Composition operators induced by Schur-Agler mappings

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Setting:

\[ \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \]

\( H^2 = \text{Hardy space on } \mathbb{D} \)

\[ k(z, w) = \frac{1}{1 - z\overline{w}} \quad \text{Szegő kernel} \]

\( \varphi : \mathbb{D} \to \mathbb{D} \) holomorphic

\[ H(\varphi) = \text{de Branges-Rovnyak space:} \]

\( \text{RKHS with kernel} \)

\[ k^\varphi(z, w) = \frac{1 - \varphi(z)\overline{\varphi(w)}}{1 - z\overline{w}} \]
Two kinds of operators:

- “Adjoints of multipliers”
  Given $f : \mathbb{D} \to \mathbb{C}$ define
  \[ M_f^* k_w = \overline{f(w)} k_w \]
  \[ h, \; fh \in H^2 \implies f(w)h(w) = \langle fh, k_w \rangle = \langle h, M_f^* k_w \rangle \]

- “Adjoints of composition operators”
  Define
  \[ C_\varphi^* k_w = k_{\varphi(w)} \]
  \[ h, \; h \circ \varphi \in H^2 \implies h(\varphi(w)) = \langle h \circ \varphi, k_w \rangle = \langle h, C_\varphi^* k_w \rangle \]
The main estimate:

If $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ is analytic and $f \in H(\varphi)$, then

$$\| C_\varphi^* M_f^* \| \leq \| f \|_{H(\varphi)}.$$

Proof:

Assume $\| f \|_{H(\varphi)} = 1$, prove $\| C_\varphi^* M_f^* \| \leq 1$:

$\| f \|_{H(\varphi)} = 1 \implies$

$$\frac{1 - \varphi(z)\overline{\varphi(w)}}{1 - z\overline{w}} - f(z)f(w) \geq 0$$

Schur product with

$$\frac{1}{1 - \varphi(z)\overline{\varphi(w)}} \implies$$

$$\frac{1}{1 - z\overline{w}} - \frac{f(z)f(w)}{1 - \varphi(z)\overline{\varphi(w)}} \geq 0$$
\[
\frac{1}{1 - zw} - \frac{f(z)f(w)}{1 - \varphi(z)\varphi(w)} \geq 0
\]

\[\iff\]
\[
\langle k_w, k_z \rangle - \langle C_\varphi \ M_f^* \ k_w, C_\varphi \ M_f^* \ k_z \rangle \geq 0
\]

\[\iff\]
\[
\|C_\varphi \ M_f^*\| \leq 1 \quad \square
\]
Corollary 1:
If \( \varphi : \mathbb{D} \to \mathbb{D} \) is holomorphic then

\[
\| C_\varphi \| \leq \left( \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{1/2}
\]

Proof: Apply estimate with

\[
f(z) = k^\varphi(z, 0) = 1 - \varphi(z)\overline{\varphi(0)}
\]

Then

\[
\| f \|_{H(\varphi)} = (k^\varphi(0, 0))^{1/2} = (1 - |\varphi(0)|^2)^{1/2}
\]

\[
\| M_{\frac{1}{f}}^* \| = \sup_{|z|<1} \left| \frac{1}{f(z)} \right| \leq \frac{1}{1 - |\varphi(0)|}
\]

Thus

\[
\| C_\varphi^* \| = \| C_\varphi^* M_f^* M_{\frac{1}{f}}^* \| \leq \left( \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{1/2} \]

\( \square \)
Remarks:

- Estimate is sharp over all $\varphi$, attained by inner functions [Nordgren 1968]
- Same argument works on weighted Bergman spaces with kernels $k(z, w)^\alpha$; exponent is $\alpha/2$
- General estimate on $H^2$:

$$ \| C_{\varphi} \| \leq \inf_{f \in H(\varphi)} \left\{ \frac{1}{f} \inf_{\infty} \| f \|_{H(\varphi)} \right\} $$

- Given a factorization

$$ \frac{1 - \varphi(z)\overline{\varphi(w)}}{1 - zw} = \sum_{j} f_j(z)\overline{f_j(w)} $$

we obtain the identity

$$ \sum_{j} M_{f_j} C_{\varphi} C^*_\varphi M^*_f = I \quad \text{(SOT)} $$
Multivariable setting:

\[ \mathbb{B}^d = \{ (z_1, \ldots, z_d) \in \mathbb{C}^d : |z_1|^2 + \cdots + |z_d|^2 < 1 \} \]

\[ \langle z, w \rangle = z_1 \overline{w_1} + \cdots + z_d \overline{w_d} \]

\[ H^2_d = \text{symmetric Fock space} \]

(not classical \( H^2 \))...RKHS with kernel

\[ k(z, w) = \frac{1}{1 - \langle z, w \rangle} \]

\[ \varphi = (\varphi_1, \ldots, \varphi_d) : \mathbb{B}^d \rightarrow \mathbb{B}^d \text{ holomorphic} \]

\[ H(\varphi) = \text{"de Branges-Rovnyak space??"} \]

not automatically...only when

\[ k^\varphi(z, w) = \frac{1 - \langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle} \geq 0 \]

that is, when \( \varphi \) belongs to the Schur-Agler class \( S_d \).
Composition operators on $H^2_d$

**Theorem:** Suppose $\varphi : \mathbb{B}^d \to \mathbb{B}^d$ belongs to $S_d$. Then $C_\varphi$ is bounded on $H^2_d$ and

$$\| C_\varphi \| \leq \left( \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{1/2}$$

**Proof:** Same proof!...almost. For $f \in H(\varphi)$, estimate

$$\| C_\varphi^* M^*_f \| \leq \| f \|_{H(\varphi)}$$

goes through. Now take $f = k^\varphi(\cdot, 0)...

