under the simplifying assumption that the leading coefficient f_0 is nonvanishing on $\mathbb{T}$ (in other words, invertible in $C(\mathbb{T})$), we obtain a necessary and sufficient condition for the invertibility of f. The *fixed point polynomials* of f, defined below, play a central role in this analysis.

Definition 4.5. *For a finite sum*

$$f = \sum_{n=0}^{N} f_n u_\varphi^n,$$

we define the fixed point polynomials ρ_f^+ *and* ρ_f^- *by*

$$\rho_f^+(z) = \sum_{n=0}^{N} f_n(\lambda_+) z^n, \qquad \rho_f^-(z) = \sum_{n=0}^{N} f_n(\lambda_-) z^n.$$

Before stating and proving the next theorem we introduce some notation. For each $n \in \mathbb{Z}$ we let $\mathbb{H}_n^+$ be the closed span of $\{\xi_k : k \geq n\}$, and Q_n^+ the orthogonal projection from $\ell^2(\mathbb{Z})$ onto $\mathbb{H}_n$. Similarly, we let $\mathbb{H}_n^-$ be the closed span of $\{\xi_k : k \leq n\}$ and Q_n^- the corresponding orthogonal projection. (Note that $Q_n^- = I - Q_{n+1}^+$.) We define a unitary operator $U_n : \mathbb{H}_n^+ \to H^2(\mathbb{T})$ by sending the orthonormal basis $\{\xi_n, \xi_{n+1}, \dots, \xi_{n+k}, \dots\}$ for $\mathbb{H}_n$ to the orthonormal basis $\{1, z, \dots z^k \dots\}$ for H^2. Similarly, V_n is the unitary operator taking $\mathbb{H}_n^-$ to H^2 via $\xi_k \to z^{n-k}$. Finally, for an operator $A \in \mathcal{B}(\ell^2(\mathbb{Z}))$, we write

$$[A]_{ij} = \langle A\xi_i, \xi_j \rangle$$

for the matrix elements of A.

Throughout this section we fix an element $f = \sum_{n=0}^{N} f_n u_\varphi^n \in C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$. We first examine matricial structure of the operator $\pi_x(f)$ with respect to the orthonormal basis $\{\xi_n\}$. From the definition of the representation π_x, the matrix elements of $\pi_x(f)$ are

$$[\pi_x(f)]_{ij} = \begin{cases} f_{i-j}(\varphi^j(x)) & 0 \leq i-j \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if x is a fixed point of φ then the matrix of $\pi_x(f)$ is a bi-infinite Toeplitz matrix, with $f_k(x)$ on the k^{th} subdiagonal. In particular, this shows that for $x = \lambda_\pm$ the operator $\pi_{\lambda_\pm}(f)$ is unitarily equivalent to multiplication by $\rho_f^\pm$ on $L^2(\mathbb{T})$, since the latter operators are represented, with respect to the basis $\{e^{2\pi i k\theta}\}$ of $L^2(\mathbb{T})$, by the same Toeplitz matrix as $\pi_{\lambda_\pm}(f)$. The following lemma tells us that for x not a fixed point, the matrix of $\pi_x(f)$ has an "asymptotic" Toeplitz structure.

Lemma 4.6. *Let* $f = \sum_{n=0}^{N} f_n u_\varphi^n \in C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$ *with fixed point polynomials* $\rho_f^\pm$. *For each* $\epsilon > 0$ *and each* $x \in \mathbb{T}$, *there exists an integer* N_x^+ *such that*

$$\|Q_n^+ \pi_x(f) Q_n^+ - U_n^* T_{\rho_f^+} U_n\| < \epsilon$$

for all $n \geq N_x^+$.

Similarly, for each $\epsilon > 0$ *and each* $x \in \mathbb{T}$ *there is an integer* N_x^- *such that*

$$\|Q_n^- \pi_x(f) Q_n^- - V_n^* T_{\rho_f^-}^* V_n\| < \epsilon$$

for all $n \leq N_x^-$.

Moreover, each of these differences is compact.

