C*-algebras generated by non-unitary group representations

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When groups appear in the thoery of operator algebras, they nearly always appear in the guise of *unitary representations:*

Definition

A *unitary representation* of a group G on a Hilbert space $\mathcal{H}$ is a homomorphism

$$\pi : G \to \mathcal{U}(\mathcal{H})$$

of G into the group of unitary operators on $\mathcal{H}$.

Example

For G discrete, let $\mathcal{H} = \ell^2(G)$. The *(left) regular representation* λ is defined by

$$\lambda_g(\xi)(h) = \xi(g^{-1}h)$$

C*-algebras generated by unitary representations appeared very early in the history of C*-algebras, e.g in the work of Gelfand, Naimark, Segal, Fell, Kaplansky,...

Definition

The reduced group C*-algebra $C_r^*(G)$ is the C*-subalgebra of $\mathcal{B}(\ell^2(G))$ generated by the image of the regular representation.

Such a C*-algebra may be viewed as a completion of the convolution algebra $\ell^1(G)$ in the C*-norm induced by the representation. We may define a maximal C*-norm on $\ell^1(G)$ by

$$\|\sum a_g g\|_{max} := \sup_{\pi} \|\sum a_g \pi(g)\| \leq \|\sum a_g g\|_1$$

This is a C*-norm, and the completion of $\ell^1(G)$ under $\|\cdot\|_{max}$ gives the full group C*-algebra $C^*(G)$.

The full group C*-algebra has the following very important universal property:

Theorem

*If π is any unitary representation of G, then there is a surjective *-homomorphism*

$$\rho : C^*(G) \to C^*(\pi(G))$$

In general, the full and reduced group C*-algebras do not coincide:

Theorem (Godement 1950)

$C^*(G) \cong C^*_r(G)$ *if and only if* G *is amenable.*

Definition

A representation $\pi : G \longrightarrow \mathcal{B}(\mathcal{H})$ is uniformly bounded if

$$\sup_{g \in G} \|\pi(g)\| < \infty$$

π is called unitarizable if there exists a unitary representation ρ and an invertible operator S such that

$$\pi(g) := S^{-1}\rho(g)S$$

for all $g \in G$.

If G is amenable then every u.b. representation is unitarizable; not so otherwise.

In this talk we are interested in the C*-algebras $C^*(\pi(G))$ generated by representations π that are not unitary, and our most interesting examples occur for π that are not even u.b. To state our main result, we first need to see how to associate a C*-algebra to the action of a group G on a topological space X.

Definition

Let X be a compact Hausdorff space and let $\alpha : G \to Homeo(X)$ be an action of G. A covariant representation of (G, X, α) on a Hilbert space $\mathcal{H}$ is a pair (π, u) where

$$\pi : C(X) \to \mathcal{B}(\mathcal{H})$$

is a *-homomorphism and

$$u : G \to \mathcal{B}(\mathcal{H})$$

is a unitary representation which satisfy

$$u_g^* \pi(f) u_g = \pi(\hat{\alpha}_g(f))$$

for all $g \in G$, $f \in C(X)$.

The Main Theorem

Theorem (J. 2006)

Let $\pi : G \to \mathcal{B}(\mathcal{H})$ be a representation of a discrete group G by invertible operators. Then there exist:

- *a compact Hausdorff space X (metrizable if G is countable),*
- *an action α of G on X, and*
- *ideals $\mathcal{J} \subset C^*(\pi(G))$, $\mathcal{I} \subset C(X) \times_\alpha G$*

such that

$$C^*(\pi(G))/\mathcal{J} \cong (C(X) \times_\alpha G)/\mathcal{I}$$

While the existence of $\mathcal{I}$ is proved via the universal property of the full crossed product, the ideal $\mathcal{J}$ is constructed explicitly.

