VALIRON'S THEOREM IN THE UNIT BALL AND SPECTRA OF COMPOSITION OPERATORS

MICHAEL T. JURY

Abstract. We prove a version of Valiron's conjugacy theorem for Schur class mappings of the unit ball of $\mathbb{C}^N$. As an application we obtain a formula for the spectral radius of composition operators on the ball with Schur class symbols.

1. Introduction

The purpose of this paper is to prove a version of the Valiron semi-conjugacy theorem for Schur class maps of the unit ball in $\mathbb{C}^N$.

Definition 1.1. *A holomorphic map $f : \mathbb{B}^N \to \mathbb{B}^N$ belongs to the* Schur class *if the Hermitian kernel*

$$\frac{1 - \langle f(z), f(w)\rangle}{1 - \langle z, w\rangle}$$

is positive semidefinite.

In dimension 1, (the case of the unit disk $\mathbb{D}$) every holomorphic map $\varphi : \mathbb{D} \to \mathbb{D}$ belongs to the Schur class, but for self-maps of the ball the inclusion is strict (that is, not every self-map of $\mathbb{B}^N$ is Schur class). However, in many respects Schur class maps are better behaved than generic self-maps of the ball; for example they satisfy a version of the Nevanlinna-Pick interpolation theorem **[1]** and Littlewood's subordination theorem **[6]**, as well as a version of the Julia-Caratheodory theorem that is stronger than what is true generically **[8]**.

To state the Valiron semi-conjugacy theorem, we first recall that for every holomorphic self-map ψ of the ball, there exists a unique point $\zeta \in \overline{\mathbb{B}^N}$ such that the iterates $\psi_n := \psi \circ \cdots \circ \psi$ (n times) converge to the constant function ζ uniformly on compact subsets of $\mathbb{B}^N$ **[9]**. Moreover, when ζ lies on the boundary, we have

$$0 < \liminf_{z \to \zeta} \frac{1 - |\psi(z)|^2}{1 - |z|^2} = \alpha \leq 1$$

Date: July 3, 2009.
Partially supported by NSF grant DMS-0701268.

This number α is called the *dilatation coefficient* of ψ at ζ, and is equal to the directional angular derivative of ψ in the ζ direction; the precise formulation is given by the Julia-Caratheodory theorem (see [**10**, Section 8.5] for the general case, and [**8**] for a strengthening when ψ is Schur class). When $\alpha < 1$, the map ψ is called *hyperbolic.* For this α, consider the Möbius transformation

$$\theta_\alpha(z) = \frac{z + \left(\frac{1-\alpha}{1+\alpha}\right)}{1 + \left(\frac{1-\alpha}{1+\alpha}\right) z}$$

Our goal is the following theorem:

Theorem 1.2 (Valiron semi-conjugacy)**.** *Let $\psi : \mathbb{B}^N \to \mathbb{B}^N$ be a hyperbolic Schur class mapping with Denjoy-Wolff point $e_1 = (1, 0, \dots 0)$ and dilatation coefficient α. Then there exists a non-constant Schur class map $\tau : \mathbb{B}^N \to \mathbb{D}$ such that*

$$\tau \circ \psi = \theta_\alpha \circ \tau$$

Roughly, the theorem says there exists a local coordinate τ such that the hyperbolic map ψ is modeled by a hyperbolic automorphism of the disk, whose first-order behavior near its Denjoy-Wolff point agrees with that of ψ (at least in the τ direction). In the case $N = 1$, every self-map of the disk is Schur class, and the above result reduces to the original theorem of Valiron. Several different proofs are available; we refer to [**5**] for background and a good survey of the 1-dimensional case. A closely related version of the Valiron semi-conjugacy theorem for $N > 1$ was proved recently in [**4**]. Instead of Schur class maps, the authors consider maps satisfying two hypotheses (see the next section for the relevant definitions):

1) If ζ is the Denjoy-Wolff point of ψ, then
$$\frac{1 - \langle \psi(z), \zeta \rangle}{1 - \langle z, \zeta \rangle}$$
has a finite K-limit (necessarily equal to α) at ζ.
2) The map ψ has a *special orbit.*

By the results of [**8**], hypothesis (1) is always satisfied by hyperbolic Schur class maps (it need *not* be satisfied by arbitrary hyperbolic maps). At this point, we do not know if (2) is always satisfied in the Schur class. (We prove in Section 4 that (2) is always satisfied by linear fractional maps (which are always Schur class), but even this result seems non-trivial.) Our proof of Theorem 1.2 follows closely that of [**4**], but requires some modifications in the absence of hypotheses (2). As a corollary to the semi-conjugacy theorem (which is the main reason for the interest in proving it in the Schur class), we obtain a theorem on the spectral radius of Schur-class composition operators (Theorem 3.2), which answers the main open question of [**7**].

