Schur class functions on the unit ball in $\mathbb{C}^n$

Michael Jury

University of Florida

October 24, 2009

Theorem

Let f be holomorphic in the disk. TFAE:

Theorem

Let f be holomorphic in the disk. TFAE:

1) $|f(z)| \leq 1$ *for all* $z \in \mathbb{D}$.

Theorem

Let f be holomorphic in the disk. TFAE:

1) $|f(z)| \leq 1$ *for all* $z \in \mathbb{D}$.

2) *The Hermitian kernel*

$$\frac{1 - f(z)\overline{f(w)}}{1 - z\overline{w}}$$

is positive semidefinite.

Theorem

Let f be holomorphic in the disk. TFAE:

1) $|f(z)| \leq 1$ *for all* $z \in \mathbb{D}$.
2) *The Hermitian kernel*

$$\frac{1 - f(z)\overline{f(w)}}{1 - z\overline{w}}$$

   is positive semidefinite.

3) $\|f(T)\| \leq 1$ *for all linear operators* T *on Hilbert space with* $\|T\| < 1$.

Theorem

Let f be holomorphic in the disk. TFAE:

1) $|f(z)| \leq 1$ *for all* $z \in \mathbb{D}$.
2) *The Hermitian kernel*

$$\frac{1 - f(z)\overline{f(w)}}{1 - z\overline{w}}$$

   is positive semidefinite.

3) $\|f(T)\| \leq 1$ *for all linear operators* T *on Hilbert space with* $\|T\| < 1$.

(2) and (3) each imply (1) trivially.

Theorem

Let f be holomorphic in the disk. TFAE:

1) $|f(z)| \leq 1$ *for all* $z \in \mathbb{D}$.

2) *The Hermitian kernel*

$$\frac{1 - f(z)\overline{f(w)}}{1 - z\overline{w}}$$

is positive semidefinite.

3) $\|f(T)\| \leq 1$ *for all linear operators* T *on Hilbert space with* $\|T\| < 1$.

(2) and (3) each imply (1) trivially.
(1) implies (2) since f multiplies H^2 into itself contractively.

Theorem

Let f be holomorphic in the disk. TFAE:

1) $|f(z)| \leq 1$ *for all* $z \in \mathbb{D}$.

2) *The Hermitian kernel*

$$\frac{1 - f(z)\overline{f(w)}}{1 - z\overline{w}}$$

is positive semidefinite.

3) $\|f(T)\| \leq 1$ *for all linear operators* T *on Hilbert space with* $\|T\| < 1$.

(2) and (3) each imply (1) trivially.
(1) implies (2) since f multiplies H^2 into itself contractively.
(1) implies (3) is von Neumann's inequality (von Neumann, 1949).

A <u>(strict) row contraction</u> is a n-tuple of commuting operators

$$T = (T_1, \dots, T_n)$$

such that

$$r^2 I - T_1 T_1^* - \cdots - T_n T_n^* \geq 0$$

for some $r < 1$.

A <u>(strict) row contraction</u> is a n-tuple of commuting operators

$$T = (T_1, \ldots, T_n)$$

such that

$$r^2 I - T_1 T_1^* - \cdots - T_n T_n^* \geq 0$$

for some $r < 1$.

Theorem (Drury, 1978)

Let f be holomorphic in the ball $\mathbb{B}^n$. TFAE:

1') ~~$|f(z)| \leq 1$ *for all* $z \in \mathbb{B}^n$.~~

2') *The Hermitian kernel*

$$\frac{1 - f(z)\overline{f(w)}}{1 - \langle z, w \rangle}$$

is positive semidefinite.

3') $\|f(T)\| \leq 1$ *for all strict row contractions* T.

Theorem (Drury, 1978)

Let f be holomorphic in the ball $\mathbb{B}^n$. TFAE:

1') ~~$|f(z)| \leq 1$ for all $z \in \mathbb{B}^n$.~~

2') *The Hermitian kernel*

$$\frac{1 - f(z)\overline{f(w)}}{1 - \langle z, w \rangle}$$

is positive semidefinite.

3') $\|f(T)\| \leq 1$ *for all strict row contractions* T.

Theorem (Drury, 1978)

Let f be holomorphic in the ball $\mathbb{B}^n$. TFAE:

1') ~~$|f(z)| \leq 1$ *for all* $z \in \mathbb{B}^n$.~~

2') *The Hermitian kernel*

$$\frac{1 - f(z)\overline{f(w)}}{1 - \langle z, w \rangle}$$

is positive semidefinite.