Proof. Since

$$\lim_{n\to\infty} f_k(\varphi^n(x)) = f_k(\lambda_+)$$

for all $x \in \mathbb{T} \setminus \{\lambda_+, \lambda_-\}$ and all $0 \leq k \leq N$, it follows that given any $\epsilon > 0$ there exists an integer N_x^+ such that

$$\max_{0\leq k\leq N} \sup_{n\geq N_x^+} |f_k(\varphi^n(x)) - f_k(\lambda_+)| < \frac{\epsilon}{N+1}. \tag{4.2}$$

Now, fix $n \geq N_x^+$. Each of the operators $Q_n^+\pi_x(f)Q_n^+$, $U_n^*T_{\rho_f^+}U_n$ has its matrix elements supported in the band $0 \leq i-j \leq N$, and within this band we have

$$[Q_n^+\pi_x(f)Q_n^+]_{ij} = f_{i-j}(\varphi^j(x)); \qquad [U_n^*T_{\rho_f^+}U_n]_{ij} = f_{i-j}(\lambda_+).$$

It follows that the difference of these operators is also supported in the band $0 \leq i-j \leq N$, and by (4.2) each of its nonzero matrix elements is less that $\epsilon/(N+1)$ in absolute value. It follows that the norm of this difference is less than ϵ. Moreover since the difference is supported in a finite band and its matrix elements tend to 0 on each subdiagonal, it is compact.

The proof for $Q_n^-\pi_x(f)Q_n^-$ is entirely analogous. □

Lemma 4.7. *If the fixed point polynomials* $\rho_f^\pm$ *do not vanish in* $\overline{\mathbb{D}}$, *then for each* $x \in \mathbb{T}$ *there exists integers* $N_x^\pm$ *such that for all* $n \geq N_x^+$, *the operator*

$$Q_n^+\pi_x(f)Q_n^+$$

is invertible, and for all $n \leq N_x^-$ *the operator*

$$Q_n^-\pi_x(f)Q_n^-$$

is invertible.

Proof. Since the fixed point polynomial ρ_f^+ does not vanish in $\overline{\mathbb{D}}$, the Toeplitz operator $T_{\rho_f^+}$ is invertible. It follows that for any n, any sufficiently small perturbation of $U_n^*T_{\rho_f^+}U_n$ will also be invertible. By the previous lemma we may choose the integer N_x^+ to make $Q_n^+\pi_x(f)Q_n^+$, for all $n \geq N_x^+$, as close in norm to $U_n^*T_{\rho_f^+}U_n$ as we like. The claimed invertibility follows, and an identical argument works for $Q_n^-\pi_x(f)Q_n^-$. □

Theorem 4.8. *Let* $f = \sum_{n=0}^{N} f_n u_\varphi^n \in C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$, *and suppose the fixed point polynomials* $\rho_f^+(z)$ *and* $\rho_f^-(z)$ *have no zeroes in the closed unit* *disk. Then for each $x \in \mathbb{T}$, $\pi_x(f)$ is invertible if and only if f_0 does not vanish on the orbit $\mathcal{O}_x$.*

Proof. Since the polynomials $\rho_f^\pm$ do not vanish in $\overline{\mathbb{D}}$, we have in particular

$$\rho_f^\pm(0) = f_0(\lambda_\pm) \neq 0,$$

that is, f_0 does not vanish at the fixed points of φ, so we assume $x \in \mathbb{T}\setminus\{\lambda_+, \lambda_-\}$ and that $f_0(y) = 0$ for some $y \in \mathcal{O}_x$; that is, $f_0(\varphi^n(x)) = 0$ for some $n \in \mathbb{Z}$.