The paper is organized as follows: in the next section we introduce the relevant background material and prove the semi-conjugacy theorem; our proof follows the lines of [**4**] (but without hypothesis (2)). Section 3 proves the spectral radius result for composition operators, and the final section shows that hypothesis (2) is satisfied by linear fractional maps.

2. Valiron semi-conjugacy

To state the Valiron theorem, we require a few preliminaries.

2.1. **Positive kernels and the Schur class.** A *positive kernel* on $\mathbb{B}^N$ is a function $K : \mathbb{B}^N \times \mathbb{B}^N \to \mathbb{C}$ with the property that

$$\sum_{i,j=1}^{n} c_i \overline{c_j} K(x_i, x_j) \geq 0$$

for all $n \geq 1$, all choices of scalars $c_1, \dots c_n$ and all choices of n points $x_1, \dots x_n$ in $\mathbb{B}^N$. Given a holomorphic function $f : \mathbb{B}^N \to \mathbb{B}^M$, we say f belongs to the Schur class $\mathcal{S}(N, M)$ if and only if the kernel

$$\frac{1 - \langle f(z), f(w)\rangle_{\mathbb{C}^M}}{1 - \langle z, w\rangle_{\mathbb{C}^N}} \tag{2.1}$$

We will only need the cases $M = N$ and $M = 1$. The kernels (2.1) defining the Schur class arise in the study of certain reproducing kernel Hilbert spaces on the ball (called the Drury-Arveson spaces) and membership in the Schur class is equivalent to being a contractive multiplier between (vector-valued versions of) these spaces [**1**]. Since a pointwise limit of positive kernels is positive, it is evident that a pointwise limit of Schur class mappings belongs to the Schur class. Since Schur class maps are of course bounded, it follows that the Schur class is compact in the topology of pointwise convergence. We shall also require the elementary fact that if f, g are Schur class maps then $f \circ g$ is also Schur class. Moreover the Schur class contains all linear fractional maps of the ball, and in particular all automorphisms (see [**7**] for these facts).

2.2. **Geometry in $\mathbb{H}^N$.** It will be convenient to privilege the first coordinate in $\mathbb{C}^N$, so we will write general points in $\mathbb{C}^N$ as (z, w) with $z \in \mathbb{C}$, $w \in \mathbb{C}^{N-1}$. We write $\pi_1(z, w) = z$ for the first coordinate mapping, and if f is a function taking values in $\mathbb{C}^N$ we write $f_1 := \pi_1 \circ f$.

The *Siegel half-space* is the domain

$$\mathbb{H}^N = \{(z, w) \in \mathbb{C}^N : \mathrm{Re}\ z > \|w\|^2\}$$

The Siegel half-space is mapped biholomorphically onto the unit ball $\mathbb{B}^N$ by

$$(z, w) \to \left(\frac{z-1}{z+1}, \frac{2w}{z+1}\right) \tag{2.2}$$

The point at infinity is taken to the boundary point $e_1 = (1, 0)$ under this map.

The Siegel half-space is equipped with a distance function (in fact a metric), the *Kobayashi distance*, defined by

$$k_{\mathbb{H}^N}((z_1, w_1), (z_2, w_2)) = \sup \rho(f(z_1, w_1), f(z_2, w_2))$$

where ρ is the hyperbolic distance in the unit disk, and the supremum is taken over all holomorphic maps $f : \mathbb{H}^N \to \mathbb{D}$. By definition, this distance is invariant under biholomorphic mappings, and is contracted by arbitrary holomorphic self-maps of $\mathbb{H}^N$.

For many purposes, the correct analogue in the ball of the non-tangential limit of a function in the disk is not a non-tangential limit but a *K-limit*. To define it, we first introduce the Koranyi regions in the ball. These are sets of the form

$$K(\zeta, c) := \{z \in \mathbb{B}^N : |1 - \langle z, \zeta\rangle| \le c(1 - |z|^2)\} \tag{2.3}$$

for fixed $\zeta \in \partial\mathbb{B}^N$ and $c > 0$. A function f in the ball has *K-limit* equal to L at ζ if $\lim f(z) = L$ whenever z approaches ζ within a Koranyi region $D_c(\zeta)$. Transporting these notions to $\mathbb{H}^N$, we find that the Koranyi regions with vertex at ∞ are regions of the form

$$K(\infty, M) := \left\{(z, w) \in \mathbb{H}^N : \|w\|^2 < \mathrm{Re}\ z - \frac{|z+1|}{M}\right\}. \tag{2.4}$$

and K-limits at ∞ are defined in the obvious way.