3') $\|f(T)\| \leq 1$ *for all strict row contractions* T.

Trivially both (2') and (3') imply (1') as before, but now (1') implies neither of the others.

Theorem (Drury, 1978)

Let f be holomorphic in the ball $\mathbb{B}^n$. TFAE:

1') ~~$|f(z)| \leq 1$ *for all* $z \in \mathbb{B}^n$.~~

2') *The Hermitian kernel*

$$\frac{1 - f(z)\overline{f(w)}}{1 - \langle z, w \rangle}$$

is positive semidefinite.

3') $\|f(T)\| \leq 1$ *for all strict row contractions* T.

Trivially both (2') and (3') imply (1') as before, but now (1') implies neither of the others.

Definition

Say f belongs to the <u>Schur class</u> if it satisfies the conditions of the theorem.

Why are (2') and (3') equivalent?

Why are (2') and (3') equivalent?

(2') implies (3'): Write the kernel as a "sum of squares" (Aronszajn/Bergman):

$$\frac{1 - f(z)\overline{f(w)}}{1 - \langle z, w \rangle} = \sum_j g_j(z)\overline{g_j(w)}$$

Why are (2') and (3') equivalent?

(2') implies (3'): Write the kernel as a "sum of squares" (Aronszajn/Bergman):

$$\frac{1 - f(z)\overline{f(w)}}{1 - \langle z, w\rangle} = \sum_j g_j(z)\overline{g_j(w)}$$

Now

$$1 - f(z)\overline{f(w)} = \sum_j g_j(z)(1 - \langle z, w\rangle)\overline{g_j(w)}$$

Why are (2') and (3') equivalent?

(2') implies (3'): Write the kernel as a "sum of squares" (Aronszajn/Bergman):

$$\frac{1 - f(z)\overline{f(w)}}{1 - \langle z, w \rangle} = \sum_j g_j(z)\overline{g_j(w)}$$

Now

$$1 - f(z)\overline{f(w)} = \sum_j g_j(z)(1 - \langle z, w \rangle)\overline{g_j(w)}$$

Functional calculus:

$$I - f(T)f(T)^* = \sum_j g_j(T)\left[1 - \sum T_i T_i^*\right] g_j(T)^*$$
$$\geq 0$$

Why are (2') and (3') equivalent?

(2') implies (3'): Write the kernel as a "sum of squares" (Aronszajn/Bergman):

$$\frac{1 - f(z)\overline{f(w)}}{1 - \langle z, w \rangle} = \sum_j g_j(z)\overline{g_j(w)}$$

Now

$$1 - f(z)\overline{f(w)} = \sum_j g_j(z)(1 - \langle z, w \rangle)\overline{g_j(w)}$$

Functional calculus:

$$I - f(T)f(T)^* = \sum_j g_j(T)\left[1 - \sum T_i T_i^*\right] g_j(T)^*$$
$$\geq 0$$

(3') implies (2'): Hahn-Banach theorem and GNS construction.

Q: What's the point?

Q: What's the point?

A: Being a "sum of squares" is a much more tractable condition than mere pointwise positivity (cf. Hilbert's seventeenth problem).

Q: What's the point?

A: Being a "sum of squares" is a much more tractable condition than mere pointwise positivity (cf. Hilbert's seventeenth problem).

Example:

Theorem (Uniqueness in the Schwarz lemma in $\mathbb{B}^n$)

If f is Schur class, $f(0) = 0$ and $Df(0) = \zeta$ with $|\zeta| = 1$, then $f(z) = \sum z_j \zeta_j$.

Q: What's the point?

A: Being a "sum of squares" is a much more tractable condition than mere pointwise positivity (cf. Hilbert's seventeenth problem).

Example:

Theorem (Uniqueness in the Schwarz lemma in $\mathbb{B}^n$)

If f is Schur class, $f(0) = 0$ and $Df(0) = \zeta$ with $|\zeta| = 1$, then $f(z) = \sum z_j \zeta_j$.

(Utterly false assuming only $\|f\|_\infty \leq 1$.)

