Composition operators induced by Schur-Agler mappings

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Setting:

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

H^2 = Hardy space on $\mathbb{D}$

$k(z, w) = \dfrac{1}{1 - z\overline{w}}$ Szegő kernel

$\varphi : \mathbb{D} \to \mathbb{D}$ holomorphic

$H(\varphi)$ = de Branges-Rovnyak space:
RKHS with kernel

$$k^{\varphi}(z, w) = \frac{1 - \varphi(z)\overline{\varphi(w)}}{1 - z\overline{w}}$$

Two kinds of operators:

- "Adjoints of multipliers"
  Given $f : \mathbb{D} \to \mathbb{C}$ define

$$M_f^* k_w = \overline{f(w)}\, k_w$$

$$h,\ fh \in H^2 \Longrightarrow$$

$$f(w)h(w) = \langle fh, k_w \rangle = \langle h, M_f^* k_w \rangle$$

- "Adjoints of composition operators"
  Define

$$C_\varphi^* k_w = k_{\varphi(w)}$$

$$h,\ h \circ \varphi \in H^2 \ \Longrightarrow$$

$$h(\varphi(w)) = \langle h \circ \varphi, k_w \rangle = \langle h, C_\varphi^* k_w \rangle$$

The main estimate:

If $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic and $f \in H(\varphi)$, then

$$\|C_\varphi^* M_f^*\| \leq \|f\|_{H(\varphi)}.$$

Proof:

Assume $\|f\|_{H(\varphi)} = 1$, prove $\|C_\varphi^* M_f^*\| \leq 1$:
$\|f\|_{H(\varphi)} = 1 \implies$

$$\frac{1 - \varphi(z)\overline{\varphi(w)}}{1 - z\overline{w}} - f(z)\overline{f(w)} \geq 0$$

Schur product with $\dfrac{1}{1 - \varphi(z)\overline{\varphi(w)}} \implies$

$$\frac{1}{1 - z\overline{w}} - \frac{f(z)\overline{f(w)}}{1 - \varphi(z)\overline{\varphi(w)}} \geq 0$$

$$\frac{1}{1 - z\overline{w}} - \frac{f(z)\overline{f(w)}}{1 - \varphi(z)\overline{\varphi(w)}} \geq 0$$

$\iff$

$$\langle k_w, k_z \rangle - \langle C_\varphi^* \ M_f^* \ k_w, C_\varphi^* \ M_f^* \ k_z \rangle \geq 0$$

$\iff$

$$\| C_\varphi^* M_f^* \| \leq 1 \qquad \square$$

Corollary 1:
If $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic then

$$\|C_\varphi\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/2}$$

Proof: Apply estimate with

$$f(z) = k^\varphi(z,0) = 1 - \varphi(z)\overline{\varphi(0)}$$

Then

- $\|f\|_{H(\varphi)} = (k^\varphi(0,0))^{1/2} = (1-|\varphi(0)|^2)^{1/2}$
- $\|M^*_{\frac{1}{f}}\| \ = \ \sup_{|z|<1} \left|\frac{1}{f(z)}\right| \ \leq \ \frac{1}{1-|\varphi(0)|}$

Thus

$$\begin{aligned} \|C^*_\varphi\| \ &= \ \|C^*_\varphi M^*_f M^*_{\frac{1}{f}}\| \\ &\leq \ \|C^*_\varphi M^*_f\| \, \|M^*_{\frac{1}{f}}\| \ \leq \ \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/2} \square \end{aligned}$$

Remarks:

- Estimate is sharp over all φ, attained by inner functions [Nordgren 1968]
- Same argument works on weighted Bergman spaces with kernels $k(z,w)^\alpha$; exponent is $\alpha/2$
- General estimate on H^2:

$$\|C_\varphi\| \leq \inf_{f\in H(\varphi)} \left\{ \left\| \frac{1}{f} \right\|_\infty \| f \|_{H(\varphi)} \right\}$$

- Given a factorization

$$\frac{1-\varphi(z)\overline{\varphi(w)}}{1-z\overline{w}} = \sum_j f_j(z)\overline{f_j(w)}$$

we obtain the identity

$$\sum_j M_{f_j} C_\varphi C_\varphi^* M_{f_j}^* = I \quad \text{(SOT)}$$

Multivariable setting:

$$\mathbb{B}^d = \left\{(z_1, \dots z_d) \in \mathbb{C}^d : |z_1|^2 + \cdots + |z_d|^2 < 1\right\}$$
$$\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_d\overline{w_d}$$

H^2_d = symmetric Fock space
(*not* classical H^2)...RKHS with kernel

$$k(z, w) = \frac{1}{1 - \langle z, w\rangle}$$

$\varphi = (\varphi_1, \dots \varphi_d) : \mathbb{B}^d \to \mathbb{B}^d$ holomorphic

$H(\varphi)$ = "de Branges-Rovnyak space??"
not automatically...only when

$$k^\varphi(z, w) = \frac{1 - \langle \varphi(z), \varphi(w)\rangle}{1 - \langle z, w\rangle} \geq 0$$

that is, when φ belongs to the *Schur-Agler class* $\mathcal{S}_d$.