Definition 2.1. *Let $(z_n, w_n) \to \infty$ in $\mathbb{H}^N$.*

a) *The convergence will be called C-*special *if there is a constant $0 \le C < \infty$ such that*

$$\limsup_{n\to\infty} k_{\mathbb{H}^N}((z_n, w_n), (z_n, 0)) = C$$

b) *The convergence is* restricted *if the sequence (z_n) converges to ∞ non-tangentially in $\mathbb{H}$.*

A sequence that is 0-special is called simply *special*. We also note that $(z_n, w_n) \to \infty$ within a Koranyi region if and only if it is restricted, and C-special for some C [**4**, Lemma 2.4].

Lemma 2.2. *Let $\varphi : \mathbb{H}^N \to \mathbb{H}^N$ be a hyperbolic map with Denjoy-Wolff point ∞ and multiplier $\lambda = \frac{1}{\alpha} > 1$. Write*

$$\varphi^n(1,0) = (z_n, w_n)$$

with $z_n = x_n + iy_n \in \mathbb{C}$ and $w_n \in \mathbb{C}^{N-1}$. Suppose that the K-limit

$$\lim_{(z,w)\to\infty} \frac{\varphi_1(z,w)}{z}$$

exists. Then:

a) *There exists a constant $C_1 \geq 0$ such that*

$$|y_n| \leq C_1 |x_n|$$

for all n.

b) *There exists a constant $\mu < 1$ such that*

$$\frac{\|w_n\|}{\sqrt{x_n}} \leq \mu$$

for all n.

c) $\displaystyle\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda.$

Proof. Items (a) and (b) are immediate consequences of the fact that $(z_n, w_n) \to \infty$ within a Koranyi region. Item (c) is our version of [[**4**], Lemma 3.3(1)]; however we note it follows just from the K-limit hypothesis; we do not need to assume a special orbit. Since the orbit (z_n, w_n) lies in a Koranyi region, we have by the K-limit assumption

$$\lim_{n\to\infty} \frac{z_{n+1}}{z_n} = \frac{\pi_1 \circ \varphi(z_n, w_n)}{z_n} = \lambda.$$

(It follows from Rudin's version of the Julia-Caratheodory theorem that if the above K-limit exists, it must equal λ.) The claim (c) now follows as in [**4**]: write $z_{n+1} = \lambda z_n + o(1)z_n$. Dividing by x_n, taking real parts, and passing to the limit, we get (c). $\square$

The theorem we now prove differs slightly from what was announced in the introduction; at this point we do not assume that φ is Schur class, but only the existence of the K-limit of $\varphi_1(z)/z$ at infinity. The existence of the intertwiner σ can be deduced solely from this. As a corollary we show that if φ is Schur class then the intertwiner produced by the theorem is also Schur class.

Theorem 2.3 (Valiron semi-conjugacy, Siegel half-space version)**.** *Let $\varphi : \mathbb{H}^N \to \mathbb{H}^N$ be a hyperbolic mapping with Denjoy-Wolff point ∞ and* *dilatation coefficient* λ*, and suppose the* K*-limit*

$$\lim_{(z,w)\to\infty} \frac{\varphi_1(z,w)}{z}$$

exits. Then there exists a non-constant holomorphic map $\sigma : \mathbb{H}^N \to \mathbb{H}$ *such that*

$$\sigma \circ \varphi = \lambda \sigma \tag{2.5}$$

Proof. Fix an *arbitrary* point $(z,w) \in \mathbb{H}^N$. We define the Valiron-like sequence

$$\sigma_n(z,w) = \frac{\pi_1 \circ \varphi^n(z,w)}{x_n}$$

Now

$$\sigma_n \circ \varphi = \frac{\pi_1 \circ \varphi^{n+1}(z,w)}{x_n} = \frac{x_{n+1}}{x_n}\sigma_{n+1} \tag{2.6}$$

We will prove that there exists a subsequence $\{\sigma_{n_k}\}$ and a map σ such that both $\{\sigma_{n_k}\}$ and $\{\sigma_{n_k+1}\}$ converge to σ.