The bidisk $\mathbb{D}^2$:

The bidisk $\mathbb{D}^2$:

Theorem (Cole-Wermer)

Let $q(z_1, z_2)$ be a rational inner function in $\mathbb{D}^2$. Then there exist rational functions $A_1, \dots A_m$ and $B_1, \dots B_n$ such that

$$1 - q(z)\overline{q(w)} = (1 - z_1\overline{w_1}) \sum A_i(z)\overline{A_i(w)} + (1 - z_2\overline{w_2}) \sum B_j(z)\overline{B_j(w)}.$$

The bidisk $\mathbb{D}^2$:

Theorem (Cole-Wermer)

Let $q(z_1, z_2)$ be a rational inner function in $\mathbb{D}^2$. Then there exist rational functions $A_1, \ldots A_m$ and $B_1, \ldots B_n$ such that

$$1-q(z)\overline{q(w)} = (1-z_1\overline{w_1})\sum A_i(z)\overline{A_i(w)}+(1-z_2\overline{w_2})\sum B_j(z)\overline{B_j(w)}.$$

This is essentially equivalent to Ando's inequality: if f is analytic in $\mathbb{D}^2$ and $\|f\|_\infty \leq 1$, then

$$\|f(T_1, T_2)\| \leq 1 \text{ for all } \|T_i\| < 1. \qquad (*)$$

The bidisk $\mathbb{D}^2$:

Theorem (Cole-Wermer)

Let $q(z_1, z_2)$ be a rational inner function in $\mathbb{D}^2$. Then there exist rational functions $A_1, \dots A_m$ and $B_1, \dots B_n$ such that

$$1-q(z)\overline{q(w)} = (1-z_1\overline{w_1})\sum A_i(z)\overline{A_i(w)}+(1-z_2\overline{w_2})\sum B_j(z)\overline{B_j(w)}.$$

This is essentially equivalent to Ando's inequality: if f is analytic in $\mathbb{D}^2$ and $\|f\|_\infty \leq 1$, then

$$\|f(T_1, T_2)\| \leq 1 \text{ for all } \|T_i\| < 1. \qquad (*)$$

Similarly, every f bounded by 1 in $\mathbb{D}^2$ admits a SOS decomposition (Agler), but not so in $\mathbb{D}^n$ for $n \geq 3$.

Admitting SOS is equivalent to the n-variable version of (*).

Definition

A function $\varphi : \mathbb{B}^d \to \mathbb{C}^d$ belongs to the Schur class on $\mathbb{B}^d$ if the Hermitian kernel

$$\frac{1 - \langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle}$$

is positive semidefinite.

Definition

A function $\varphi : \mathbb{B}^d \to \mathbb{C}^d$ belongs to the Schur class on $\mathbb{B}^d$ if the Hermitian kernel

$$\frac{1 - \langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle}$$

is positive semidefinite.

If φ is Schur class, then φ is automatically holomorphic and bounded by 1.

The converse holds when $d = 1$ but fails when $d > 1$.

Definition

A function $\varphi : \mathbb{B}^d \to \mathbb{C}^d$ belongs to the Schur class on $\mathbb{B}^d$ if the Hermitian kernel

$$\frac{1 - \langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle}$$

is positive semidefinite.

If φ is Schur class, then φ is automatically holomorphic and bounded by 1.

The converse holds when $d = 1$ but fails when $d > 1$.

E.g. $\varphi(z_1, z_2) = (2z_1 z_2, 0)$ is bounded by 1, but NOT Schur class.

Definition

A function $\varphi : \mathbb{B}^d \to \mathbb{C}^d$ belongs to the Schur class on $\mathbb{B}^d$ if the Hermitian kernel

$$\frac{1 - \langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle}$$

is positive semidefinite.

If φ is Schur class, then φ is automatically holomorphic and bounded by 1.

The converse holds when $d = 1$ but fails when $d > 1$.

E.g. $\varphi(z_1, z_2) = (2z_1z_2, 0)$ is bounded by 1, but NOT Schur class.

Every linear fractional map of $\mathbb{B}^d$ is Schur class.

$$\varphi(z) = \frac{Az + B}{\langle z, C \rangle + D}$$

Define

$$C_\varphi f := f \circ \varphi$$

Define

$$C_\varphi f := f \circ \varphi$$

Theorem (Littlewood, 1925)

Let φ be a holomorphic self-map of $\mathbb{D}$ with $\varphi(0) = 0$. Then

$$\int_0^{2\pi} |f(\varphi(e^{i\theta}))|^2 \, d\theta \leq \int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta$$

for all $f \in H^2$. Equivalently,

$$\|C_\varphi\| \leq 1.$$

Define

$$C_\varphi f := f \circ \varphi$$

Theorem (Littlewood, 1925)

Let φ be a holomorphic self-map of $\mathbb{D}$ with $\varphi(0) = 0$. Then

$$\int_0^{2\pi} |f(\varphi(e^{i\theta}))|^2\, d\theta \leq \int_0^{2\pi} |f(e^{i\theta})|^2\, d\theta$$

for all $f \in H^2$. Equivalently,

$$\|C_\varphi\| \leq 1.$$

The analogous result is utterly false when $d > 1$; in fact C_φ is not even bounded on $H^2(\mathbb{B}^d)$ in general, and even for very nice φ.