...need to estimate the multiplier norm of $\frac{1}{f}$.

Not too hard: since $I - \sum M_{z_j} M^*_{z_j} \succeq 0$,

$$\left\| \sum_{j=1}^d M_{z_j} \cdot \overline{w_j} \right\| \leq |w| \quad \forall \ |w| \leq 1$$
\[ \left\| \sum_{j=1}^{d} M_{zj} \cdot \overline{w_j} \right\| \leq |w| \quad \forall \ |w| \leq 1 \]

Take \( w = \varphi(0) \), then the series

\[ \sum_{n=0}^{\infty} \left( \sum_{j=1}^{d} M_{zj} \varphi_j(0) \right)^n \]

converges in norm to \( M_{1 \overline{f}} \), and

\[ \| M_{1 \overline{f}} \| \leq \sum_{n=0}^{\infty} |\varphi(0)|^n = (1 - |\varphi(0)|)^{-1} \]

Rest of the argument goes through. \( \square \)
Remarks:

- Estimate is sharp over $\varphi \in S_d$; upper bound is obtained on automorphisms of $\mathbb{B}^d$.

- Estimates go through for spaces $H_{d,\alpha}^2$ with kernel $(1 - \langle z, w \rangle)^{-\alpha}$; exponent $\alpha/2$—includes classical Hardy ($\alpha = d$) and Bergman ($\alpha = d + 1$)

- Unfortunately, $C_\varphi$ bounded $\not\Rightarrow \varphi \in S_d$. Example:

  $$\varphi_r(z_1, z_2) = (2rz_1z_2, 0)$$

  is bounded on $H_2^2$ iff $r < 1$. But

  $$k^{\varphi_r} \geq 0 \ (\forall \ r < 1) \implies k^{\varphi_1} \geq 0$$
Linear fractional maps
Given \(d + 1 \times d + 1\) matrices

\[
T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} I_{d \times d} & 0 \\ 0 & -1 \end{pmatrix}
\]

define

\[
\varphi(z) = \frac{Az + B}{\langle z, C \rangle + D}
\]

Then \(\varphi : \mathbb{B}^d \to \mathbb{B}^d\) if and only if for some scalar \(m\)

\[
J - |m|^2 T^* J T \geq 0.
\]

[Cowen-MacCluer (2000)] \(C_\varphi\) is bounded on classical Hardy & Bergman spaces. (Indirect proof—no norm estimates)

[Bayart (2007)] gives an estimate for special class of “parabolic” \(\varphi\) on the classical Hardy space

\[
\|C_\varphi\| \leq \frac{C(d, \varphi)}{(1 - |\varphi(0)|)^{d/2}}
\]
**Proposition:** Every LFT $\varphi : \mathbb{B}^d \rightarrow \mathbb{B}^d$ belongs to $S_d$.

**Proof:** Factor

$$J - T^*JT = X^*X$$

Put

$$L(z) = X \begin{pmatrix} z \\ 1 \end{pmatrix}$$

Then

$$k_\varphi(z, w) = \frac{1 - \langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle}$$

$$= \frac{1}{\langle z, C \rangle + D} \left( 1 + \frac{L(z)L(w)^*}{1 - \langle z, w \rangle} \right) \frac{1}{\langle w, C \rangle + D}$$

**Corollary:** For every LFT $\varphi$, $C_\varphi$ is bounded on $H^2_{d,\alpha}$ and

$$\left( \frac{1}{(1 - |\varphi(0)|^2)} \right)^{\alpha/2} \leq \| C_\varphi \| \leq \left( \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{\alpha/2}$$
Spectral radii
If \( \varphi \) fixes a point of \( B^d \) (hence called elliptic) then \( r(C_\varphi) = 1 \).
In the case of no interior fixed point, we have the Denjoy-Wolff point \( \zeta \in \partial B^d \); and
\[
\lim_{r \to 1} D_{\zeta} \varphi(r\zeta) = \alpha
\]
for some \( 0 < \alpha \leq 1 \), called the dilatation coefficient (analogous to the angular derivative when \( d = 1 \)).
- \( \alpha = 1 \) parabolic
- \( \alpha < 1 \) hyperbolic
From the previous norm estimate we get a prototype spectral radius formula on $H^2_d$:

**Proposition:** For $\varphi \in S_d$

$$r(C\varphi) = \lim \|C\varphi^n\|^{1/n} = \lim(1 - |\varphi_n(0)|)^{-1/2n}.$$

**Theorem:** [Cowen (1983)] For $\varphi : \mathbb{D} \to \mathbb{D}$, on $H^2$ we have

- $r(C\varphi) = 1$ (elliptic)
- $r(C\varphi) = \alpha^{-1/2}$ (otherwise)
Theorem: [J. (2007)] \((d \geq 1)\) For \(\varphi \in S_d\), on \(H_d^2\) we have
- \(r(C_\varphi) = 1\) (elliptic)
- \(r(C_\varphi) = 1 = \alpha^{-1/2}\) (parabolic)

If \(\varphi\) is an LFT, then we also have
- \(r(C_\varphi) = \alpha^{-1/2}\) (hyperbolic)

Conjecture: Formula holds for all hyperbolic \(\varphi \in S_d\).