We first observe that if σ is any subsequential limit of $\{\sigma_n\}$, then σ is not constant. Indeed, we have

$$\text{Re } \sigma_n(1,0) = 1 \quad \text{ for all } n,$$

while

$$\text{Re } \sigma_n(\varphi(1,0)) = \frac{x_{n+1}}{x_n} \to \lambda > 1 \quad \text{ as } n \to \infty.$$

Next we show that

$$\lim_{n\to\infty} k_{\mathbb{H}^N}(\sigma_n(z,w), \sigma_{n+1}(z,w)) = 0. \tag{2.7}$$

To see this, we first show that for each $(z,w) \in \mathbb{H}^N$, the set $\{\sigma_n(z,w)\}$ is precompact in $\mathbb{H}^N$. For each n let L_n denote the automorphism of $\mathbb{H}^N$ defined by

$$L_n(z,w) = \left(\frac{z}{x_n}, \frac{w}{\sqrt{x_n}}\right)$$

Let $S_n(z,w) = L_n \circ \varphi^n(z,w)$. For $\epsilon > 0$, define

$$D_\epsilon = \{(z,w) \in \mathbb{H}^N : \text{Re } z - \|w\|^2 \geq \epsilon\}.$$

Now consider the sequence $\{S_n(1,0)\}$. We have by definition

$$S_n(1,0) = \left(\sigma_n(1,0), \frac{w_n}{\sqrt{x_n}}\right) = \left(1 + i\frac{y_n}{x_n}, \frac{w_n}{\sqrt{x_n}}\right)$$

It follows from Lemma 2.2(b) that $\{S_n(1,0)\}$ lies in D_ϵ for all n, with $\epsilon = 1 - \mu^2$. On the other hand, from Lemma 2.2(a) and (b) we see that $\{S_n(1,0)\}$ is bounded. Since the intersection of any D_ϵ with any closed, bounded subset of $\mathbb{C}^N$ is compact in $\mathbb{H}^N$, it follows that $\{S_n(1,0)\}$ is precompact in $\mathbb{H}^N$.

Since $\{S_n(1,0)\}$ is precompact, there exists a constant $C < \infty$ such that

$$k_{\mathbb{H}^N}(S_n(1,0),(1,0)) \leq C$$

for all n. Thus for any fixed $(z,w) \in \mathbb{H}^N$, we have

$$k_{\mathbb{H}^N}(S_n(z,w),(1,0)) \leq k_{\mathbb{H}^N}(S_n(z,w),S_n(1,0)) + k_{\mathbb{H}^N}(S_n(1,0),(1,0)) \tag{2.8}$$

$$\leq k_{\mathbb{H}^N}((z,w),(1,0)) + C \tag{2.9}$$

by the triangle inequality and the fact that holomorphic maps contract the Kobayashi distance. It follows that for any (z,w), the sequence $\{S_n(z,w)\}$ is precompact in $\mathbb{H}^N$, and hence $\{\sigma_n(z,w)\}$ is precompact in $\mathbb{H}$.

With this precompactness result established, to prove (2.7) it suffices to prove that

$$\rho_n := \pi_1 \circ L_{n+1} \circ \varphi \circ L_n^{-1} \to \pi_1 \quad \text{as } n \to \infty.$$

This follows exactly as in [**4**]; we have

$$\rho_n(z,w) = \frac{\pi_1(\varphi(x_n z, \sqrt{x_n} w))}{x_n z} \frac{x_n z}{x_{n+1}}.$$

Observe that the sequence $(x_n z, \sqrt{x_n} w)$ is C-special and restricted: indeed, by [**4**, Equation (2.2)]

$$k_{\mathbb{H}^N}(x_n z, \sqrt{x_n} w), (x_n z, 0)) = \tanh^{-1} \frac{\|w\|}{\sqrt{\text{Re } z}} < \infty$$

Thus, $(x_n z, \sqrt{x_n} w)$ approaches ∞ within a Koranyi region; hence by Lemma (2.2)(c) and (d) we get $\rho_n(z,w) \to z$.

Finally, if we let σ_{n_k} be any convergent subsequence of σ_n, by (2.7) the sequence σ_{n_k+1} has the same limit σ. Taking the limit along the subsequence n_k in (2.6) and applying Lemma 2.2(c), we see that this σ solves the Schroeder equation (2.5). □