E.g. $\varphi(z_1, z_2) = (2z_1 z_2, 0)$ (More examples: Cima-Wogen et al.)

Proof (J., 2007):

Two observations:

- Since $|\varphi(z)| < 1$ for all $z \in \mathbb{D}$, the kernel

$$\frac{1}{1 - \varphi(z)\overline{\varphi(w)}}$$

is positive semidefinite.

Proof (J., 2007):

Two observations:

- Since $|\varphi(z)| < 1$ for all $z \in \mathbb{D}$, the kernel

$$\frac{1}{1 - \varphi(z)\overline{\varphi(w)}}$$

  is positive semidefinite.

- Since φ is a contractive multiplier of H^2, the kernel

$$k_\varphi(z, w) = \frac{1 - \varphi(z)\overline{\varphi(w)}}{1 - z\overline{w}}$$

  is positive semidefinite; and since $\varphi(0) = 0$ the kernel

$$k_\varphi(z, w) - \frac{k_\varphi(z, 0)k_\varphi(0, w)}{k_\varphi(0, 0)} = \frac{1 - \varphi(z)\overline{\varphi(w)}}{1 - z\overline{w}} - 1$$

  is also positive (Schur complement theorem)

Proof, continued.

Consider the Szegő kernel

$$k_w(z) = \frac{1}{1 - z\overline{w}}$$

Note that $C_\varphi^* k_w = k_{\varphi(w)}$.

Proof, continued.

Consider the Szegő kernel

$$k_w(z) = \frac{1}{1 - z\overline{w}}$$

Note that $C_\varphi^* k_w = k_{\varphi(w)}$.
We must show $\|C_\varphi\| \leq 1$; or equivalently $I - C_\varphi C_\varphi^* \geq 0$. Test against k:

$$\begin{aligned}
\langle (I - C_\varphi C_\varphi^*) k_w, k_z \rangle &= \langle k_w, k_z \rangle - \langle k_{\varphi(w)}, k_{\varphi(z)} \rangle \\
&= \frac{1}{1 - z\overline{w}} - \frac{1}{1 - \varphi(z)\overline{\varphi(w)}} \\
&= \left(\frac{1}{1 - \varphi(z)\overline{\varphi(w)}} \right) \cdot \left(\frac{1 - \varphi(z)\overline{\varphi(w)}}{1 - z\overline{w}} - 1 \right)
\end{aligned}$$

which is positive, so done. □

Let $H^2_{d,m}$ denote the RKHS with kernel

$$k_m(z,w) = \frac{1}{(1-\langle z,w\rangle)^m} \quad m = 1,2,\dots$$

The above proof generalizes immediately to the ball, provided φ belongs to the Schur class:

Let $H^2_{d,m}$ denote the RKHS with kernel

$$k_m(z,w) = \frac{1}{(1-\langle z,w\rangle)^m} \quad m = 1,2,\dots$$

The above proof generalizes immediately to the ball, provided φ belongs to the Schur class:

Theorem (J., 2007)

Let φ be a Schur class mapping of $\mathbb{B}^d$ and $\varphi(0)=0$. Then $\|C_\varphi\| \leq 1$ on $H^2_{d,m}$. In particular (for $m=d$)

$$\int_{\partial\mathbb{B}^d} |f\circ\varphi|^2\, d\sigma \leq \int_{\partial\mathbb{B}^d} |f|^2\, d\sigma$$

for all f in the classical Hardy space $H^2(\mathbb{B}^d)$.
(σ = surface measure on $\partial\mathbb{B}^d$)

the case $m = 1$.

As in the disk case, the kernels

$$\frac{1}{1-\langle\varphi(z),\varphi(w)\rangle}, \qquad \frac{1-\langle\varphi(z),\varphi(w)\rangle}{1-\langle z,w\rangle} - 1$$

are positive (since φ is Schur class!!!)