Since f_0 is continuous, it is nonvanishing in neighborhoods of λ_+ and λ_-. Hence for all $x \in \mathbb{T} \setminus \{\lambda_+, \lambda_-\}$,

$$\lim_{n\to\pm\infty} [\pi_x(f)]_{nn} = \lim_{n\to\pm\infty} f_0(\varphi^n(x)) = f_0(\lambda_\pm) \neq 0$$

and thus $[\pi_x(f)]_{nn} = f_0(\varphi^n(x))$ is nonzero for all n sufficiently large in absolute value. It follows that $\pi_x(f)$ has only finitely many zeroes on the diagonal, and hence there is some largest m such that $[\pi_x(f)]_{mm} = 0$. By Lemma 4.7, the compression $Q_M^+\pi_x(f)Q_M^+$ is invertible for all sufficiently large $M > m$. We may now write $Q_m^+\pi_x(f)Q_m^+$ in block diagonal form as

$$Q_m^+\pi_x(f)Q_m^+ = \begin{pmatrix} A & 0 \\ K & B \end{pmatrix}$$

where $B = Q_M^+\pi_x(f)Q_M^+$ is invertible and A is an $(M-m)\times(M-m)$ lower triangular matrix whose first row is 0. Thus we can find $v \in \mathbb{H}_m^+ \ominus \mathbb{H}_M^+$ such that $Av = 0$. It follows that the vector $v \oplus (-B^{-1}Kv)$ lies in the kernel of $Q_m^+\pi_x(f)Q_m^+$, and hence $0 \oplus v \oplus (-B^{-1}Kv)$ lies in the kernel of $\pi_x(f)$ and hence $\pi_x(f)$ is not invertible.

Conversely, if f does not vanish on $\mathcal{O}_x$ then we may decompose $\pi_x(f)$ into a block lower triangular form with each diagonal block invertible. Indeed choose M according to Lemma 4.7 so that $Q_M^+\pi_x(f)Q_M^+$ is invertible, and similarly we choose $L \in \mathbb{Z}, L < M$ so that $Q_L^-\pi_x(f)Q_L^-$ is invertible. The operator $\pi_x(f)$ thus admits a block triangular decomposition

$$\pi_x(f) = \begin{pmatrix} Q_L^-\pi_x(f)Q_L^- & 0 & 0 \\ * & C & 0 \\ * & * & Q_M^+\pi_x(f)Q_M^+ \end{pmatrix},$$

where the matrix C is an $(M-L)\times(M-L)$ lower triangular matrix which does not vanish on the diagonal, and is hence invertible. Thus $\pi_x(f)$ admits a block triangular form with invertible diagonal blocks, and is hence invertible. ∎

Theorem 4.9. *Let* $T = \sum_{n=0}^{N} T_{f_n} U_\varphi^n + K$*. If the fixed point polynomials* ρ_f^+ *and* ρ_f^- *have no zeroes in the closed unit disk, then* T *is Fredholm if and only if* T_{f_0} *is Fredholm. Furthermore, in this case* $ind(T) = ind(T_{f_0})$*.*

Proof. T is Fredholm if and only if its symbol

$$\chi(T) = \sum_{n=0}^{N} f_n u_\varphi^n$$

is invertible. By Theorems 4.8 and 4.4, $\chi(T)$ is invertible if and only if f_0 does not vanish on $\mathbb{T}$. This is the case if and only if the Toeplitz operator T_{f_0} is Fredholm.

To prove the statement about the indices, we will construct an explicit homotopy between the symbols f and f_0 through invertible elements of $C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$. Since T and T_{f_0} are liftings of f and f_0 respectively, the equality of the indices follows.

To construct the homotopy, we first observe that since the fixed point polynomials have no zeroes in $\overline{\mathbb{D}}$, each is homotopic to a nonzero constant polynomial, and the homotopy may be taken through polynomials without zeroes in $\overline{\mathbb{D}}$. Indeed, for $0 \leq t \leq 1$ put

$$p_t^\pm(z) = \rho_f^\pm(tz)$$

and observe that if the zeroes of $\rho_f^\pm$ have modulus greater than one then the same is true for each $p_t^\pm$. We now define the homotopy between f and f_0 by

$$g(t) = \sum_{n=0}^{N} f_n t^n u_\varphi^n.$$

For each t, the fixed point polynomials of $g(t)$ are p_t^+ and p_t^- respectively. Thus, by Theorem 4.8, $g(t)$ is invertible for all t. By construction $g(0) = f_0$ and $g(1) = f$. □

This result can be restated as a spectral inclusion theorem:

Corollary 4.10. *Let* $f = \sum_{n=0}^{N} f_n u_\varphi^n \in C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$*, with* φ *hyperbolic. Then*

$$\sigma(f) \subseteq f_0(\mathbb{T}) \cup \rho_f^+(\overline{\mathbb{D}}) \cup \rho_f^-(\overline{\mathbb{D}}).$$

Proof. Suppose $\lambda \in \mathbb{C}$ does not belong to the above union of sets. Then the leading coefficient $f_0 - \lambda$ of $f - \lambda$ has no zeroes on the circle, and since the fixed point polynomials of $f - \lambda$ are $\rho_f^\pm - \lambda$, these have no zeroes in the closed disk. Therefore $f - \lambda$ is invertible. □

In general this inclusion is strict (in the example following Theorem 4.12, we have $0 \in \rho_f^+(\overline{\mathbb{D}}) \cup \rho_f^-(\overline{\mathbb{D}})$ but f is invertible). A more refined spectral inclusion will follow from that theorem.

Under the assumption that f_0 is invertible, but now allowing zeroes of the fixed point polynomials in $\mathbb{D}$, we characterize the invertibility of $f = \sum f_n u_\varphi^n$ in terms of the invertibility of an additional matrix-valued function on $\mathbb{T}$. Again, we fix $x \in \mathbb{T}$ and examine invertibility in the representation π_x.

First, we observe that since ρ_f^+ is an analytic polynomial with d zeroes in $\mathbb{D}$, the Toeplitz operator $T_{\rho_f^+}$ has a d-dimensional cokernel and trivial kernel (and hence Fredholm index $-d$). By Lemma 4.6 the same will be true of $Q_M^+\pi_x(f)Q_M^+$ for all sufficiently large M.

We now fix some notation: for a fixed M as above, put

$$F_+(x) = Q_M^+\pi_x(f)Q_M^+, \qquad F_-(x) = Q_M^-\pi_x(f)Q_M^-, \qquad K(x) = Q_M^+\pi_x(f)Q_M^-$$

and note that $K(x)$ is finite rank and hence compact.

With this notation $\pi_x(f)$ has the block triangular form

$$\pi_x(f) = \begin{pmatrix} F_-(x) & 0 \\ K(x) & F_+(x) \end{pmatrix}.$$

Observe also that by compactness, we may choose a single M so that this decomposition is valid for all $x \in I_1 \cup I_2$ (see subsection 4.1).

Lemma 4.11. *Suppose the fixed point polynomials ρ_f^+, ρ_f^- are nonvanishing on $\partial\mathbb{D}$ and have d_+ and d_- zeroes in $\mathbb{D}$, respectively. Then for each $x \in \partial\mathbb{D}\backslash\{\lambda_+, \lambda_-\}$, the operator $\pi_x(f)$ is Fredholm and ind$(\pi_x(f)) = d_- - d_+$. Furthermore, if f_0 does not vanish on $\mathcal{O}_x$ then* $\ker F_-(x)^* = \{0\}$.

Proof. By Lemma 4.6 and the above triangular form of $\pi_x(f)$, we see that $\pi_x(f)$ is unitarily equivalent to a compact perturbation of the operator

$$\begin{pmatrix} T_{\rho_f^-}^* & 0 \\ 0 & T_{\rho_f^+} \end{pmatrix}$$

on $H^2 \oplus H^2$. The assumption on the zeroes of $\rho_f^\pm$ then implies that the operators $T_{\rho_f^-}^*$ and $T_{\rho_f^+}$ are Fredholm of index d_- and $-d_+$, respectively. This proves the first statement.

