UFID: ____________________

1. Find a polynomial function that has zeros in $x = -1$,$x = \sqrt{2}$,$x = 3$ and $x = 4$.
Solution : Since this promising polynomial has roots in above points, then it has the form'

$$(x - (-1))(x - \sqrt{2})(x - 3)(x - 4) \tag{1}$$

2. Use long division algorithm to simplify below expression.

$$\frac{x^4 + 9x^3 - 5x^2 - 36x + 4}{x^2 - 4} \tag{2}$$

Solution:

$$x^4 + 9x^3 - 5^2 - 36x + 4 \quad = \quad (x^2 - 4)(x^2 + 9x - 1) \tag{3}$$

3. Use quadratic formula to find the complex roots of the equation $x^2 + 6x + 10 = 0$.
Solution: If we consider our quadratic equation as $ax^2 + bx + c$, then roots x_1 and x_2 will be obtained as follow:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{4}$$

and

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{5}$$

Now, if we consider $a = 1$,$b = 6$, and $c = 10$ then we have

$$x_1 = \frac{-6 + \sqrt{-4}}{2} \tag{6}$$

$$x_2 = \frac{-6 - \sqrt{-4}}{2} \tag{7}$$

Since, $\sqrt{-1} = i$ then one can simplify roots and rewrite them as

$$x_1 = -3 + 2i \qquad\qquad x_2 = -3 - 2i \tag{8}$$

4. Write the quotient in standard form.

$$\frac{1 - 2i}{1 + 2i} \tag{9}$$

Solution :

$$\frac{1 - 2i}{1 + 2i} \quad = \quad \frac{1 - 2i}{1 + 2i} \times \frac{1 - 2i}{1 - 2i} \quad = \quad \frac{(1 - 2i)^2}{(1 + 2i)(1 - 2i)} \tag{10}$$

$$= \quad \frac{1^2 - 2(2i)(1) + (2i)^2}{1^2 - (2i)^2} \tag{11}$$

Here, I use two important identities

$$(a+b)(a-b) = a^2 - b^2 \qquad and (a-b)^2 = a^2 - 2ab + b^2 \tag{12}$$

On the other hand

$$(2i)^2 = (2)^2(i)^2 = -4 \tag{13}$$

Therefore, by considering above facts and putting them in 10 we have

$$\frac{1-2i}{1+2i} \times \frac{1-2i}{1-2i} \quad = \quad \frac{1-4i+(-4)}{1-(-4)} \quad = \frac{-3-4i}{5} \tag{14}$$

5. If $1-\sqrt{3}i$ is the root of $h(x) = 3x^3 - 4x^2 + 8x + 8$,find all other real roots.
Hint: Use the fact that if a polynomial with a complex root has only real coefficients, then the complex conjugate of that root is also a root of that polynomial.

Solution : By using above fact, since all coefficients of the given polynomial are all reals and $1-\sqrt{3}i$ is one of its complex roots, therefore the conjugate of this complex root, $1+\sqrt{3}i$, is the other root of the given polynomial. Hence,

$$(x-(1-\sqrt{3}i))(x-(1+\sqrt{3}i)) \quad = \quad (x-1)^2 - (3i)^2 \quad = \quad x^2-2x+1+3 \quad = \quad x^2-2x+4 \tag{15}$$

divides $h(x) = 3x^3 - 4x^2 + 8x + 8$. Here, I use identity from equation 12 to simplify 15. Just consider $x-1$ as (a) and $\sqrt{3}i$ as (b).Now, if one divides $3x^3 - 4x^2 + 8x + 8$ by $x^2 - 2x + 4$ then

$$\frac{3x^3 - 4x^2 + 8x + 8}{x^2 - 2x + 4} = 3x + 2 \tag{16}$$

Therefore, according to division algorithm

$$3x^3 - 4x^2 + 8x + 8 \quad = \quad (x^2 - 2x + 4)(3x + 2) \tag{17}$$

As we saw $x^2 - 2x + 4$ includes two complex roots of $3x^3 - 4x^2 + 8x + 8$. Therefore, the only real root of this polynomial is the root of

$$3x + 2 = 0 \tag{18}$$

which is

$$x = \frac{-2}{3} \tag{19}$$