Composition operators on $\mathbf{H^2_d}$

Theorem: Suppose $\varphi : \mathbb{B}^d \to \mathbb{B}^d$ belongs to $\mathcal{S}_d$. Then C_φ is bounded on H^2_d and

$$\| C_\varphi \| \ \leq \ \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{1/2}$$

Proof: Same proof!...almost. For $f \in H(\varphi)$, estimate

$$\| C_\varphi^* M_f^* \| \ \leq \ \| f \|_{H(\varphi)}$$

goes through. Now take $f = k^\varphi(\cdot, 0)$...

...need to estimate the *multiplier norm* of $\dfrac{1}{f}$.

Not too hard: since $I - \sum M_{z_j} M_{z_j}^* \geq 0$,

$$\left\| \sum_{j=1}^{d} M_{z_j} \cdot \overline{w_j} \right\| \ \leq |w| \qquad \forall \ |w| \leq 1$$

$$\left\| \sum_{j=1}^{d} M_{z_j} \cdot \overline{w_j} \right\| \leq |w| \qquad \forall\ |w| \leq 1$$

Take $w = \varphi(0)$, then the series

$$\sum_{n=0}^{\infty} \left(\sum_{j=1}^{d} M_{z_j} \overline{\varphi_j(0)} \right)^n$$

converges in norm to $M_{\frac{1}{f}}$, and

$$\| M_{\frac{1}{f}} \| \leq \sum_{n=0}^{\infty} |\varphi(0)|^n \ = \ (1 - |\varphi(0)|)^{-1}$$

Rest of the argument goes through. $\square$

Remarks:

- Estimate is sharp over $\varphi \in \mathcal{S}_d$; upper bound is obtained on automorphisms of $\mathbb{B}^d$.
- Estimates go through for spaces $H^2_{d,\alpha}$ with kernel $(1-\langle z,w\rangle)^{-\alpha}$; exponent $\alpha/2$—includes classical Hardy $(\alpha = d)$ and Bergman $(\alpha = d+1)$
- Unfortunately, C_φ bounded $\not\Rightarrow \varphi \in \mathcal{S}_d$. Example:

$$\varphi_r(z_1, z_2) = (2rz_1z_2, 0)$$

is bounded on H^2_2 iff $r < 1$. But

$$k^{\varphi_r} \geq 0 \ (\forall\ r < 1) \implies k^{\varphi_1} \geq 0$$

Linear fractional maps

Given $d+1 \times d+1$ matrices

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} I_{d\times d} & 0 \\ 0 & -1 \end{pmatrix}$$

define

$$\varphi(z) = \frac{Az+B}{\langle z, C\rangle + D}$$

Then $\varphi : \mathbb{B}^d \to \mathbb{B}^d$ if and only if for some scalar m

$$J - |m|^2 T^* J T \geq 0.$$

[Cowen-MacCluer (2000)] C_φ is bounded on classical Hardy & Bergman spaces . (Indirect proof—no norm estimates)
[Bayart (2007)] gives an estimate for special class of "parabolic" φ on the classical Hardy space

$$\|C_\varphi\| \leq \frac{C(d,\varphi)}{(1-|\varphi(0)|)^{d/2}}$$

Proposition: Every LFT $\varphi : \mathbb{B}^d \to \mathbb{B}^d$ belongs to $\mathcal{S}_d$.
Proof: Factor

$$J - T^*JT = X^*X$$

Put

$$L(z) = X \begin{pmatrix} z \\ 1 \end{pmatrix}$$

Then

$$k^\varphi(z,w) = \frac{1 - \langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle}$$

$$= \frac{1}{\langle z, C \rangle + D} \left(1 + \frac{L(z)L(w)^*}{1 - \langle z, w \rangle} \right) \frac{1}{\overline{\langle w, C \rangle + D}}$$

Corollary: For every LFT φ, C_φ is bounded on $H^2_{d,\alpha}$ and

$$\left(\frac{1}{(1 - |\varphi(0)|^2)} \right)^{\alpha/2} \leq \| C_\varphi \| \leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{\alpha/2}$$

Spectral radii

If φ fixes a point of $\mathbb{B}^d$ (hence called *elliptic*) then $r(C_\varphi) = 1$.
In the case of no interior fixed point, we have the *Denjoy-Wolff* point $\zeta \in \partial\mathbb{B}^d$; and

$$\lim_{r \to 1} D_\zeta \varphi(r\zeta) = \alpha$$

for some $0 < \alpha \leq 1$, called the *dilatation coefficient* (analogous to the angular derivative when $d = 1$).

- $\alpha = 1$ *parabolic*
- $\alpha < 1$ *hyperbolic*

From the previous norm estimate we get a prototype spectral radius formula on H^2_d:

Proposition: For $\varphi \in \mathcal{S}_d$

$$r(C_\varphi) = \lim \|C_\varphi^n\|^{1/n} = \lim(1 - |\varphi_n(0)|)^{-1/2n}.$$

Theorem: [Cowen (1983)] For $\varphi : \mathbb{D} \to \mathbb{D}$, on H^2 we have

- $r(C_\varphi) = 1$ (elliptic)
- $r(C_\varphi) = \alpha^{-1/2}$ (otherwise)

Theorem: [J. (2007)] ($d \geq 1$) For $\varphi \in \mathcal{S}_d$, on H^2_d we have

- $r(C_\varphi) = 1$ (elliptic)
- $r(C_\varphi) = 1 = \alpha^{-1/2}$ (parabolic)

If φ is an LFT, then we also have

- $r(C_\varphi) = \alpha^{-1/2}$ (hyperbolic)

Conjecture: Formula holds for all hyperbolic $\varphi \in \mathcal{S}_d$.