Thus

$$\langle (I - C_\varphi C_\varphi^*)k_w, k_z\rangle = \left(\frac{1}{1-\langle\varphi(z),\varphi(w)\rangle}\right)\cdot\left(\frac{1-\langle\varphi(z),\varphi(w)\rangle}{1-\langle z,w\rangle} - 1\right)$$

is a positive kernel. □

If 0 is not fixed, we have:

Theorem

Let φ be a Schur class mapping of $\mathbb{B}^d$. Then C_φ is bounded on each of the spaces $H^2_{d,m}$, and

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{m/2} \leq \|C_\varphi\|_m \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{m/2}$$

If 0 is not fixed, we have:

Theorem

Let φ be a Schur class mapping of $\mathbb{B}^d$. Then C_φ is bounded on each of the spaces $H^2_{d,m}$, and

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{m/2} \leq \|C_\varphi\|_m \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{m/2}$$

Two nice things:

- both sides are roughly the same size, $\sim (1-|\varphi(0)|)^{-m/2}$
- the inequality iterates...

The norm inequality

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{m/2} \leq \|C_\varphi\|_m \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{m/2}$$

iterates to give

$$\|C_\varphi^n\|_m \sim (1-|\varphi_n(0)|)^{-m/2}$$

[here $\varphi_n = \varphi \circ \cdots \circ \varphi$, n times], and hence

The norm inequality

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{m/2} \leq \|C_\varphi\|_m \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{m/2}$$

iterates to give

$$\|C_\varphi^n\|_m \sim (1-|\varphi_n(0)|)^{-m/2}$$

[here $\varphi_n = \varphi \circ \cdots \circ \varphi$, n times], and hence

Corollary (Spectral radius)

Let φ be a Schur class mapping of the ball. The spectral radius of C_φ acting on $H^2_{d,m}$ is

$$r(C_\varphi) = \lim_{n\to\infty} (1-|\varphi_n(0)|)^{-m/2n}$$

Can we evaluate this limit?

Theorem (MacCluer, 1983)

Let φ be a holomorphic self-map of $\mathbb{B}^d$. Then:

1. *There exists a unique point $\zeta \in \overline{\mathbb{B}^d}$ (the Denjoy-Wolff point) such that*
$$\varphi_n(z) \to \zeta$$
locally uniformly in $\mathbb{B}^d$.
2. *If $\zeta \in \partial\mathbb{B}^d$, then*
$$0 < \liminf_{z\to\zeta} \frac{1-|\varphi(z)|^2}{1-|z|^2} = \alpha \leq 1$$

Theorem (MacCluer, 1983)

Let φ be a holomorphic self-map of $\mathbb{B}^d$. Then:

1. *There exists a unique point $\zeta \in \overline{\mathbb{B}^d}$ (the Denjoy-Wolff point) such that*

$$\varphi_n(z) \to \zeta$$

locally uniformly in $\mathbb{B}^d$.

2. *If $\zeta \in \partial\mathbb{B}^d$, then*

$$0 < \liminf_{z\to\zeta} \frac{1-|\varphi(z)|^2}{1-|z|^2} = \alpha \leq 1$$

If the Denjoy-Wolff point ζ lies in $\mathbb{B}^d$, then φ is called elliptic.

If $\zeta \in \partial\mathbb{B}^d$, the number α is called the dilatation coefficient of φ.

The map φ is called parabolic if $\alpha = 1$, and hyperbolic if $\alpha < 1$.

We want to evaluate

$$\lim_{n\to\infty} (1 - |\varphi_n(0)|)^{-1/2n}$$

If φ is elliptic or parabolic, it is not hard to show this limit is 1.

In one dimension we have:

We want to evaluate

$$\lim_{n\to\infty}(1-|\varphi_n(0)|)^{-1/2n}$$

If φ is elliptic or parabolic, it is not hard to show this limit is 1.

In one dimension we have:

Theorem (C. Cowen, 1983)

If φ is an elliptic or parabolic self-map of $\mathbb{D}$, then the spectral radius of C_φ (on H^2) is 1.

If φ is hyperbolic with dilatation coefficient α, then the spectral radius is $\alpha^{-1/2}$.

We want to evaluate

$$\lim_{n\to\infty} (1 - |\varphi_n(0)|)^{-1/2n}$$

If φ is elliptic or parabolic, it is not hard to show this limit is 1.

In one dimension we have:

Theorem (C. Cowen, 1983)

If φ is an elliptic or parabolic self-map of $\mathbb{D}$, then the spectral radius of C_φ (on H^2) is 1.

If φ is hyperbolic with dilatation coefficient α, then the spectral radius is $\alpha^{-1/2}$.

Goal: Extend this theorem to Schur class mappings of $\mathbb{B}^d$.

The elliptic and parabolic cases go through (with identical proofs). The hyperbolic case takes work...