Outline of the construction

Consider the polar decomposition of $\pi(g)$:

$$\pi(g) = v_g |\pi(g)|$$

where

$$|\pi(g)| := (\pi(g)^* \pi(g))^{1/2}$$

$$v_g = \pi(g)|\pi(g)|^{-1}$$

Note that each v_g is unitary.
We have

$$C^*(\pi(G)) = C^*\{v_g, |\pi(g)| \ : \ g \in G\}$$

Definition

Let

$$\mathcal{P} = C^*\{|\pi(g)| \ : \ g \in G\}$$

$$\mathcal{J} = \text{ ideal of } C^*(\pi(G)) \text{ generated by } \{pq - qp \ : \ p, q \in \mathcal{P}\}$$

In general, $v_g v_h \neq v_{gh}$. However:

Theorem

For all $g, h \in G$,

$$v_g v_h - v_{gh} \in \mathcal{J}$$

Let $\rho : C^*(\pi(G)) \to C^*(\pi(G))/\mathcal{J}$ be the quotient map. Then:

- $C^*(\pi(G))/\mathcal{J}$ is generated by $\rho(\mathcal{P})$ and the unitaries $u_g := \rho(v_g)$
- Since $\mathcal{J} \supset [\mathcal{P}, \mathcal{P}]$ we have

$$\rho(\mathcal{P}) = \rho(\mathcal{P} + \mathcal{J}) = \frac{\mathcal{P} + \mathcal{J}}{\mathcal{J}} \cong \frac{\mathcal{P}}{\mathcal{J} \cap \mathcal{P}}$$

  is *commutative*; put $X = \sigma(\rho(\mathcal{P}))$.
- $u_g \rho(\mathcal{P}) u_g^* \subset \rho(\mathcal{P})$; G-action is deduced from this.

The first example shows that the space X can be trivial:

Example

If π is a unitary representation, then

- $\mathcal{P} = \mathbb{C}$
- $\mathcal{J} = \{0\}$
- $X = \{\mathrm{pt}\}$

In this case, the theorem tells us only that $C^*(\pi(G))$ is a quotient of the full group C*-algebra $C^*(G)$.

It can happen that X is not trivial but the action of G is:

Example

- $G = \mathbb{Z}$
- $T \in \mathcal{B}(\mathcal{H})$ invertible and normal, i.e. $T^*T = TT^*$
- $\pi(n) = T^n$

Then since $C^*(T)$ is commutative, we have

- $\mathcal{P} = C^*(|T|) \subset C^*(T) \implies \mathcal{J} = 0$ and $X = \sigma(|T|)$
- $v_n \in C^*(T) \implies v_n p v_n^* = p\ \forall n, \forall p \in \mathcal{P} \implies \alpha = id.$

However, it turns out that the normality of the operators $\pi(g)$ in the last two examples is essentially the only way to obtain trivial X or trivial G-action:

Theorem

Suppose $[\pi(g)^*, \pi(g)] \notin \mathcal{J}$ for some $g \in G$. Then $\exists x \in X$ such that $g \cdot x \neq x$.

The main example: Fuchsian groups

From now on G will be a Fuchsian group, i.e. a discrete group of conformal automorphisms of the open unit disk $\mathbb{D} \subset \mathbb{C}$.

Examples

- $G = \pi_1(M)$, M a compact Riemann surface of genus $g \geq 2$
- $G \subset PSL(2, \mathbb{Z})$
- $G \cong \mathbb{F}_d$, the free group on d generators

We consider representations of G on the Hardy space:

Definition

$H^2 := \{f = \sum a_n z^n : \sum |a_n^2| < \infty\}$ with $\|f\| = \left(\sum |a_n|^2\right)^{1/2}$

Theorem (Fatou)

If $f \in H^2$ *then*

$$\tilde{f}(e^{i\theta}) := \lim_{r \to 1} f(re^{i\theta})$$

exists a.e. on $\partial\mathbb{D}$ *and* $\|f\|_{H^2} = \|\tilde{f}\|_{L^2(\partial\mathbb{D})}$

H^2 is a reproducing kernel Hilbert space: for each $w \in \mathbb{D}$, the function $k_w(z) := (1 - \overline{w}z)^{-1}$ safisfies

$$\langle f, k_w \rangle = f(w) \qquad \forall f \in H^2$$

We represent G on H^2 by

$$\pi(g)(f)(z) = f(g^{-1}(z))$$

Easy estimates show $\pi(g)$ is bounded $\forall g$, though not uniformly (in fact $\|\pi(g)\| = \frac{1}{1-|g(0)|^2}$).