For the second statement, we consider a further block decomposition of $F_-(x)$. By Lemma 4.6 we can choose $L < M$ so that $Q_L^-\pi_x(f)Q_L^-$ is unitarily equivalent to an arbitrarily small perturbation of $T_{\rho_f^-}^*$. Since this latter operator has trivial cokernel, we can choose L so that $Q_L^-\pi_x(f)Q_L^-$

also has trivial cokernel. We then obtain a block triangular form for $F_-(x)$:

$$F_-(x) = \begin{pmatrix} Q_L^-\pi_x(f)Q_L^- & 0 \\ X & C \end{pmatrix}$$

where X is finite rank and C is an $(M-L)\times(M-L)$ lower triangular matrix which does not vanish on the diagonal, and is hence invertible. We now consider $\ker F_-(x)^*$: suppose v, w are vectors such that

$$\begin{pmatrix} (Q_L^-\pi_x(f)Q_L^-)^* & X^* \\ 0 & C^* \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then $w = 0$ since C^* is invertible, and therefore $v = 0$ since $(Q_L^-\pi_x(f)Q_L^-)^*$ has trivial kernel. Thus $\ker F_-(x)^*$ is trivial. □

We now return to the triangular form of $\pi_x(f)$. Assume now that the fixed point polynomials each have d zeroes in $\mathbb{D}$ (and are nonvanishing on $\partial\mathbb{D}$) and that f_0 does not vanish on $\mathcal{O}_x$. In the block decomposition of $\pi_x(f)$, the operator $F_+(x)$ has trivial kernel and, by the previous lemma, $F_-(x)$ has trivial cokernel. Since these operators are also Fredholm, they have closed range and we can define two projection-valued functions on $I_1 \cup I_2$ by

$$P_+(x) \equiv 1 - F_+(F_+^*F_+)^{-1}F_+^*; \qquad P_-(x) \equiv 1 - F_-^*(F_-F_-^*)^{-1}F_-$$

(here we have suppressed the x-dependence on the right-hand side). In other words, $P_+(x)$ is the projection onto the cokernel of $F_+(x)$, and $P_-(x)$ is the projection onto the kernel of $F_-(x)$. We are now ready to state and prove our main result about the invertibility of $\pi_x(f)$.

Theorem 4.12. *Let $f = \sum f_n u_\varphi^n$ with f_0 nonvanishing and suppose the fixed point polynomials $\rho_f^\pm$ each have d zeroes in $\mathbb{D}$ (and are nonvanishing on the boundary). Then f is invertible if and only if for all $x \in \partial\mathbb{D} \setminus \{\lambda_+, \lambda_-\}$, the matrix-valued function*

$$D(x) \equiv P_+(x)K(x)P_-(x)$$

is invertible from $\ker F_-(x)$ *to* $\ker F_+(x)^*$.

Proof. By the assumptions on $\rho_f^\pm$, Lemma 4.11 shows that $\pi_x(f)$ is Fredholm of index 0. It is therefore invertible if and only if it has trivial kernel. Considering the block triangular form of $\pi_x(f)$, suppose a nonzero vector $v \oplus w$ lies in the kernel of $\pi_x(f)$. Then, since $\ker F_+$ is trivial, $v \in \ker F_-$ must be nonzero, and we must have $Kv \in \operatorname{ran} F_+$. In other words, v must be a nonzero vector in the kernel of $P_+(x)K(x)P_-(x)$. Conversely, given any such v we may solve for w such that $Kv + F_+(x)w = 0$, whence $v \oplus w$ lies in $\ker\pi_x(f)$. Thus $\pi_x(f)$

is invertible if and only if $D(x)$ is invertible. The theorem then follows from Theorem 4.4. □

Exactly as in the case of Theorem 4.9, this result can be stated as a spectral inclusion. To account for the necessary condition that the fixed point polynomials have the same number of zeroes in $\mathbb{D}$, let $W_\rho \subset \mathbb{C}$ denote the set of all points w such that either 1) ρ_f^+ and ρ_f^- have different winding numbers about w, or 2) $w \in \rho_f^+(\mathbb{T}) \cup \rho_f^-(\mathbb{T})$ (that is, at least one of the winding numbers is undefined). It is not hard to see that W_ρ is compact. Finally, let W_D denote the subset of the complement of $f_0(\mathbb{T}) \cup W_\rho$ such that the matrix-valued function D constructed from$f - w$ in the manner above is not invertible.