Definition

Given a point $\zeta \in \partial\mathbb{B}^d$ and a real number $c > 0$, the Koranyi region $D_c(\zeta)$ is the set

$$D_c(\zeta) = \left\{ z \in \mathbb{B}^d : |1 - \langle z, \zeta \rangle| \leq \frac{c}{2}(1 - |z|^2) \right\}$$

A function f has K-limit equal to L at ζ if

$$\lim_{z \to \zeta} f(z) = L$$

whenever $z \to \zeta$ within a Koranyi region.

Definition

Given a point $\zeta \in \partial\mathbb{B}^d$ and a real number $c > 0$, the Koranyi region $D_c(\zeta)$ is the set

$$D_c(\zeta) = \left\{ z \in \mathbb{B}^d : |1 - \langle z, \zeta \rangle| \leq \frac{c}{2}(1 - |z|^2) \right\}$$

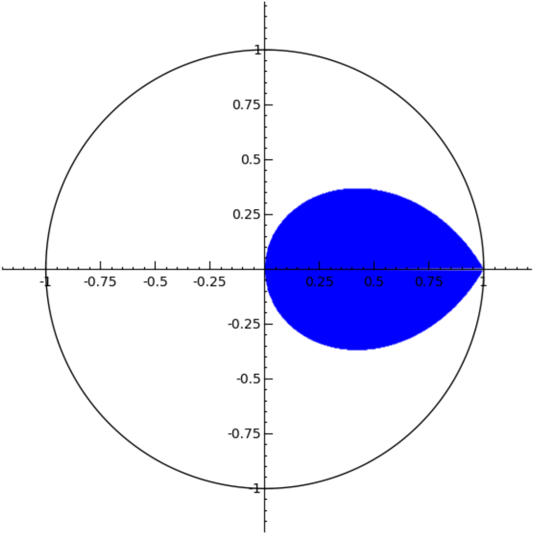
A function f has K-limit equal to L at ζ if

$$\lim_{z \to \zeta} f(z) = L$$

whenever $z \to \zeta$ within a Koranyi region.

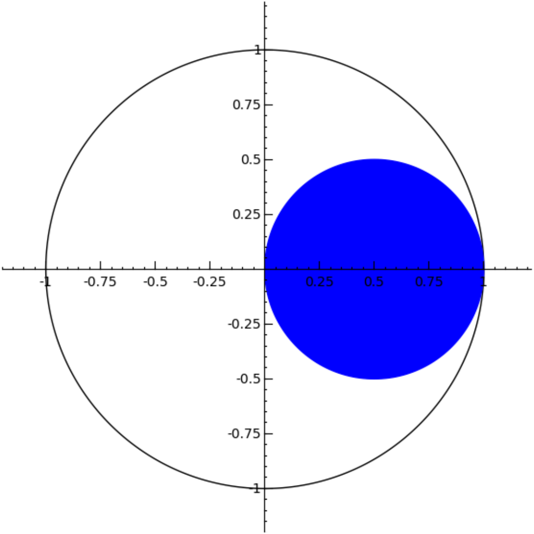
When $d = 1$ (the disk), K-limit is the same as non-tangential limit. Not so in the ball...

Slice of a Koranyi region with vertex at $(1, 0)$ in $\mathbb{B}^2$:



$z_2 = 0$

Slice of a Koranyi region with vertex at $(1,0)$ in $\mathbb{B}^2$:



$$\mathrm{Im} z_1 = \mathrm{Im} z_2 = 0$$

$$\alpha = \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} < \infty. \tag{C}$$

$$\alpha = \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} < \infty. \qquad \text{(C)}$$

Theorem (Rudin, 1980)

Suppose $\varphi = (\varphi_1, \ldots \varphi_d)$ is a holomorphic mapping from $\mathbb{B}^d$ to itself satisfying condition (C)at e_1. The following functions are then bounded in every Koranyi region with vertex at e_1:

(i) $\frac{1-\varphi_1(z)}{1-z_1}$

(ii) $(D_1\varphi_1)(z)$

(iii) $\frac{1-|\varphi_1(z)|^2}{1-|z_1|^2}$

(iv) $\frac{1-|\varphi(z)|^2}{1-|z|^2}$

Moreover, each of these functions has restricted K-limit α at e_1.

$$\alpha = \liminf_{z\to\zeta} \frac{1-|\varphi(z)|^2}{1-|z|^2} < \infty. \qquad \text{(C)}$$

Theorem (Rudin, 1980)

Suppose $\varphi = (\varphi_1, \dots \varphi_d)$ is a holomorphic mapping from $\mathbb{B}^d$ to itself satisfying condition (C)at e_1. The following functions are then bounded in every Koranyi region with vertex at e_1:

(i) $\frac{1-\varphi_1(z)}{1-z_1}$

(ii) $(D_1\varphi_1)(z)$

(iii) $\frac{1-|\varphi_1(z)|^2}{1-|z_1|^2}$

(iv) $\frac{1-|\varphi(z)|^2}{1-|z|^2}$

Moreover, each of these functions has restricted K-limit α at e_1.