Corollary 2.4. *Let ψ be a hyperbolic Schur class mapping of $\mathbb{B}^N$, with dilatation coefficient α. Then there exists a non-constant Schur class mapping $\tau : \mathbb{B}^N \to \mathbb{D}$ such that*

$$\tau \circ \psi = \theta_\alpha \circ \tau.$$

Proof. By [**8**], Schur class maps satisfy the K-limit hypothesis, and so the intertwiner τ exists by conjugating Theorem 2.3 back to the ball. To see that τ belongs to the Schur class, we return to $\mathbb{H}^N$ and refer to the maps σ_n in the proof of Theorem 2.3. To see that each σ_n belongs to the Schur class (that is, is conjugate to a Schur class map of the ball), observe that we can write

$$\sigma_n = \pi_1 \circ L_n \circ \varphi^n$$

Since π_1, L_n and φ all belong to the Schur class, σ_n does as well, since the Schur class is closed under composition. Since the Schur class is also closed under pointwise limits, the intertwiner σ is Schur class. □

Several remarks about the proof are in order. First, we cannot quite claim that "Valiron's construction works" in the sense of [**4**], since our proof does not show that the sequence σ_n is convergent; we can extract only a convergent subsequence. On the other hand, the above proof works for *any* orbit of φ; we do not need to assume it is special. Indeed the existence of such orbits (even for Schur class maps) is still an open question; though we show in the final section that linear fractional maps always admit such an orbit. Moreover we do not seem to be able to deduce any reasonable smoothness at infinity for our intertwiner σ.

3. Spectra of composition operators

The existence of the Valiron map τ (belonging to the Schur class) allows us to prove Theorem 3.2 below on the spectrum of the composition operator C_ψ, which answers the main open question of [**7**]. For integers $\beta \geq 1$, we denote by $H^2_{N,\beta}$ the Hilbert space of holomorphic functions on $\mathbb{B}^N$ with reproducing kernel

$$K^\beta(z,w) = \frac{1}{(1-\langle z,w\rangle)^\beta}$$

We first show that Schur class mappings from the ball to the disk induce bounded composition operators between appropriate pairs of Hilbert function spaces. The following is a simple generalization (for maps from the ball to the disk, rather than self-maps of the ball) of the main boundedness result of [**6**]:

Proposition 3.1. *Suppose $\tau : \mathbb{B}^N \to \mathbb{D}$ belongs to the Schur class; that is, the kernel*

$$\frac{1-\tau(z)\overline{\tau(w)}}{1-\langle z,w\rangle}$$

is positive semidefinite. Then the map $C_\tau f := f\circ\tau$ defines a bounded operator from $H^2_{1,\beta}$ to $H^2_{N,\beta}$.

Proof. It is easy to verify that the space $H^2_{1,\beta}$ is Möbius invariant, so we may assume $\tau(0) = 0$. The kernel

$$\left(\frac{1-\tau(z)\overline{\tau(w)}}{1-\langle z, w\rangle}\right)^\beta$$

is positive semidefinite (since τ is Schur class and β is a positive integer); since $\tau(0) = 0$ the kernel

$$\left(\frac{1-\tau(z)\overline{\tau(w)}}{1-\langle z, w\rangle}\right)^\beta - 1$$

is also positive, by the Schur complement theorem. Now let

$$K_w(z) = \frac{1}{(1-\langle z, w\rangle_N)^\beta}, \quad k_w(z) = \frac{1}{(1-z\overline{w})^\beta}$$

be the reproducing kernels for $H^2_{N,\beta}$ and $H^2_{1,\beta}$ respectively. A simple calculation shows that $C_\tau^* K_w = k_{\tau(w)}$, and thus

$$\begin{aligned}
\langle (I - C_\tau C_\tau^*)K_w, K_z\rangle_{H^2_{N,\beta}} &= \langle K_w, K_z\rangle_{H^2_{N,\beta}} - \langle k_{\tau(w)}, k_{\tau(z)}\rangle_{H^2_{1,\beta}} \\
&= \frac{1}{(1-\langle z, w\rangle)^\beta} - \frac{1}{(1-\tau(z)\overline{\tau(w)})^\beta} \\
&= \frac{1}{(1-\tau(z)\overline{\tau(w)})^\beta}\left(\left(\frac{1-\tau(z)\overline{\tau(w)}}{1-\langle z, w\rangle}\right)^\beta - 1\right).
\end{aligned}$$

Since the kernel on the last line is positive, it follows that $I - C_\tau C_\tau^*$ is a positive operator, so $\|C_\tau\| \leq 1$. □

Theorem 3.2. *Let ψ be a hyperbolic Schur class map of $\mathbb{B}^N$ with dilatation coefficient α. Then the spectral radius of C_ψ acting on $H^2_{N,\beta}$ is $\alpha^{-\beta/2}$. Moreover every complex number λ in the annulus*

$$\alpha^{\beta/2} < |\lambda| < \alpha^{-\beta/2}$$

is an eigenvalue of C_ψ of infinite multiplicity.