Corollary 4.13. *Let* $f = \sum_{n=0}^{N} f_n u_\varphi^n \in C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$. *Then*

$$\sigma(f) \subseteq f_0(\mathbb{T}) \cup W_\rho \cup W_D.$$

Unlike in Theorem 4.8, it is possible to have the symbol of a Fredholm operator satisfy the hypotheses of Theorem 4.12 but $ind(T) \neq ind(T_{f_0})$:

Example: Let $0 < r < 1$ and put

$$T = rT_z + U_\varphi.$$

Then $f = \chi(T) = rz + u_\varphi$ is invertible, since u_φ is unitary and $\|f - u_\varphi\| = \|rz\|_\infty < 1$. We have $f_0(z) = rz$ nonvanishing on the unit circle, and the fixed point polynomials are

$$\rho_f^+(w) = r\lambda_+ + w, \qquad \rho_f^-(w) = r\lambda_- + w;$$

each of which has one zero in $\mathbb{D}$ and is nonvanishing on the unit circle. Thus f satisfies the hypotheses of Theorem 4.12. However, since $\|f - u_\varphi\| < 1$, the symbol f is homotopic to u_φ. It follows that $ind(T) = ind(U_\varphi) = 0$ while $ind(T_{f_0}) = ind(T_{rz}) = -1$.

If a Toeplitz operator T_g is Fredholm and $ind(T_g) = 0$, then T_g is invertible. We do not know if the corresponding statement is true in $\mathcal{TC}_\varphi$, even for the Fredholm operators described by the theorems in this section:

Question: If $T \in \mathcal{TC}_\varphi$ is Fredholm and $ind(T) = 0$, is T invertible?

5. Parabolic automorphisms

For the present purposes, it is best to view a parabolic automorphism as a degenerate hyperbolic automorphism, for which the attracting and repelling fixed points coalesce. Taking this point of view, the main theorems of the previous section are easily modified to apply to the parabolic case, with essentially the same proofs. We will therefore only sketch the proofs in this section, taking care to highlight the necessary changes.

We let φ now denote a parabolic automorphism of $\mathbb{D}$, with fixed point $\lambda \in \mathbb{T}$. As before we consider operators of the form

$$f = \sum_{n=0}^{N} f_n u_\varphi^n$$

and define the *fixed point polynomial* ρ_f by

$$\rho_f(z) = \sum_{n=0}^{N} f_n(\lambda) z^n.$$

In the hyperbolic case, the fixed points λ_+ and λ_- are attracting for φ and φ^{-1}, respectively. Now that φ is parabolic, the point λ is attracting for both φ and φ^{-1}, in the sense that for any $x \in \mathbb{T}$, $x \neq \lambda$, we have

$$\lim_{n\to\infty} \varphi^n(x) = \lim_{n\to\infty} \varphi^{-n}(x) = \lambda.$$

With this fact in mind, the obvious modifications to the proof of Lemma 4.6 show that the conclusion of that lemma holds for parabolic φ if we replace both ρ_f^+ and ρ_f^- with ρ_f. The statement of Lemma 4.7 is then also valid in the parabolic case. With these substitute lemmas in hand, the following theorem is proved in the same way as Theorem 4.8, *mutatis mutandis.*

Theorem 5.1. *For a parabolic automorphism φ, let $f = \sum_{n=0}^{N} f_n u_\varphi^n \in C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$, and suppose the fixed point polynomial $\rho_f(z)$ has no zeroes in the closed unit disk. Then for each $x \in \mathbb{T}$, $\pi_x(f)$ is invertible if and only if f_0 does not vanish on the orbit $\mathcal{O}_x$.*

In turn, using this result, the following analog of Theorem 4.9 is seen to be true, by letting $\rho_f^+ = \rho_f^- = \rho_f$ in its proof:

Theorem 5.2. *Let $T = \sum_{n=0}^{N} T_{f_n} U_\varphi^n + K$. If the fixed point polynomial ρ_f has no zeroes in the closed unit disk, then T is Fredholm if and only if T_{f_0} is Fredholm. Furthermore, in this case ind$(T) =$ ind(T_{f_0}).*