KEY FACT: For all $g \in G$ and all $w \in \mathbb{D}$,

$$\pi(g)^* k_w = k_{g^{-1}(w)}$$

It follows that if we fix a base point $z_0 \in \mathbb{D}$ then the subspace

$$M = \overline{span}\{k_{g(z_0)} : g \in G\} \subset H^2$$

is invariant for the operators $\pi(g)^*$. We thus get a representation $\tilde{\pi}$ on M.
Finally, let

$$\Lambda = \overline{\{g(z_0) : g \in G\}} \cap \partial\mathbb{D}$$

be the limit set of G; Λ is closed and G-invariant. It is either finite, a Cantor set, or all of $\partial\mathbb{D}$.

Theorem (J. 2005)

For the representations $\tilde{\pi}$ we have:

- $\mathcal{P}=$ *the compression of the Toeplitz algebra* $\mathcal{T}$ *to* M
- $\mathcal{J} = \mathcal{K}(M)$
- $X = \Lambda$ *with the natural* G*-action*

But we can say more...

Combining the previous theorem with our main theorem says that there is an ideal $\mathcal{I} \subset C(\Lambda) \times_\alpha G$ such that

$$C^*(\tilde{\pi}(G))/\mathcal{K} \cong (C(\Lambda) \times_\alpha G)/\mathcal{I}$$

But it is known that the crossed product $C(\Lambda) \times_\alpha G$ is simple; i.e. has no non-trivial ideals. We then deduce:

Theorem

For a non-elementary Fuchsian group G, there is an exact sequence of C-algebras*

$$0 \to \mathcal{K}(M) \to C^*(\tilde{\pi}(G)) \to C(\Lambda) \times_\alpha G \to 0$$

It can be shown that this exact sequence is semisplit, i.e. the quotient map admits a completely positive section. It follows that $\tilde{\pi}$, via this construction, determines a class

$$[\tilde{\pi}] \in K_1(C(\Lambda) \times G)$$

in the (odd) analytic K-homology of the crossed product $C(\Lambda) \times G$. (With more work we can show that we actually get a class in the equivariant Kasparov group $KK^1_G(C(\Lambda), \mathbb{C})$).

Sketch of proof

When $M = H^2$, the proof rests on the fact that $\mathcal{P}$ is equal to the Toeplitz algebra $\mathcal{T} = C^*(M_z)$.
Using the fundamental identity

$$(1 - |g(0)|^2)\pi(g)\pi(g)^* = 1 - \overline{g(0)}M_z + g(0)M_z^* + |g(0)|^2 M_z M_z^*$$

and choosing a sequence $g_n(0) \to \lambda \in \partial\mathbb{D}$ shows that $\mathcal{P}$ contains $\overline{\lambda}M_z + \lambda M_z^*$ for all $|\lambda| = 1$ and hence M_z. The identification $\mathcal{J} = \mathcal{K}(H^2)$ follows, and we can identify the G-action using the fact that

$$M_{g^{-1}(z)}\pi(g) = \pi(g)M_z$$

When $M \neq H^2$ (e.g. when the limit set of G is a Cantor set) we must replace M_z with $P_M M_z P_M$, use Beurling's theorem to write

$$M = \Theta H^2, \qquad \Theta = \text{ Blaschke prod. with zeroes on orbit of the origin}$$

to get $P_M = I - M_\Theta M_\Theta^*$. From here we need to know

$$[M_\Theta, M_z^*] \in \mathcal{K}(H^2) \qquad \text{(Hartman's theorem)}$$

and use the Livšic-Moeller theorem to prove

$$\sigma_e(P_M M_z P_M) = \Lambda$$