Proof. By [**7**, Corollary 5], the spectral radius of C_ψ on $H^2_{N,\beta}$ is

$$\lim_{n\to\infty} (1 - |\psi(z_n)|)^{-\beta/2n} \tag{3.1}$$

where $(z_n) = \varphi^n(z_0)$ is any orbit of ψ. Using this fact, the inequality

$$r(C_\psi) \leq \alpha^{-\beta/2} \tag{3.2}$$

is a straightforward consequence of the Julia-Caratheodory theorem: if (z_n) is any orbit of φ, then we have

$$\liminf_{n\to\infty} \frac{1-|\psi(z_{n+1})|}{1-|\psi(z_n)|} \geq \alpha. \tag{3.3}$$

It follows that

$$\begin{aligned}\lim_{n\to\infty} (1-|\psi(z_n)|)^{1/n} &= \lim_{n\to\infty} \left(\prod_{k=0}^{n-1} \frac{1-|\psi(z_{k+1})|}{1-|\psi(z_k)|}\right)^{1/n} \\ &\geq \alpha \quad \text{by (3.3)},\end{aligned}$$

which proves (3.2).

For the reverse inequality, it suffices to prove the claim about the eigenvalues; for this use the Valiron conjugacy τ to transfer eigenfunctions of C_{θ_α} to C_ψ. Formally, if F is a holomorphic function satisfying $F\circ\psi_\alpha = \lambda F$ for some λ, then by the Valiron conjugacy we have

$$F\circ\tau\circ\psi = F\circ\theta_\alpha\circ\tau = \lambda F\circ\tau$$

So $F\circ\tau$ is a holomorphic eigenfunction for C_ψ; it only needs to be checked that for λ in the asserted range, we can find F (indeed infinitely many such) such that $F\circ\tau$ belongs to $H^2_{N,\beta}$. This will depend crucially on the fact that the conjugacy τ belongs to the Schur class.

Indeed, since $\tau:\mathbb{B}^N\to\mathbb{D}$ belongs to the Schur class, it will induce a bounded composition operator from $H^2_{1,\beta}$ to $H^2_{N,\beta}$. For each real s in the range $-\beta/2 < s < \beta/2$ and all real t, the function

$$F(z) = \exp\left[(s+it)\log\left(\frac{1+z}{1-z}\right)\right]$$

belongs to $H^2_{1,\beta}$ and satisfies $F\circ\theta_\alpha = \exp(-(s+it)\log\alpha)F$. By the boundedness of C_τ, the computation above proves that $F\circ\tau$ is an eigenfunction of C_ψ. For each λ in the annulus $\mathbb{A}_\beta$, there are infinitely many $s+it$ such that $\lambda = \exp(-(s+it)\log\alpha)$. Thus λ is an eigenvalue of C_ψ of infinite multiplicity. In particular, $\lambda\in\sigma(C_\psi)$, so $r(C_\psi)\geq \alpha^{-\beta/2}$. □

Remark: As noted in the proof, the spectral radius of C_ψ is a certain power of

$$\lim_{n\to\infty}(1-|\psi^n(z_0)|)^{1/n}; \tag{3.4}$$

we may take as a corollary of the theorem that this limit exists and equals the dilatation coefficient α. However we do not know a "direct" proof of this fact, even in the Schur class. More than that, we do not know if this fact is true for maps outside the Schur class; it is not even clear that the limit (3.4) exists for general hyperbolic maps.

Question 3.3. *Let ψ be a hyperbolic self-map of $\mathbb{B}^N$ with dilatation coefficient α. Is it always the case that the limit (3.4) exists and equals α?*

The answer to the question is "yes" if ψ is Schur class, or if ψ has a special orbit.

4. Special orbits for linear fractional maps

As has already been mentioned, the question of whether or not every hyperbolic self-map of the ball admits a special orbit remains open, even if we also insist that the map be Schur class. The existence of special orbits for maps satisfying certain smoothness conditions at the Denjoy-Wolff point has been established by Bracci and Gentili in [**3**]. However this is deduced only from a stronger Valiron-type intertwining theorem at the Denjoy-Wolff point, so it would be of interest to find a direct proof that did not already appeal to the existence of an intertwiner. We give such a proof in this section in the case of linear fractional maps. Even for these very special maps the result does not seem trivial, and hopefully will shed some light on the general case. It is easy to see that every hyperbolic automorphism has a special orbit, since each such map admits an invariant slice. However there exist hyperbolic linear fractional maps which do not possess invariant slices, so the existence of special orbits for these maps seems to be a more delicate matter.