Continuing as in the discussion following Theorem 4.9 for each $x \in \mathbb{T}$, $x \neq \lambda$, we have the block triangular form for the matrix

$$\pi_x(f) = \begin{pmatrix} F_-(x) & 0 \\ K(x) & F_+(x) \end{pmatrix}.$$

From this follows a simpler version of Lemma 4.11:

Lemma 5.3. *Suppose the fixed point polynomial ρ_f is nonvanishing on $\partial\mathbb{D}$. Then for each $x \in \partial\mathbb{D} \setminus \{\lambda\}$, the operator $\pi_x(f)$ is Fredholm and ind$(\pi_x(f)) = 0$. Furthermore, if f_0 does not vanish on $\mathcal{O}_x$ then* $\ker F_-(x)^* = \{0\}$.

Proof. Using the block triangular form of $\pi_x(f)$ and the parabolic version of Lemma 4.6, we find that $\pi_x(f)$ is unitarily equivalent to a compact perturbation of

$$\begin{pmatrix} T^*_{\rho_f} & 0 \\ 0 & T_{\rho_f} \end{pmatrix}$$

on $H^2 \oplus H^2$. This operator is Fredholm of index 0. The second part of the lemma is proved as in the hyperbolic case. □

Finally, since we have now shown that $\pi_x(f)$ is always Fredholm of index 0 when φ is parabolic, using the same notation as in Theorem 4.12 we have a parabolic version of that theorem and its corresponding spectral inclusion:

Theorem 5.4. *Let $f = \sum f_n u_\varphi^n$ with f_0 nonvanishing and suppose the fixed point polynomial ρ_f is nonvanishing on $\mathbb{T}$. Then f is invertible if and only if for all $x \in \mathbb{T}$, $x \neq \lambda$, the matrix-valued function*

$$D(x) \equiv P_+(x)K(x)P_-(x)$$

is invertible from $\ker F_-(x)$ *to* $\ker F_+(x)^*$.

Since we have only a single fixed point polynomial, the corresponding spectral inclusion theorem is somewhat simpler in the parabolic case. We here let W_D denote those points w of the complement of $f_0(\mathbb{T}) \cup \rho_f(\mathbb{T})$ for which the function D associated to $f - w$ is not invertible.

Corollary 5.5. *Let $f = \sum f_n u_\varphi^n \in C(\mathbb{T}) \rtimes_\varphi \mathbb{Z}$. Then*

$$\sigma(f) \subseteq f_0(\mathbb{T}) \cup \rho_f(\mathbb{T}) \cup W_D.$$

6. Non-automorphic linear fractional maps

In this section we consider the extension

$$0 \to \mathcal{K} \to \mathcal{TC}_\varphi \to \mathcal{D} \to 0 \tag{6.1}$$

constructed by Kriete, MacCluer and Moorhouse [8]. We first recall the notation and results of [8]. Throughout this section, φ is a non-automorphic linear fractional map $\varphi : \mathbb{D} \to \mathbb{D}$ such that $\varphi(\zeta) = \eta$ for some $\zeta \neq \eta \in \partial\mathbb{D}$. The compact Hausdorff space Λ consists of the disjoint union of $\partial\mathbb{D}$ and the closed interval $[0, 1]$ with the points ζ, η

and 0 identified, so that Λ is homeomorphic to a figure eight with a closed interval attached by one endpoint to the vertex, which we denote $\mathbf{p}$.

The C*-algebra $\mathcal{D}$ consists of all functions $b : \Lambda \to M_2(\mathbb{C})$ satisfying the following conditions: there exist $w \in C(\partial\mathbb{D})$ and $G \in M_2(C([0,1]))$ such that

- For $\lambda \in C(\partial\mathbb{D}) \setminus \{\zeta, \eta\}$,
$$b(\lambda) = \begin{pmatrix} w(\lambda) & 0 \\ 0 & w(\lambda) \end{pmatrix}.$$
- For $\lambda \in [0,1]$, $b(\lambda) = G(\lambda)$.
- At the vertex $\mathbf{p}$,
$$b(\mathbf{p}) = G(0) = \begin{pmatrix} w(\zeta) & 0 \\ 0 & w(\eta) \end{pmatrix}.$$

The set of such functions is a C*-algebra when equipped with the supremum norm, pointwise operations, and the obvious involution. We will denote elements of $\mathcal{D}$ by (w, G) where w and G are as above. Note that in this description products and adjoints are taken entrywise: $(w_1, G_1) \cdot (w_2, G_2) = (w_1 w_2, G_1 G_2)$, and $(w, G)^* = (\overline{w}, G^*)$.