What is a restricted K-limit?

Fix a point $\zeta \in \partial \mathbb{B}^d$ and consider a curve $\Gamma : [0,1) \to \mathbb{B}^n$ such that $\Gamma(t) \to \zeta$ as $t \to 1$. Let $\gamma(t) = \langle \Gamma(t), \zeta \rangle \zeta$ be the projection of Γ onto the complex line through ζ. The curve Γ is called special if

$$\lim_{t \to 1} \frac{|\Gamma - \gamma|^2}{1 - |\gamma|^2} = 0 \qquad (1)$$

and restricted if it is special and in addition

$$\frac{|\zeta - \gamma|}{1 - |\gamma|^2} \leq A \qquad (2)$$

for some constant $A > 0$.

Fix a point $\zeta \in \partial\mathbb{B}^d$ and consider a curve $\Gamma : [0,1) \to \mathbb{B}^n$ such that $\Gamma(t) \to \zeta$ as $t \to 1$. Let $\gamma(t) = \langle \Gamma(t), \zeta \rangle \zeta$ be the projection of Γ onto the complex line through ζ. The curve Γ is called special if

$$\lim_{t\to 1} \frac{|\Gamma - \gamma|^2}{1 - |\gamma|^2} = 0 \tag{1}$$

and restricted if it is special and in addition

$$\frac{|\zeta - \gamma|}{1 - |\gamma|^2} \leq A \tag{2}$$

for some constant $A > 0$.

Definition

We say that a function $f : \mathbb{B}^d \to \mathbb{C}$ has restricted K-limit L at ζ if $\lim_{z\to\zeta} f(z) = L$ along every restricted curve.

We have

K-limit $\implies$ restricted K-limit $\implies$ non-tangential limit
and each implication is strict when $d > 1$.

One more fact:

Theorem

Let φ be a hyperbolic, holomorphic self-map of $\mathbb{B}^d$ with Denjoy-Wolff point ζ. Then

$$\varphi_n(z_0) \to \zeta$$

within a Koranyi region for every $z_0 \in \mathbb{B}^d$.

C. Cowen 1981 ($d = 1$)
Bracci, Poggi-Corradini 2003 ($d > 1$)

One more fact:

Theorem

Let φ be a hyperbolic, holomorphic self-map of $\mathbb{B}^d$ with Denjoy-Wolff point ζ. Then

$$\varphi_n(z_0) \to \zeta$$

within a Koranyi region for every $z_0 \in \mathbb{B}^d$.

C. Cowen 1981 ($d = 1$)
Bracci, Poggi-Corradini 2003 ($d > 1$)

If we knew that some orbit $\{\varphi_n(z_0)\}$ approached the Denjoy-Wolff point restrictedly, this combined with Rudin's Julia-Caratheodory theorem would imply

$$\lim_{n\to\infty} (1 - |\varphi_n(z_0)|)^{-1/2n} = \alpha^{-1/2}$$

It is not known if such orbits always exist. (Yes, if φ is an LFT.)

Indeed, if we know some orbit $z_n := \varphi_n(z_0)$ approaches ζ restrictedly, then

$$\lim_{n\to\infty} \frac{1 - |\varphi(z_n)|}{1 - |\varphi(z_{n-1})|} = \alpha$$

and hence

$$\lim_{n\to\infty} (1 - |\varphi_n(z_0)|)^{1/n} = \lim_{n\to\infty} \left(\prod_{k=1}^{n} \frac{1 - |\varphi(z_k)|}{1 - |\varphi(z_{k-1})|} \right)^{1/n}$$

$$= \alpha$$

$$\alpha = \liminf_{z\to\zeta} \frac{1-|\varphi(z)|^2}{1-|z|^2} < \infty. \tag{C}$$

Theorem (J., 2008)

Let φ be a Schur class map and $\zeta \in \partial\mathbb{B}^d$. Then the following are equivalent:

1. *Condition (C).*
2. *There exists $\xi \in \partial\mathbb{B}^d$ such that the function*

   $$h(z) = \frac{1-\langle\varphi(z),\xi\rangle}{1-\langle z,\zeta\rangle}$$

   belongs to $\mathcal{H}(\varphi)$.
3. *Every $f \in \mathcal{H}(\varphi)$ has a finite K-limit at ζ.*

$d = 1$ case: Sarason 1994

Theorem (J., 2008)

Suppose $\varphi = (\varphi_1, \dots \varphi_d)$ is a ~~holomorphic~~ Schur class mapping from $\mathbb{B}^d$ to itself satisfying condition (C) at e_1. The following functions are then bounded in every Koranyi region with vertex at e_1:

(i) $\frac{1-\varphi_1(z)}{1-z_1}$

(ii) $(D_1\varphi_1)(z)$

(iii) $\frac{1-|\varphi_1(z)|^2}{1-|z_1|^2}$

(iv) $\frac{1-|\varphi(z)|^2}{1-|z|^2}$

Moreover, each of these functions (i)-(iii) has ~~restricted~~ K-limit α at e_1.

Unfortunately, our attempted argument works only if (iv) has a K-limit at e_1, which is not true in general even if φ is Schur class. Nonetheless, this theorem is sufficient to solve the spectral radius problem, in a more indirect way....

For $0 < \alpha < 1$, let θ_α denote the disk automorphism

$$\theta_\alpha(z) = \frac{z + \left(\frac{1-\alpha}{1+\alpha}\right)}{1 + \left(\frac{1-\alpha}{1+\alpha}\right) z}$$

Theorem (J., 2009)

Let φ be a hyperbolic Schur class self-map of $\mathbb{B}^d$ with dilatation coefficient α. Then there exists a nonconstant Schur class map $\sigma : \mathbb{B}^d \to \mathbb{D}$ such that

$$\sigma \circ \varphi = \theta_\alpha \circ \sigma$$

Proof needs[1] strengthened Julia-Caratheodory theorem.

($d = 1$: Valiron, 1931; also Pommerenke 1979, C. Cowen 1981)
($d > 1$, under different assumptions:
Bracci, Gentili, Poggi-Corradini, 2007)

[1] Probably.

Given a Schur class solution σ to the Abel-Schroeder equation

$$\sigma \circ \varphi = \theta_\alpha \circ \sigma$$

we can compute the spectral radius of C_φ via "transference:"

Given a Schur class solution σ to the Abel-Schroeder equation

$$\sigma \circ \varphi = \theta_\alpha \circ \sigma$$

we can compute the spectral radius of C_φ via "transference:" Suppose F is holomorphic in $\mathbb{D}$ and $\lambda \in \mathbb{C}$ satisfies

$$F \circ \theta_\alpha = \lambda F$$

Then

$$\begin{aligned} F \circ \sigma \circ \varphi &= F \circ \theta_\alpha \circ \sigma \\ &= \lambda F \circ \sigma \end{aligned}$$

Formally, σ transfers eigenfunctions of C_{θ_α} to eigenfunctions of C_φ.

Given a Schur class solution σ to the Abel-Schroeder equation

$$\sigma \circ \varphi = \theta_\alpha \circ \sigma$$

we can compute the spectral radius of C_φ via "transference:" Suppose F is holomorphic in $\mathbb{D}$ and $\lambda \in \mathbb{C}$ satisfies

$$F \circ \theta_\alpha = \lambda F$$

Then

$$\begin{aligned} F \circ \sigma \circ \varphi &= F \circ \theta_\alpha \circ \sigma \\ &= \lambda F \circ \sigma \end{aligned}$$

Formally, σ transfers eigenfunctions of C_{θ_α} to eigenfunctions of C_φ.

KEY FACT: If $F \in H^2(\mathbb{D})$ and σ Schur class, then $F \circ \sigma \in H^2_{d,1}$.

The transference technique then proves:

Theorem (J., 2009)

Let φ be a hyperbolic Schur class map of $\mathbb{B}^d$ with dilatation coefficient α. Then the spectral radius of C_φ acting on $H^2_{d,m}$ is $\alpha^{-m/2}$. Moreover every complex number λ in the annulus

$$\alpha^{m/2} < |\lambda| < \alpha^{-m/2}$$

is an eigenvalue of C_φ of infinite multiplicity.

($d = 1$ case: C. Cowen, 1983)

If $d > 1$ and φ is an automorphism, the closure of this annulus is equal to the spectrum of C_φ (MacCluer, 1984)