Our construction of special orbits requires the following "shadowing lemma" for orbits of contractive linear maps T of $\mathbb{C}^m$; roughly it says that any sequence of vectors (y_n), which behaves asymptotically like an orbit of T, is "shadowed" by a genuine orbit $(T^n x)$.

Lemma 4.1 (Shadowing Lemma)**.** *Let T be a contractive $m \times m$ matrix. Suppose (y_n) is a sequence of vectors in $\mathbb{C}^m$ such that*

$$\sum_{n=1}^{\infty} \|y_{n+1} - Ty_n\| < \infty. \tag{4.1}$$

Then there exists a vector $x \in \mathbb{C}^m$ such that

$$\lim_{n\to\infty} \|y_n - T^n x\| = 0.$$

Proof. Define $a_n := \|y_{n+1} - Ty_n\|$. We first observe that for all n and $k \geq 1$,

$$\|y_{n+k} - T^k y_n\| \leq \sum_{j=n}^{n+k-1} a_j \tag{4.2}$$

Indeed, this holds for $k = 1$ by definition. We proceed by induction: supposing the result holds for all n and $1 \leq k \leq m$, we have

$$\|y_{n+m+1} - T^{m+1} y_n\| = \|y_{n+m+1} - Ty_{n+m} + Ty_{n+m} - T^{m+1} y_n\| \tag{4.3}$$

$$\leq \|y_{n+m+1} - Ty_{n+m}\| + \|y_{n+m} - T^m y_n\| \tag{4.4}$$

$$\leq \sum_{j=n}^{n+m} a_j \tag{4.5}$$

where the first inequality uses the fact that $\|T\| \leq 1$.

We now reduce to the case where T is unitary. Suppose λ is an eigenvalue of T with $|\lambda| = 1$ and eigenvector v. An elementary linear algebra argument shows that since $\|T\| \leq 1$, the space $\{v\}^\perp$ must be reducing for T. By induction, it follows that T can be decomposed into an orthogonal direct sum $T = U \oplus V$, where U is a unitary matrix, and V has spectral radius $r(V) < 1$. Decompose $y_n = s_n \oplus t_n$ across these subspaces. Then

$$\|y_{n+1} - Ty_n\|^2 = \|s_{n+1} - Us_n\|^2 + \|t_{n+1} - Vt_n\|^2$$

Now it follows from (4.2) that for each fixed m,

$$\lim_{n\to\infty} \|t_{n+m} - V^m t_n\| = 0$$

On the other hand, $V^m \to 0$ as $m \to \infty$ since $r(V) < 1$. It follows that $t_n \to 0$. Thus if there exists a vector s such that

$$\|s_n - U^n s\| \to 0$$

then setting $x = s \oplus 0$ works. The problem is thus reduced to the case where $T = U$ is unitary.

Let $\epsilon > 0$. Let N be an integer such that

$$\sum_{j=N}^{\infty} a_j < \epsilon.$$

Then for all $m \geq n \geq N$, we have by (4.2) (and the assumption that T is unitary)

$$\|T^{-m} y_m - T^{-n} y_n\| = \|y_m - T^{m-n} y_n\| < \epsilon$$

This proves that the sequence $T^{-n}y_n$ is Cauchy, and hence converges to a vector x; using again the assumption that T is unitary we conclude $\|y_n - T^n x\| \to 0$. $\square$

Theorem 4.2. *Every hyperbolic linear fractional map of* $\mathbb{B}^n$ *has a special orbit.*

Proof. We will work on the Siegel upper half-space, where every hyperbolic linear fractional map of the ball is conjugate to a map of the form

$$\varphi(z,w) = \frac{1}{\alpha}(z + c + \langle w, b\rangle, Aw + d)$$

for scalar c, vectors $b, d \in \mathbb{C}^{n-1}$, and $(n-1)\times(n-1)$ matrix A. These four parameters satisfy the conditions derived in [**2**], in particular $\|A\| \le \sqrt{\alpha} < 1$. Letting $(z_n, w_n) := \varphi^n(z,w)$, to prove the existence of a special orbit amounts to finding (z,w) such that

$$\lim_{n\to\infty} \frac{\|w_n\|^2}{\operatorname{Re} z_n} = 0.$$

We will obtain fairly explicit expressions for z_n and w_n in general, and then show that the above is satisfied for suitable choices of z and w.