For the next theorem, we recall two definitions: first, if A and B are C*-algebras and $\rho, \sigma : A \to B$ are *-homomorphisms, we say that ρ and σ are *homotopic* if there exists a path of *-homomorphisms $\rho_t : A \to B$, $0 \leq t \leq 1$, with $\rho = \rho_0$ and $\sigma = \rho_1$. The path is required to be continuous in the sense that for each $a \in A$, the map $t \to \rho_t(a)$ is continuous from $[0,1]$ to B (equipped with the norm topology). Secondly, a pair of C*-algebras A and B are called *homotopy equivalent* if there exist *-homomorphisms $\theta : A \to B$ and $\psi : B \to A$ such that the compositions $\theta \circ \psi$ and $\psi \circ \theta$ are homotopic to id_A and id_B respectively.

Theorem 6.1. *The C*-algebras $\mathcal{D}$ and $C(\mathbb{T})$ are homotopy equivalent.*

Proof. For $w \in C(\mathbb{T})$, let W be the $M_2(\mathbb{C})$-valued function on $[0,1]$ which is identically equal to $\mathrm{diag}(w(\zeta), w(\eta))$. We define *-homomorphisms $\theta : C(\mathbb{T}) \to \mathcal{D}$ and $\psi : \mathcal{D} \to C(\mathbb{T})$ by

$$\theta(w) = (w, W), \qquad \psi(w, G) = w.$$

Obviously $\psi \circ \theta = \mathrm{id}_{C(\mathbb{T})}$. We have

$$\theta \circ \psi(w, G) = (w, W).$$

Note that $W(\lambda) = G(0)$ for all $\lambda \in [0,1]$. For $t \in [0,1]$ define the *-homomorphism $\rho_t : \mathcal{D} \to \mathcal{D}$ by

$$\rho_t(w, G) = (w, G_t)$$

where $G_t(\lambda) = G(t\lambda)$. Thus ρ_t is a homotopy between $\rho_1 = \mathrm{id}_{\mathcal{D}}$ and $\rho_0 = \theta \circ \psi$. $\square$

Theorem 6.2. *Let $T = T_w + C + K \in \mathcal{TC}_\varphi$. If T is Fredholm then T_w is Fredholm and $ind(T) = ind(T_w)$.*

Proof. Since the image of T in $\mathcal{D}$ is of the form (w, G), T is Fredholm if and only if w and G are pointwise invertible, so if T is Fredholm then w is invertible and the Toeplitz operator T_w is Fredholm. To prove the index statement, it suffices to prove that the images of T and T_w are homotopic through invertibles in $\mathcal{D}$. Using the homotopy (w, G_t) of the previous theorem, we see that (w, G) is homotopic to (w, W), and since G is assumed invertible each G_t is invertible, and hence (w, G_t) is invertible in $\mathcal{D}$ for all $t \in [0, 1]$. Finally, the element (w, W) is by construction the image of T_w in $\mathcal{D}$. $\square$

Combining the previous two theorems we get the following corollary:

Corollary 6.3. *The group $Ext(\mathcal{D})$ is isomorphic to $\mathbb{Z}$ and is generated by the class of the extension (6.1).*

Proof. The first statement follows from Theorem 6.1 and the homotopy invariance of the Ext functor. Since $Ext(\mathbb{T}) \cong \mathbb{Z}$ is generated by the class of an extension for which the function $w(z) = z$ lifts to an operator of index ± 1, the group $Ext(\mathcal{D})$ will be generated by an extension for which the element $\theta(w) = (w, W)$ lifts to an operator of index ± 1. By Theorem 6.2 the extension (6.1) has this property ((w, W) lifts to the unilateral shift). $\square$

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