A straightforward induction shows that (z_n, w_n) may be expressed as

$$(z_n, w_n) = \frac{1}{\alpha^n}(z + c_n + \langle w, b_n\rangle, A_n w + d_n)$$

where the parameters A_n, b_n, c_n, d_n satisfy the recursion formulas

$$c_{n+1} = c_n + \alpha^n c + \langle d_n, b\rangle \tag{4.6}$$

$$b_{n+1} = b_n + A_n^* b \tag{4.7}$$

$$A_{n+1} = AA_n \tag{4.8}$$

$$d_{n+1} = Ad_n + \alpha^n d \tag{4.9}$$

It is evident that $A_n = A^n$, and this is the only recurrence we will need to solve explicitly. W e must find a point $(z,w) \in \mathbb{H}^n$ so that

$$\lim_{n\to\infty} \frac{1}{\alpha^n} \frac{\|A^n w + d_n\|^2}{\operatorname{Re}(z + c_n + \langle w, b_n\rangle)} = 0. \tag{4.10}$$

We will choose w first, then a suitable z. Let $T = \frac{1}{\sqrt{\alpha}}A$; note $\|T\| \le 1$. We also define

$$e_n = \frac{1}{\sqrt{\alpha}^n} d_n$$

and using (4.9) we see that e_n satisfies the recursion

$$e_{n+1} = Te_n + \sqrt{\alpha}^{n-1} d \tag{4.11}$$

The expression (4.10) is then equivalent to

$$\lim_{n\to\infty} \frac{\|T^n w + e_n\|^2}{\mathrm{Re}(z + c_n + \langle w, b_n\rangle)} = 0. \tag{4.12}$$

Now observe that T and e_n satisfy the hypotheses of the Shadowing Lemma: indeed, by (4.11) we have

$$\sum_{n=1}^{\infty} \|Te_n - e_{n+1}\| = \|d\| \sum_{n=1}^{\infty} \sqrt{\alpha}^{n-1} < \infty$$

Thus, there exists a vector $x \in \mathbb{C}^{N-1}$ so that $\|T^n x - e_n\| \to 0$; we put $w = -x$ so the numerator of (4.12) goes to 0. If we now choose any z so that $(z, w) \in \mathbb{H}^N$, the denominator of (4.12) must remain nonnegative; increasing the real part of z if necessary we find that (4.12) holds, and thus the orbit of (z, w) is special. □

References

[1] Joseph A. Ball, Tavan T. Trent, and Victor Vinnikov. Interpolation and commutant lifting for multipliers on reproducing kernel Hilbert spaces. In *Operator theory and analysis (Amsterdam, 1997)*, volume 122 of *Oper. Theory Adv. Appl.*, pages 89–138. Birkhäuser, Basel, 2001.

[2] Filippo Bracci, Manuel D. Contreras, and Santiago Díaz-Madrigal. Classification of semigroups of linear fractional maps in the unit ball. *Adv. Math.*, 208(1):318–350, 2007.

[3] Filippo Bracci and Graziano Gentili. Solving the Schröder equation at the boundary in several variables. *Michigan Math. J.*, 53(2):337–356, 2005.

[4] Filippo Bracci, Graziano Gentili, and Pietro Poggi-Corradini. Valiron's construction in higher dimension. *Rev. Mat. Iber.*, to appear.

[5] Filippo Bracci and Pietro Poggi-Corradini. On Valiron's theorem. In *Future trends in geometric function theory*, volume 92 of *Rep. Univ. Jyväskylä Dep. Math. Stat.*, pages 39–55. Univ. Jyväskylä, Jyväskylä, 2003.

[6] Michael T. Jury. Reproducing kernels, de Branges-Rovnyak spaces, and norms of weighted composition operators. *Proc. Amer. Math. Soc.*, 135(11):3669–3675 (electronic), 2007.

[7] Michael T. Jury. Norms and spectral radii of linear fractional composition operators on the ball. *J. Funct. Anal.*, 254(9):2387–2400, 2008.

[8] Michael T. Jury. An improved Julia-Caratheodory theorem for Schur-Agler mappings of the unit ball. *Complex Variables and Elliptic Equations*, to appear.

[9] Barbara D. MacCluer. Iterates of holomorphic self-maps of the unit ball in $\mathbf{C}^N$. *Michigan Math. J.*, 30(1):97–106, 1983.

[10] Walter Rudin. *Function theory in the unit ball of* $\mathbf{C}^n$, volume 241 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]*. Springer-Verlag, New York, 1980.

Department of Mathematics, University of Florida, Gainesville, Florida 32611-1805

E-mail address: mjury@math.ufl.edu